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Nonextensivity and Quantum Groups

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Abstract

Classical set theory is nonextensive since the mathematical definition of a set excludes the possibility that more than one copy of the same element can be in a set. We show that the description of random sets in terms of an algebra of creation operators and their hermitean conjugates yields an interpretation of the unitary quantum group $SU_q(d)$ as the symmetry associated with the construction of d random sets from a given source set of $M = (1-q)^{-1}$ elements.

Standard statistical mechanics is extensive due to the large number of particles or subsystems in the system. Thus when two systems are joined, extensive statistical properties such as energy, volume, entropy are added. As an example consider a gas in volume V consisting of N particles and imagine dividing this system into two volumes of $V/2$. Then volume $V/2$ approximately contains $N/2$ particles. Since the fluctuation of this number is given by $\sqrt{N/2}$, as a measure of nonextensivity of the system we obtain

$$nonextensivity \approx \frac{\sqrt{N/2}}{N/2} \sim \frac{1}{\sqrt{N}} \quad . \tag{1}$$

For $N \approx 10^{20}$, $N^{-1/2} \approx 10^{-10} \approx 0$ and for all practical purposes the system is extensive.

Another source of nonextensivity is due to the Heisenberg uncertainity principle of quantum mechanics. Consider a free particle in a cube of volume $V = L^3$. Since the uncertainity in the position of the particle is

$$\Delta x < L \quad , \tag{2}$$

the Heisenberg uncertainity principle

$$\Delta p \, \Delta x \approx \hbar \tag{3}$$

says that the minimum uncertainity Δp in the momentum of the particle is

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$$\Delta p \;\approx\; \frac{\hbar}{L} \qquad . \tag{4}$$

This gives rise to an uncertainty ΔE in the energy of the particle by

$$\begin{aligned} E &= \frac{p^2}{2m} \\ \Delta E &= \frac{p\,\Delta p}{m} \qquad . \end{aligned} \tag{5}$$

For N particles we have

$$\begin{aligned} E_N &= N\,\frac{p^2}{2m} \\ \Delta E_N &= \sqrt{N}\,\Delta E = \sqrt{N}\;\frac{p\,\Delta p}{m} \qquad . \end{aligned} \tag{6}$$

As a measure of nonextensivity in energy we can take

$$nonextensivity \;\approx\; \frac{\Delta E_N}{E_N} = \frac{1}{\sqrt{N}}\,\frac{\Delta p}{p} \qquad . \tag{7}$$

As $T \to 0$ the minimum momentum a particle can have is Δp . This gives

$$nonextensivity \;\stackrel{T\to 0}{\longrightarrow}\; \frac{1}{\sqrt{N}} \qquad . \tag{8}$$

On the other hand at high temperatures $E = kT/2$ and we have

$$nonextensivity \;\approx\; \frac{1}{\sqrt{N}}\,\frac{\hbar}{L}\,\frac{1}{\sqrt{2mE}} \;=\; \frac{1}{\sqrt{N}}\,\frac{\hbar}{L\sqrt{mkT}} \qquad . \tag{9}$$

Thus the nonextensivity is again inversely proportional to the square root of the number of particles.

Admittedly these considerations are rather crude and apparently do not yield a precise mathematical formulation of nonextensivity. The question is whether one can find or define a simple system which is nonextensive and which will yield a mathematically precise definition of this concept. A natural candidate for such a system is classical set theory with the nonextensive property of number of elements in a set. Let A , B denote sets and $m(A)$ denote the number of elements in set A . Then

$$m(A \cup B) = m(A) + m(B) - m(A \cap B) \qquad . \tag{10}$$

Thus if the intersection $A \cap B$ of A and B is not empty the number of elements in a set is nonextensive in the sense that when A and B are joined the number of elements do not add up. However this property is still too complicated since the nonextensivity depends on the detail which elements are common to both A and B . We can get rid of this dependence on the specific elements in A and B by defining random sets with the following properties :

1 . A and B are random subsets of a source set S of M elements.

2. The only relevant property of A or B is the average number of elements in the set.

These definitions endow the M elements of the set S with two properties which we name the fermionic and the bosonic. A set can only have one copy of the same element. This is the fermionic property. On the other hand only the number of elements in S , A or B is relevant, thus in this sense all elements are identical. This is the bosonic property.

Using standard rules of probability and denoting probabilities by the letter P, for any random subset X of S and for any element x of S we have

$$P(x \epsilon X) = \frac{m_X}{M} \tag{11}$$

where m_X is the (average) number of elements in X. If A and B are independent random sets we have

$$P(x \epsilon A \cap B) = P(x \epsilon A).P(x \epsilon B) \tag{12}$$

which yields

$$\frac{m_{A\cap B}}{M} = \frac{m_A}{M}.\frac{m_B}{M} \qquad . \tag{13}$$

Thus for random sets $m_{A\cap B}$ does not depend on the details of the sets A and B but only on the (average) number of elements they contain. Combining (10) and (13) we obtain an operation which we will denote by $*$. The star operation defined by

$$m_{A\cup B} = m_A * m_B \tag{14}$$

satisfies

$$m_A * m_B = m_A + m_B - \frac{1}{M}\, m_A.m_B \qquad . \tag{15}$$

Thus for random sets (10) yields (15) . The $*$ operation defined by (15) for all real numbers m_A , m_B is commutative and associative. For $M = \infty$ it becomes ordinary addition. When random sets are joined their number of elements are “starred”. Equation (15) with M replaced by $(1-q)^{-1}$ is well known in Tsallis generalized thermodynamics [1] where it is the law of “addition” of entropy. The origin of nonextensivity can be traced to [2] the distribution of the free energy over the lattice sites in a system with exact discrete dilatation symmetry.

The simplest nonempty random set contains one element. We can build other random sets by joining such random sets one by one. Denoting the empty random set thus built in n steps by A_n we have

$$m_{A_1} = 1$$
$$m_{A_2} = m_{A_1} * m_{A_1} = 1 * 1 = 1 + 1 - \frac{1}{M} = 1 + q$$
$$m_{A_3} = m_{A_2} * m_{A_1} = 1 * 1 * 1 = 1 + q + q^2$$

$$m_{A_n} = \frac{1 - q^n}{1 - q} \equiv [n] \qquad , \qquad q = 1 - \frac{1}{M} \qquad . \tag{16}$$

Thus the average number of elements in random set A_n built in n steps from the source set S is given by the Jackson [3] basic number $[n]$. The exact probability distribution describing the probability that A_n has m elements can be calculated [4]. It is given by

$$P_{nm}^{(M)} = S_{nm} \frac{M!}{(M - m)! M^n} \tag{17}$$

where $S_{n,m}$ are Stirling numbers of the second kind satisfying the recurrence relation

$$S_{n+1,m} = m S_{nm} + S_{n,m-1} \tag{18}$$

with the initial conditions $S_{00} = 1$, $S_{10} = 0$ and $S_{11} = 1$. The m_{A_n} in (16) is the mean of the random variable $\hat{m}$.

$$m_{A_n} = \mu(\hat{m}) = \sum_{m=0}^{n} m P_{nm}^{(M)} = [n] \qquad . \tag{19}$$

In standard quantum mechanics the random set A_n can be described by a density matrix entailing the probabilities P_{nm} . In this approach the pure states are the (nonrandom) bosonic set of m elements described by the vectors $\mid m >$ in a Fock space created from the vacuum which corresponds to the empty set.

$$\begin{aligned} \hat{m} \mid m > &= m \mid m > \\ a^* \mid m > &= \alpha_m \mid m + 1 > \\ < m' \mid m > &= \delta_{m'm} \\ \hat{m} &= a^* a \end{aligned} \tag{20}$$

where $\hat{m}$ is the number operator and a^* is the creation operator. These relations yield the bosonic commutation relation

$$a a^* - a^* a = 1 \qquad . \tag{21}$$

The mixed state A_n is described by the density matrix

$$\rho^{(n)} = \sum_{m=0}^{n} \mid m > P_{nm} < m \mid \qquad , \tag{22}$$

and the average number of elements in A_n is calculated by

$$m_{A_n} = tr(\hat{m}\rho^{(n)}) = \frac{1-q^n}{1-q} \qquad . \tag{23}$$

The density matrix formalism is the incorporation of standard probability theory in an algebra of operators on a Hilbert space of states and thus will not yield any “physics” beyond what can be obtained by just probability theory alone.

It can be argued that the pure states of the Hilbert space describing random sets should be directly associated with the random sets A_n rather than the sets of definite number of elements. Thus we postulate that corresponding to the random sets A_n there exists a complete set of pure states which we denote by $\mid n >$ and there exists an average number operator μ whose eigenvalues for the states $\mid n >$ are $[n]$. Thus (20) are replaced by

$$\mu \mid n > = \frac{1-q^n}{1-q} \mid n > \tag{24}$$

$$a^* \mid n > = \alpha_n \mid n+1 > \tag{25}$$

$$< n \mid n' > = \delta_{nn'} \tag{26}$$

$$\mu = a^* a \qquad . \tag{27}$$

Taking the hermitian conjugate of (25) and replacing n by $n'-1$ yields

$$< n'-1 \mid a = \bar{\alpha}_{n'-1} < n' \mid \tag{28}$$

$$< n'-1 \mid a \mid n > = \bar{\alpha}_{n'-1}\, \delta_{n'n} \ = \ \bar{\alpha}_{n-1} < n'-1 \mid n-1 >$$

for all n and n' . Hence

$$a \mid n > = \bar{\alpha}_{n-1} \mid n-1 > \tag{29}$$

$$a^* a \mid n > = \mid \alpha_{n-1} \mid^2 \ \mid n > \tag{30}$$

which by (24) and (27) yield

$$\mid \alpha_{n-1} \mid^2 = \frac{1-q^n}{1-q} \qquad . \tag{31}$$

Equations (25), (29) and (31) now completely determine the algebra satisfied by the operators a and a^* by

$$(aa^* - q\, a^* a) \ \mid n> = \ (\mid \alpha_n \mid^2 - q \mid \alpha_{n-1} \mid^2) \mid n> = \mid n> \qquad . \tag{32}$$

Thus

$$aa^* - q\, a^* a = 1 \qquad . \tag{33}$$

This is the q-oscillator [5] which has gained importance in recent years following the discovery of quantum groups [6].

To exhibit the relevance of quantum groups for random sets, let us consider d independent random sets $A^{(1)}, A^{(2)}, \cdots A^{(d)}$ built from the same source set S of $M = (1-q)^{-1}$ elements. The respective creation operators for these random sets will be denoted by $a_1^*, a_2^*, \cdots a_d^*$. They will satisfy the d-dimensional q-oscillator algebra

$$\begin{aligned} a_k a_k{}^* - q\, a_k{}^* a_k &= 1 \\ [\, a_k, a_l \,] &= 0 \\ [\, a_k, a_l{}^*] &= 0 \qquad k \neq l \qquad . \end{aligned} \tag{34}$$

The average number of elements in random set $A^{(k)}$ will be given by the eigenvalues of the operator

$$\mu_k = a_k^* a_k \qquad . \tag{35}$$

Now consider the random set A which the union of the d random sets $A^{(k)}$, $k = 1, 2, \cdots d$.

$$A = A^{(1)} \cup A^{(2)} \cup \cdots \cup A^{(d)} \tag{36}$$

and denote the average number operator of A by μ . Due to the nonextensivity of random sets, μ will not be given by the sum of μ_k . Instead it is given by the formula

$$\mu = \mu_1 + q^{\hat{n}_1} \mu_2 + q^{\hat{n}_1 + \hat{n}_2} \mu_3 + \cdots + q^{\hat{n}_1 + \hat{n}_2 + \cdots + \hat{n}_{d-1}} \mu_d \tag{37}$$

where

$$q^{\hat{n}_k} = 1 - (1-q)\mu_k \qquad . \tag{38}$$

This expression is not unique due to the d ! ways of labelling the d random sets $A^{(k)}$. The possibility of these different labelings can be taken to correspond to in which order the union (36) is taken. Thus the first random set taken is denoted by $A^{(1)}$ and the average number of elements in it is given by μ_1 . When $A^{(2)}$ is joined to $A^{(1)}$, due to nonextensivity, μ_2 has to be multiplied by q^{n_1} and so forth. (37) can be put in a simple form by defining new creation operators by

$$
\begin{aligned}
&c_1^* = a_1^* \\
&c_2^* = q^{\hat{n}_1/2} a_2^* \\
&c_3^* = q^{(\hat{n}_1+\hat{n}_2)/2} a_3^* \\
&\vdots \\
&c_d^* = q^{(\hat{n}_1+\hat{n}_2+\cdots+\hat{n}_{d-1})/2} a_d^* \qquad .
\end{aligned}
\tag{39}
$$

Then (37) becomes

$$
\mu = c_1^* c_1 + c_2^* c_2 + \cdots + c_d^* c_d \qquad , \tag{40}
$$

and the algebra (34) is replaced by

$$
\begin{aligned}
c_1 c_1^* - q\, c_1^* c_1 &= 1 \\
c_k {c_k}^* - q\, {c_k}^* c_k &= [\, c_{k-1}, {c^*}_{k-1} \,] \qquad , \qquad k = 2, \cdots, d \\
c_k c_l &= q^{1/2} c_l c_k \qquad , \qquad l > k \\
c_k c_l^* &= q^{1/2} c_l^* c_k \qquad , \qquad l \neq k \qquad .
\end{aligned}
\tag{41}
$$

These equations define the quantum covariant q-oscillators which were first derived [7] using the unitary quantum group $SU_q(d)$ [8] which acts on the c_k by

$$
c_k \to \alpha_{kl}\, c_l \qquad . \tag{42}
$$

α_{kl} , the elements of a d x d matrix belonging to $SU_q(d)$ satisfy nontrivial commutation relations among themselves but commute with the c_k . Under such a transformation (40) and (41) remain invariant. The parameter q of this talk is usually referred to as q^2 in most of the literature on quantum groups. Thus the unitary quantum group is the "symmetry group" of the algebra of creation operators of random sets.

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