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The Milne and the Constant Source Problems for the FBIS Kernel

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Abstract

Transport solutions to the monoenergetic constant source and the Milne problems for the forward-backward-isotropic scattering (FBIS) kernel are obtained by method of singular expansion modes. The monoenergetic linear transport equation for extremely anisotropic scattering with constant source is reduced to a transport equation with isotropic scattering. The expansion coefficient is obtained in a form involving exact integral equations and determined by the accompanying new boundary conditions and the half-range orthogonality relations. In the zeroth order approximation, the analytical expressions for neutron density and the emergent angular distributions are also obtained as a function of forward scattering.

1. Introduction

The Milne as a constant source problem, involve the search for angular density distribution functions in a half space, through which neutrons diffuse from a source at $+\infty$ and from a uniform isotropic source present in the right half-space, respectively. In both problems half of space is a vacuum and therefore a vacuum boundary condition should be imposed. Solutions to problems involving angular density and for isotropic scattering are well known [1].

The problem of finding the angular density for the FBIS kernel, which is used in many physical problems, requires the solution of the following equation:

$$\mu\frac{\partial\Psi(x,\mu)}{\partial x}+\Psi(x,\mu)=\frac{ac}{2}\int_{-1}^{+1}\Psi(x,\mu')d\mu'+bc\Psi(x,\mu)+dc\Psi(x,-\mu)+S, \qquad (1)$$

where

$$\Psi(0,\mu)=0, \qquad \mu>0. \qquad (2)$$

$\Psi(x,\mu)$ is the one-speed angular density with distances is measured in units of mean-free path, c is the number of secondaries after collision, and a, b, d are the isotropic, forward and backward scattering parameters.

From the normalization of the scattering kernel,

$$a + b + d = 1. \tag{3}$$

The scattering parameters a, b, d assume the following values for three important scattering conditions:

1. $a = 1 - \alpha,\ \ b = \alpha,\ \ d = 0$ isotropic scattering with forward scattering;
2. $a = 1 - \alpha,\ \ b = 0,\ \ d = \alpha$ isotropic scattering with backward scattering;
3. $a = 1 - 2k,\ \ b = k + \alpha,\ \ d = k - \alpha$ forward scattering with backward scattering,

where $0 \le \alpha \le 1,\ \ -\alpha \le k \le \alpha$ in each case.

Eq. (1) and the boundary condition in Eq. (2) can be written as

$$\mu \frac{\partial \Phi(x,\mu)}{\partial x} + \Phi(x,\mu) = \frac{ac}{2} \int_{-1}^{+1} \Phi(x,\mu') d\mu' + bc\Phi(x,\mu) + dc\Phi(x,-\mu), \tag{4}$$

where

$$\Phi(0,\mu) = -k, \qquad \mu > 0, \tag{5}$$

and

$$\Psi(x,\mu) = k + \Phi(x,\mu), \qquad k = \frac{s}{1-c}. \tag{6}$$

The solution of the form

$$\Phi(x,\mu) = e^{\frac{-x}{\nu}} \Phi(\mu) \tag{7}$$

leads to

$$\left(1 - \frac{\mu}{\nu}\right) \phi_\nu(\mu) - bc\phi_\nu(\mu) = \frac{ac}{2} + cd\phi_\nu(-\mu), \tag{8}$$

where ϕ_ν denotes the eigenfunctions corresponding to the eigenvalues ν, normalized by

$$\int_{-1}^{+1} \phi_\nu(\mu) d\mu = 1. \tag{9}$$

The solution to Eq. (8) for the eigenfunctions $\phi_\nu(\mu)$ may easily be obtained in terms of the eigenfunctions of the isotropic scattering:

$$\phi_\nu(\mu) = A\phi'_\nu(\mu) + B\phi'_\nu(-\mu), \tag{10}$$

where

$$\phi'_\nu(\mu) = \frac{c'\nu'}{2} \frac{1}{\nu' - \mu} \tag{11}$$

and

$$c' = \frac{ac}{1-(b+d)c}, \tag{12}$$

$$\nu' = [(1-bc)^2 - c^2d^2]^{\frac{1}{2}}\nu, \tag{13}$$

$$A = \frac{1}{2}\left[1 + \left(\frac{1-bc-cd}{1-bc+cd}\right)^{\frac{1}{2}}\right], \tag{14}$$

$$B = \frac{1}{2}\left[1 - \left(\frac{1-bc-cd}{1-bc+cd}\right)^{\frac{1}{2}}\right]. \tag{15}$$

Thus we can rewrite the angular flux as

$$\Phi_\nu(x,\mu) = A\Phi'_\nu(x',\mu) + B\phi'_\nu(x',-\mu), \tag{16}$$

and

$$\Phi_\nu(x,-\mu) = A\Phi'_\nu(x',-\mu) + B\phi'_\nu(x',\mu), \tag{17}$$

where

$$\Phi'(x',\mu) = \Phi^{is,c'}_{\nu'}(x',\mu) \tag{18}$$

describes the isotropic scattering and

$$x' = [(1-bc)^2 - c^2d^2]^{\frac{1}{2}}x. \tag{19}$$

Note that Eqs. (16) and (17) can also be written as

$$\Phi'(x',\mu) = \frac{1}{A-B}[A\Phi_\nu(x,\mu) - B\Phi_\nu(x,-\mu)] \tag{20}$$

$$\Phi'(x',-\mu) = \frac{1}{A-B}[A\Phi_\nu(x,-\mu) - B\Phi_\nu(x,\mu)]. \tag{21}$$

Using the boundary condition in Eq. (5) and Eqs. (20 and (21) we obtain

$$\Phi'(0,\mu) = \frac{1}{A-B}[-Ak - B\Phi_\nu(0,-\mu)], \tag{22}$$

$$\Phi'(0,-\mu) = \frac{1}{A-B}[A\Phi_\nu(0,-\mu) + Bk], \tag{23}$$

or

$$\frac{\Phi'(0,\mu) + \frac{A}{A-B}k}{\Phi'(0,-\mu) - \frac{B}{A-B}k} = -\frac{B}{A}. \tag{24}$$

Eq. (24) express the new boundary condition. That is, the constant source boundary condition for extremely anisotropic scattering. If $a = 1$ and $b = d = 0$, Eq. (24), reduces to the usual constant source boundary condition for isotropic scattering [1]. On the other hand if $k = 0$, it becomes a boundary condition for the Milne problem for isotropic scattering with backward scattering, and for $a = 1, b = d = 0$ reduces to the usual boundary condition for isotropic scattering. Now the problem is to solve Eq. (4) with the new boundary condition given in Eq. (25) and the transformed Eqs. (20) and (21).

2. Expansion Coefficients for the Milne and the Constant Source Problems

Solutions to the Milne and constant source problems can be taken as the linear combination of solutions which vanish at infinity plus $q\Phi'_{0-}(x,\mu)$, or the transformed singular normal modes, [1, 37, 38]:

$$\Phi'(x',\mu) = q\Phi'_{0-}(x',\mu) + a_{0+}\Phi'_{0+}(x',\mu) + \int_0^{+1} C(\nu')\Phi'_{\nu'}(x',\mu)d\nu'. \tag{25}$$

$(q,k) = (1,0), (0,k)$ denote the conditions for the Milne and the constant source problems, respectively. Substituting Eq. (25) into Eq. (24), we obtain

$$(qA + Ba_{0+})\Phi'_{0-}(\mu) + (qB + Aa_{0+})\Phi'_{0+}(\mu) + \int_0^{+1} C(\nu')[A\Phi'(\mu) + B\Phi'(-\mu)]d\nu' + k = 0. \tag{26}$$

The expantion coefficients a_{0+} and $C(\nu')$ may be evaluated from the orthogonality relations [1] as

$$\begin{aligned} a_{0+} \quad = \quad & q\frac{BX'(\nu_0) - AX'(-\nu_0)}{BX'(-\nu - 0) - AX'(\nu_0)} - \frac{B}{\nu'_0}\frac{\int_0^1 C(\nu')\nu' X'(-\nu)d\nu'}{BX'(-\nu_0) - AX'(\nu_0)} \\ & - \frac{2k}{c'\nu'_0}\frac{1}{BX'(-\nu_0) - AX'(\nu_0)} \end{aligned} \tag{27}$$

and

$$\begin{aligned} qAc'\nu''\nu'_0X'(-\nu_0)\Phi'_{0-}(\nu'') \quad & + \quad qB\frac{BX'(\nu_0) - AX'(-\nu_0)}{BX'(-\nu_0) - AX'(\nu_0)}c'\nu''\nu'_0X'(-\nu_0)\Phi'_{0-}(-\nu'') \\ & + \quad \int_0^1 C(\nu')\left[\frac{Bc'\nu''}{2}\Phi'_{-\nu}(\nu'')(\nu'_0 + \nu'')X'(-\nu)\right. \\ & - \quad \left. B^2\frac{c'\nu'_0\nu'\nu''X'(-\nu)X'(-\nu_0)}{BX'(-\nu_0) - AX'(\nu_0)}\Phi'_{0-}(\nu'')\right] d\nu' \\ & + \quad \frac{AC(\nu'')W(\nu'')N(\nu'')}{\nu''} - 2k\frac{B\nu''\nu'_0X'(-\nu_0)\Phi'_{0-}(-\nu'')}{BX'(-\nu_0) - AX'(\nu_0)} \\ & + \quad k\frac{c'\nu''}{2} = 0 \end{aligned} \tag{28}$$

where

$$\begin{aligned} X'(\nu_0) &= X(c', \nu_0'), \\ X(z) &= \int_0^1 \frac{\gamma(\mu)}{\mu - z} d\mu, \\ \gamma(\nu') &= \frac{w(\nu')}{\nu_0' - \nu'}, \\ w(\nu') &= \frac{c'\nu'}{2} \frac{1}{X'(-\nu)(\nu_0' + \nu')(1 - c')}, \\ N(\nu') &= \nu'[(1 - c'\nu' \tanh^{-1} \nu')] + \frac{c'^2 \pi^2 \nu'^2}{4}. \end{aligned} \tag{29}$$

To a first order approximation, neglecting the integral term in Eq. (28), we obtain

$$\begin{aligned} C(\nu') = & -\frac{\nu'}{AW(\nu')N(\nu')} \left[qAc'\nu'\nu_0'X'(-\nu_0)\Phi_{0-}'(\nu) + Bc'\nu'\nu_0'X'(-\nu_0)\Phi_{0-}'(\nu) \right. \\ & \left. q \left[\frac{BX'(\nu_0) - AX'(-\nu_0)}{BX'(-\nu_0) - AX'(\nu_0)} \right] - k \left[\frac{2B\nu'X'(-\nu_0)\Phi_{0-}'(\nu)}{BX'(-\nu_0) - AX'(\nu_0)} - \frac{c'\nu'}{2} \right] \right]. \end{aligned} \tag{30}$$

Substituting Eq. (30) into Eq. (27), the coefficient a_{0+} takes the form

$$\begin{aligned} a_{0+} = & \frac{1}{BX'(-\nu_0) - AX'(\nu_0)} [qBX'(\nu_0) - qAX'(-\nu_0) + qBc'(1 - c')\nu_0'X'(-\nu_0) \\ & \left[1 + q\frac{B}{A} \frac{BX'(\nu_0) - AX'(-\nu_0)}{BX'(-\nu_0) - AX'(\nu_0)} \right] \int_0^1 \frac{\nu'X'^2(-\nu)}{(1 - c'\nu' \tanh^{-1} \nu')^2 + (\frac{c'\pi\nu'}{2})^2} d\nu' \\ & -\frac{B^2}{A} k \int_0^1 \frac{2\nu'X'(-\nu_0)X'^2(-\nu)(1 - c')}{[BX'(-\nu_0) - AX'(\nu_0)][(1 - c' \tanh^{-1} \nu')^2 + (\frac{c'\pi\nu'}{2})^2]} d\nu' \\ & \frac{B}{A} \frac{k}{\nu_0'} \int_0^1 \frac{\nu'(\nu_0' + \nu')X'^2(-\nu)(1 - c')}{(1 - c' \tanh^{-1} \nu')^2 + (\frac{c'\pi\nu'}{2})^2} d\nu' - \frac{2k}{c'\nu_0'}. \end{aligned} \tag{31}$$

As a further approximation, neglecting the integral terms in Eq. (31), we obtain

$$a_{0+} = \frac{1}{BX'(-\nu_0) - AX'(\nu_0)} \left[BqX'(\nu_0) - AqX'(-\nu_0) - \frac{2k}{c'\nu_0'} \right]. \tag{32}$$

For isotropic scattering with forward scattering, that is for $d = 0$ $(A = 1, B = 0) a_{0+}$ in Eq. (32) turns to be

$$a_{0+} = q\frac{X'(-\nu_0)}{X'(\nu_0)} + k\frac{2}{c'\nu_0'X'(\nu_0)}. \tag{33}$$

3. The Density and the Emergent Angular Distributions

From Eqs. (6), (16), (25) for the constant source problem $(q = 0)$ and for forward scattering with isotropic scattering $(a = 1 - \alpha,\ \ b = \alpha,\ \ d = 0)$ we obtain

$$\Psi(x, \mu) = k + a_{0+}\Phi'_{0+}(x', \mu) + \int_0^{+1} C(\nu')\Phi'_{\nu'}(x', \mu)d\nu'. \tag{34}$$

the boundary conditions in Eq. (2) leads to

$$\Psi(0, \mu) = \frac{k}{X'(\mu)(\nu'_0 - \mu)}, \quad \mu < 0. \tag{35}$$

Similarly, for $q \neq 0,\ k \neq 0$ the same boundary conditions gives

$$\Psi(0, \mu) = \frac{c'\nu'_0}{2}\frac{a_{0+}}{\nu'_0 - \mu}\frac{X'(\nu_0)}{X'(\mu)} + \frac{c'\nu'_0}{2}\frac{1}{\mu + \nu'_0}\frac{X'(-\nu_0)}{X'(\mu)}, \quad \mu < 0. \tag{36}$$

Replacing a_{0+} in Eq. (33) into Eq. (36), $\Psi(0, \mu)$ expresses as

$$\Psi(0, \mu) = \frac{k}{X'(\mu)(\nu'_0 - \mu)} + q\frac{c'\nu'^2}{\nu_0'^2 - \mu^2}\frac{X'(-\nu_0)}{X'(\mu)}, \quad \mu < 0. \tag{37}$$

Neutrons streaming parallel to the interface can be obtained from Eq. (37) using the identity

$$X'^2(0) = \frac{1}{\nu_0'^2(1 - c')} \tag{38}$$

as

$$\Psi(0, 0) = \frac{s\nu'_0}{(1 - c')^{\frac{1}{2}}(\nu'_0 - \mu)} + qc'\nu'_0(1 - c')^{\frac{1}{2}}X'(-\nu_0). \tag{39}$$

Similarly, using Eq. (37) for the density at the interface

$$\rho(0) = 2\pi\int_{-1}^{0}\Psi_0(0, \mu)d\mu \tag{40}$$

we obtain

$$\frac{\rho(0)}{4\pi} = \frac{s}{c}\left[\frac{1}{(1 - c')^{\frac{1}{2}}} - 1\right] + q\nu'_0X'(-\nu_0)(1 - c')^{\frac{1}{2}}. \tag{41}$$

4. Conclusion

We have studied the solution of the linear transport equation for the monoenergetic constant source and the Milne problems for forward-backward isotropic scattering kernel. For isotropic scattering, the solutions of the two problems are known. For isotropic scattering with a backward leak the solution of the Milne problem is also studied. In this work for the FBIS scattering kernel using the method of elementary solutions and the transformed new boundary conditions, the expansion coefficients for the angular density for both Milne and the constant source problem are calculated in the form of exact integral equations. The density and the angular distributions are also obtained for the FIS kernel.

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