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Quaternionic Roots of E_8 Related Coxeter Graphs and Quasicrystals*

Mehmet KOCA, Nazife Özdeş KOCA
*Department of Physics, College of Science,
Sultan Qaboos University, P.O. Box 36, Al-Khod 123
Muscat, Sultanate of Oman, Department of Physics,
Çukurova University, 01330 Adana - TURKEY*
Ramazan KOÇ
*Department of Physics, Çukurova University,
01330 Adana - TURKEY*

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Abstract

The lattice matching of two sets of quaternionic roots of F_4 leads to quaternionic roots of E_8 which has a decomposition $H_4 + \sigma H_4$ where the Coxeter graph H_4 is represented by the 120 quaternionic elements of the binary icosahedral group. The 30 pure imaginary quaternions constitute the roots of H_3 which has a natural extension to $H_3 + \sigma H_3$ describing the root system of the Lie algebra D_6. It is noted that there exist three lattices in 6-dimensions whose point group $W(D_6)$ admits the icosahedral symmetry H_3 as a subgroup, the roots of which describe the mid-points of the edges of an icosahedron. A natural extension of the Coxeter group H_2 of order 10 is the Weyl group $W(A_4)$ where $H_2 + \sigma H_2$ constitute the root system of the Lie algebra A_4. The relevance of these systems to quasicrystals are discussed.

1. Introduction

Emergence of the noncrystallographic Coxeter graph H_3 in SU(3) conformal field theory [1] motivates further studies of the noncrystallographic Coxeter graphs H_2, H_3, and H_4 and the associated Coxeter-Dynkin diagrams which naturally admit them as subgraphs. They are the respective Weyl groups $W(A_4), W(D_6)$ and $W(E_8)$ generated by reflections. The Lie groups derived from the diagrams of A_4 and E_8 are closely related

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to physical phenomena where supersymmetric SU(5) seems to be the best candidate for the grand unification of he quark-lepton interactions, and E_8 plays the fundamental role in Heterotic string theory [2].

The dihedral group of order 10 and the symmetry group Y_h of icosahedron are interesting symmetries, the latter associated with many physical phenomena. They are respectively generated by the Coxeter graphs H_2 and H_3. It was noted earlier that the interpretation of the experimental data of the quasi crystals with Y_h symmetry requires embedding of Y_h into a crystallographic group in higher dimensions [3]. That requires the crystallographic structures in 6-dimensional Euclidean space with icosahedral symmetry. It was pointed out that there exist three types of crystals; simple cubic (SC), face-centered cubic (FCC), and body centered cubic (BCC) compatible with the Y_h symmetry.

In what follows, we prove that the required crystals in 6-dimension are generated from the root lattice of the Lie algebra $D_6 \approx SO(12)$. There is a profound mathematical structure relating H_3 to D_6 symmetry. We study their root systems using the quaternion constructions and reconcile the two apparently different approaches [4] describing the same phenomena. We show that there lies a beautiful mathematics directly related to E_8 and a magic square associated with it.

One of us (M.K) [5] had, several years ago, constructed the root system of E_8 in terms of quaternions. The method is based on the lattice matching of two sets of quaternionic roots of F_4. In Section 2 we briefly summarize the related results of reference [5] and indicate how the root systems $H_4 + \sigma H_4$, $H_3 + \sigma H_3$ and $H_2 + \sigma H_2$ describe, respectively, the root system of E_8, D_6 and A_4 with $\sigma = \frac{1}{2}(1-\sqrt{5})$. Here, H_4 denotes the 120 quaternionic roots (elements of the binary icosahedral group) of the associated Coxeter graph H_4. The H_3 and H_2 stand for the subroot systems. We also point out the descriptions of certain root systems by the quaternionic representations of the finite subgroups of SU(2). In Section 3 we discuss the quaternionic root systems of H_3 and D_6 and the constructions of the lattices associated with D_6. Embedding of Y_h in the Weyl group $W(D_6)$ is also explicitly discussed. A similar discussion on the (H_2, A_4) system is presented. Generation of the associated root systems by quaternion multiplications is emphasized. Finally, in Section 4, we make further remarks concerning the quaternionic descriptions of the relevant root systems and associated physical phenomena.

2. Quaternions and the Related Root Systems

Hamilton's quaternion algebra has the complex basis units $1, e_1, e_2, e_3$, where the imaginary units satisfy the relations

$$\begin{aligned} e_i e_j &= -\delta_{ij} + \epsilon_{ijk} e_k, \quad i,j,k = 1,2,3 \\ \overline{e}_i &= -e_i. \end{aligned} \tag{1}$$

Here, δ_{ij} and ϵ_{ijk} are usual Kronecker and Levi-Civita symbols, respectively. Quaternionic units can be represented by the Pauli matrices where, for instance, $e_1 = i\sigma_1, e_2 =$

$i\sigma_2, e_3 = -i\sigma_3$ and the real unit is 2×2 unit matrix I. Any real quaternion q can be written as

$$q = q_0 + q_i e_i \tag{2a}$$

and its conjugate as

$$\overline{q} = q_0 - q_i e_i, \tag{2b}$$

where q_0 and q_i are real numbers. The quaternions of unit norm $q\overline{q} = \overline{q}q = 1$ form a group isomorphic to SU(2). The finite subgroups of SU(2) can then be described as the discrete elements of the unit quaternions. The finite subgroups of SU(2), also known as the binary polyhedral group [6], come in five major classes: the cyclic groups $\langle n, n, 1\rangle$ of order $2n$, the dicyclic groups $\langle n, 2, 2\rangle$ of order $4n$, the binary tetrahedral group $\langle 3, 3, 2\rangle$ of order 24, the binary octahedral group $\langle 4, 3, 2\rangle$ of order 48, and finally the binary icosahedral group $\langle 5, 3, 2\rangle$ of order 120. The latter quaternions are directly related with the root system of H_4. The character tables of the binary polyhedral groups are the eigenvectors of the incidence matrices (2I-Cartan matrix) of the ADE series of affine Lie algebras [7]. Another interesting aspect is that the quaternionic group elements represent the scaled version of certain root systems of the Coxeter graphs. This fact, which is relatively less familiar to the physics community, can be summarized as follows [8]. The $2n$ group elements of the cyclic group C_{2n} generated by the quaternion $\exp(e_1 \frac{\pi}{n})$ can be used to describe the root system of the Coxeter graph $I_2(n)$:

$$I_2(n) : \overset{n}{-} \quad n = 1, 2, \ldots . \tag{3}$$

Similarly, the $4n$ elements of the dicyclic group $\langle n, 2, 2\rangle$ describe the two copies of $I_2(n)$ as

$$\overset{n}{-} + \overset{n}{-}, \tag{4}$$

where the $4n$ elements can be generated by the quaternions

$$\exp\left(e_1 \frac{\pi}{n}\right) \quad \text{and} \quad e_2 \exp\left(e_1 \frac{\pi}{n}\right). \tag{5}$$

Our notations for the Coxeter graphs are those of Humphreys [9].

For $n = 3$ relation (5) corresponds to a scaled SU(3) $\times$ SU(3) root system. The $n = 2$ case is the well known quaternion group of elements $\pm 1, \pm e_1, \pm e_2, \pm e_3$ which represent the four copies of SU(2) roots where $SU(2)^4$ is a maximal Lie algebra in SO(8). Indeed, the next larger group, $\langle 3, 3, 2\rangle$, with 24 elements composed of

$$A_0 : \pm 1, \pm e_1, \pm e_2, \pm e_3, \ \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \tag{6}$$

describe the root system of SO(8). They are also known as the units of the Hurwitz integers [10]. It has been also noted that the F_4 root system corresponding to the 48 quaternions [11] can be described by the set A_0 and the weights of the three eight-dimensional irreducible representations of SO(8) denoted by the sets A_1, A_2 and A_3:

$$F_4 : A_0, \begin{array}{c} A_1 \\ \hline \frac{1}{2}(\pm 1 \pm e_1) \\ \frac{1}{2}(\pm e_2 \pm e_3) \end{array}, \begin{array}{c} A_2 \\ \hline \frac{1}{2}(\pm 1 \pm e_2) \\ \frac{1}{2}(\pm e_3 \pm e_1) \end{array}, \begin{array}{c} A_3 \\ \hline \frac{1}{2}(\pm 1 \pm e_3) \\ \frac{1}{2}(\pm e_1 \pm e_2) \end{array}. \tag{7}$$

We note that the group elements of the binary octahedral groups $\langle 4,3,2\rangle$ is nothing other than the set of quaternions

$$\langle 4,3,2\rangle : \; A_0 + \sqrt{2}(A_1 + A_2 + A_3). \tag{8}$$

Finally, the 120 quaternionic elements of the binary icosahedral group $\langle 5,3,2\rangle$

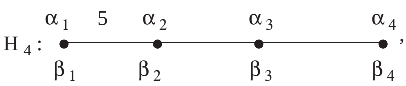
$$\pm 1, \pm e_1, \pm e_2, \pm e_3, \quad \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \tag{9a}$$

$$\frac{1}{2}(\pm 1 \pm \tau e_1 \pm \sigma e_3),\ \frac{1}{2}(\pm \tau \pm e_1 \pm \sigma e_2),\ \frac{1}{2}(\pm \sigma \pm \tau e_2 \pm e_3),\ \frac{1}{2}(\pm \sigma e_1 \pm e_2 \pm \tau e_3) \tag{9b}$$

$$\frac{1}{2}(\pm 1 \pm \tau e_2 \pm \sigma e_1),\ \frac{1}{2}(\pm \tau \pm e_2 \pm \sigma e_3),\ \frac{1}{2}(\pm \sigma \pm \tau e_3 \pm e_1),\ \frac{1}{2}(\pm \sigma e_2 \pm e_3 \pm \tau e_1) \tag{9c}$$

$$\frac{1}{2}(\pm 1 \pm \tau e_3 \pm \sigma e_2),\ \frac{1}{2}(\pm \tau \pm e_3 \pm \sigma e_1),\ \frac{1}{2}(\pm \sigma \pm \tau e_1 \pm e_2),\ \frac{1}{2}(\pm \sigma e_3 \pm e_1 \pm \tau e_2) \tag{9d}$$

describe the root system of the noncrystallographic Coxeter graph H_4,



(10)

where the simple roots $\alpha_i (i = 1,2,3,4)$ can be chosen as

$$\alpha_1 = -e_2, \;\; \alpha_2 = \frac{1}{2}(e_1 + \tau_2 + \sigma e_3), \;\; \alpha_3 = -e_1, \;\; \alpha_4 = \frac{1}{2}(\sigma + e_1 + \tau e_3) \tag{11}$$

and $\sigma = \frac{1}{2}(1 - \sqrt{5})$, $\tau = \frac{1}{2}(1 + \sqrt{5})$; with $\sigma + \tau = 1$, $\sigma\tau = -1$, $\sigma^2 = \sigma + 1$, $\tau^2 = \tau + 1$. Note that (9c-d) can be obtained from (9b) by the cyclic permutation $e_1 \to e_2 \to e_3 \to e_1$. Here, we use the scalar product

$$(p,q) = \frac{1}{2}(\overline{p}q + \overline{q}p) = p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3, \tag{12}$$

which is a real number. When p, q are chosen from the set of elements in (9), then it takes the value

$$(p,q) = a + \sigma b, \tag{13}$$

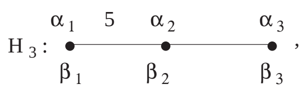
where a and b are $0, \pm\frac{1}{2}, \pm 1$. Later, we will also use a "reduced scalar product" defined by [12]

$$a + \sigma b \rightarrow (p, q)' = a \tag{14}$$

with one can construct the quaternionic root systems in higher dimensions. The reduced scalar product of relation (14) is used for the quaternionic construction of Leech lattice. The root systems of interest can be generated by reflections through the hyperplanes represented by the simple roots. The reflection of an arbitrary quaternion λ through one of the simple root α_i can be written as a triple product of quaternions:

$$R_i\lambda = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i = -\alpha_i\overline{\lambda}\alpha_i. \tag{15}$$

The group generated by the reflections in graph (10) is of order 14400 whose properties are beyond the scope of this paper. It has 34 conjugacy classes with irreducible representations of dimensions and multiplicities $\underline{1}(2), \underline{4}(4), \underline{6}(2), \underline{8}(2), \underline{9}(4), \underline{10}, \underline{16}(5), \underline{18}, \underline{24}(4), \underline{25}(2), \underline{30}(2)$, $\underline{36}(2), \underline{40}$ and $\underline{48}$. The H_4 admits H_3, the noncrystallographic Coxeter graph in 3-dimension, as subgraph



(16)

whose simple roots are pure imaginary quaternions of the roots in (11). The H_3, with these simple roots, generate 30 pure imaginary roots given in the last column of equations (9a-d). The group generated by reflections in (16) is isomorphic to $Y_h \approx 2 \times A_5$, the icosahedron symmetry including inversions, with A_5 being the group of even permutations of 5 letters. The 30 roots of H_3 represent the vectors joining the mid-points of the edges of an icosahedron to the origin. When one describes the C_{60} molecule by a truncated icosahedron, then these edges represent the double bonds binding the adjacent carbons. The H_3 and its quaternionic root system will be further studied in the next section, for it has far-reaching physical applications.

A very interesting phenomena is lattice matching [13] which is used to obtain the root system of higher rank Lie algebras. The principle is that two sets of orthogonal short roots add up to produce long roots. The relevant lattice matching is given in the magic square described in Table 1.

Table 1. The magic square of lattice matching

	$SU(3)$	$SP(3)$	F_4
$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
$SP(3)$	$SU(6)$	$SO(12)$	E_7
F_4	E_4	E_7	E_8

It was already noted in [5] that the matching of two sets of quaternionic F_4 root system in the form

$$E_8 : (A_0, 0) + (0, A_0) + (A_1, A_2) + (A_2, A_3) + (A_3, A_1) \tag{17}$$

leads to the quaternionic root system of E_8 provided the reduced scalar product of (14) is employed. The use of parenthesis in relation (17) denotes

$$(A_0, 0) = A_0, \quad (0, A_0) = \sigma A_0, \quad (A_i, A_j) = A_i + \sigma A_j, \;\; i \neq j, \tag{18}$$

with $\sigma = \frac{1}{2}(1 - \sqrt{5})$. It is interesting to note that quaternions $1, e_i, \sigma e_i (i = 1, 2, 3)$ form an orthonormal basis in 8-dimension when the reduced scalar product is invoked. It is not difficult to show that the roots of E_8 in (17) can be regrouped as the union of roots of H_4 given in relation (9) and their σ multiple as

$$E_8 : H_4 + \sigma H_4. \tag{19}$$

There is also a simpler interpretation of (19) in terms of the Coxeter-Dynkin diagram of E_8 which results from the Coxeter diagram of H_4 given in (10). It is explained in Figure 1 in terms of the simple roots $\alpha_i (i = 1, 2, 3, 4)$ of H_4 and $-\sigma\alpha_i$.

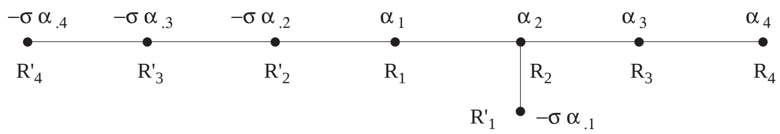


Figure 1. The Coxeter-Dynkin diagram of E_8 with the simple roots of H_4

To prove that the roots in Figure 1 constitute a set of simple roots of E_8 provided the reduced scalar product is applied is simple. Since $\alpha_i (i = 1, 2, 3, 4)$ are the simple roots of H_4 given by (11) they satisfy the relations

$$(\alpha_1, \alpha_2) = -\frac{\tau}{2}, \;\; (\alpha_2, \alpha_3) = (\alpha_3, \alpha_4) = -\frac{1}{2}. \tag{20}$$

When we use the reduced scalar product (14) we simply have $(\alpha_1, \alpha_2)' = (\alpha_2, \alpha_3)' = (\alpha_3, \alpha_4)' = -\frac{1}{2}$. The others follow similar calculations. We have the relations

$$\begin{aligned}
(-\sigma\alpha_4, -\sigma\alpha_3) &= \frac{\sigma^2}{2} = -\frac{1}{2} - \frac{\sigma}{2} \rightarrow (-\sigma\alpha_4, -\sigma\alpha_3)' = -\frac{1}{2} \\
(-\sigma\alpha_3, -\sigma\alpha_2) &= -\frac{1}{2} - \frac{\sigma}{2} \rightarrow (-\sigma\alpha_3, -\sigma\alpha_2)' = -\frac{1}{2} \\
(-\sigma\alpha_2, \alpha_1) &= -\frac{1}{2} \rightarrow (-\sigma\alpha_2, \alpha_1)' = -\frac{1}{2}
\end{aligned} \tag{21}$$

which proves that Figure 1 represents the E_8 diagram under the reduced scalar product. This indicates that when $\alpha_i (i = 1, 2, 3, 4)$ serve as simple roots of H_4 under the ordinary scalar product, the roots α_i and $-\sigma\alpha_i (i = 1, 2, 3, 4)$ serve as simple roots of E_8 under the reduced scalar product. This helps us to understand why E_8 roots can be written in the form of relation (19). It is possible to obtain the reflection generators of H_4 by folding [14] the diagram in Figure 1 leading to the generators $\beta_1 = R'_1 R_1,\ \beta_2 = R'_2 R_2,\ \beta_3 = R'_3 R_3$, and $\beta_4 = R'_4 R_4$. The $\beta_i (i = 1, 2, 3, 4)$ generate the group H_4 of order 14400 with the root system given in (9). Although the actions of R_i and R'_i on the simple roots require the reduced scalar product, the action of $\beta_i = R'_i R_i$ amounts to the ordinary scalar product. It is then clear that under the generators β_i of the roots of H_4 and σH_4 are separately left invariant. The projection of the root system of E_8 to H_4 takes a simpler form: just identify the quaternionic elements of the binary icosahedral group constituting the root system of H_4. We will later prove that it suffices to take only the quaternionic roots $\alpha_1, \alpha_2, \alpha_3$ to achieve all these properties. There are a great number of interesting properties of E_8 and H_4 related to the quaternionic constructions of their root systems which rely on the charge of $\sigma \leftrightarrow \tau$. They may perhaps constitute an independent investigation and remain outside of the context of this paper.

3. Quaternionic Roots of the System (H_3, D_6) and the Quasicrystals

We have already noted in Section 2 that the 30 pure imaginary quaternions in relation (9) constitute the root system of H_3. One can readily show that the root system of D_6 is nothing other than the set of quaternions $H_3 + \sigma H_3$. This follows immediately if one excludes the roots α_4 and $-\sigma\alpha_4$ in Figure 1, where the remaining diagram represents the Coxeter-Dynkin diagram of D_6. One can also obtain the roots of D_6 from magic square of Table 1 by matching two sets of quaternionic roots of SP(3). Being a subalgebra of F_4, the quaternionic roots of SP(3) can be represented by the set

$$SP(3) : \frac{B_0}{\pm e_1, \pm e_2, \pm e_3} \frac{B_1}{\pm e_2, \pm e_3} \frac{B_2}{\pm e_3, \pm e_1} \frac{B_3}{\pm e_1, \pm e_2}. \tag{22}$$

Lattice matching (SP(3), SP(3)) in the form of (17)

$$D_6 : (B_0, 0) + (0, B_0) + (B_1, B_2) + (B_2, B_3) + (B_3, B_1) \tag{23}$$

will lead to the quaternionic root system of D_6 provided we use the definition in (14) for the scalar product. The roots scaled by $\sqrt{2}$ are given by the pure imaginary quaternions:

$$\begin{array}{c} H_3 \\ \hline \pm e_1, \pm e_2, \pm e_3 \\ \frac{1}{2}(\pm\sigma e_1 \pm e_2 \pm \tau e_3) \\ \frac{1}{2}(\pm\sigma e_2 \pm e_3 \pm \tau e_1) \\ \frac{1}{2}(\pm\sigma e_3 \pm e_1 \pm \tau e_2) \end{array} \tag{24a}$$

$$\begin{array}{c} \sigma H_3 \\ \hline \pm\sigma e_1, \pm\sigma e_2, \pm\sigma e_3 \\ \frac{\sigma}{2}(\pm\sigma e_1 \pm e_2 \pm \tau e_3) \\ \frac{\sigma}{2}(\pm\sigma e_2 \pm e_3 \pm \tau e_1) \\ \frac{\sigma}{2}(\pm\sigma e_3 \pm e_1 \pm \tau e_2). \end{array} \tag{24b}$$

For the rest of our discussion, let us define the vectors

$$\begin{aligned} \ell_1 &= \frac{1}{\sqrt{2}}(e_1+\sigma e_2);\ \ \ell_3 = \frac{1}{\sqrt{2}}(e_2+\sigma e_3);\ \ \ell_5 = \frac{1}{\sqrt{2}}(e_3+\sigma e_1) \\ \ell_2 &= \frac{1}{\sqrt{2}}(e_1-\sigma e_2);\ \ \ell_4 = \frac{1}{\sqrt{2}}(e_2-\sigma e_3);\ \ \ell_6 = \frac{1}{\sqrt{2}}(e_3-\sigma e_2). \end{aligned} \tag{25}$$

The vectors $\pm\ell_i (i = 1, 2, \ldots, 6)$ denote that the 12 vertices of an icosahedron with edge length $|\sigma| = \frac{1}{2}(\sqrt{5}-1)$. The vectors ℓ_i have positive projections along ℓ_6. Now, if we the reduced scalar product (14) the vectors ℓ_i form an orthonormal basis in 6 dimensions. The Coxeter-Dynkin diagram of D_6 obtained from Figure 1 by deleting α_4 and $-\sigma\alpha_4$ can be illustrated with the use of vectors ℓ_i, as shown in Figure 2.

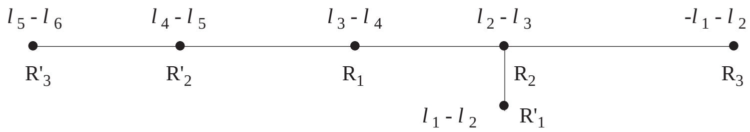


Figure 2. The Coxeter-Dynkin diagram of D_6

The roots of D_6 derived from Figure 2 are of the form

$$\pm\ell_i \pm \ell_j \qquad i \neq j\ \ (i, j = 1, 2, \ldots, 6). \tag{26}$$

The roots in (26) are $\sqrt{2}$ times those of the quaternions given in (24a-b). Using the vectors of (25) representing the vertices of the icosahedron, one can verify that the quaternions in (24a) are the mid-points of the edges of the icosahedron. For sure, the generators $R_i'(i = 1, 2, 3)$ generate the Weyl group $W(D_6)$ of the order $2^5 6!$. By folding the D_6 diagram in Figure 2 one can reproduce the Coxeter graph of H_3 where the reflection generators are given by [15]

$$\beta_1 = R_1' R_1, \quad \beta_2 = R_2' R_2, \quad \beta_3 = R_3' R_3. \tag{27}$$

Using the definition of the generation relations $(\beta_i\beta_j)^{m_{ij}} = 1$ of the Coxeter group, one can prove that they generate the group $2 \times A_5$ isomorphic to Y_h. Here A_5 is the group of even permutations of five letters and isomorphic to the proper icosahedral group of Y of order $60 \cdot 2$ stands for the inversion group. To prove this, let us define generators

$$A = \beta_1\beta_2 = R_1'R_1R_2'R_2,\ B = \beta_2\beta_3 = R_2'R_2R_3'R_3,\ C = (\beta_1\beta_2\beta_3)^5 = (R_1'R_1R_2'R_2R_3'R_3)^5, \tag{28}$$

where $D = \beta_1\beta_2\beta_3$ has period 10 and is the Coxeter element of H_3 as well as of D_6. One can readily show that

$$A^5 = B^3 = (AB)^2 = 1, \tag{29}$$

which is the generation relation of the proper icosahedral group A_5 [6]. From the fact that the $\beta_1\beta_2\beta_3$ is the Coxeter element implying $C^2 = 1$ and C commutes with A and B, one can conclude that the group generated by $\beta_i (i = 1, 2, 3)$ is the well celebrated icosahedral group $2 \times A_5$ followed by an inversion. This also indicates that the group $2 \times A_5$ is an important subgroup of the Weyl group $W(D_6)$ which is the point group of lattices generated by root lattice of D_6. We will say more later on the lattice structures based on D_6.

We now discuss the 6×6 matrix representations of the generators A, B and C as to how the icosahedral symmetry is embedded in D_6. It is straightforward to compute the matrix representations of the generators when they act on the orthogonal set ℓ_i. One should not forget to use the reduced scalar product while the generators R'_i and $R_i (i = 1, 2, 3)$ are acting on this basis. Then the matrix representations A, B, and C read

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad C = -I, \tag{30}$$

where I is the 6×6 unit matrix. The Coxeter element D of $W(D_6)$ in this basis is given by the matrix

$$D = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D^5 = C. \tag{31}$$

Note that matrix A represents the five-fold rotation around the vertex ℓ_6 permuting the other five vertices having positive projections on ℓ_6.

We now return to the Coxeter graph H_6 given by (16). One can compute the matrix representations of β_i acting on the orthonormal basis $e_i (i = 1, 2, 3)$. The action of β_i can be written as the triple product of quaternions

$$\beta_i : e_j \rightarrow \alpha_i e_j \alpha_i \ \ (i, j = 1, 2, 3). \tag{32}$$

Note here we use the scalar product (12), consequently relation (15). The matrix representations of β_i take the simple forms

$$\beta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta_2 = \frac{1}{2} = \frac{1}{2} \begin{pmatrix} 1 & -\tau & -\sigma \\ -\tau & \sigma & 1 \\ -\sigma & 1 & \tau \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{33}$$

The generators of $2 \times A_5$ can be written as $a = \beta_1\beta_2$, $b = \beta_2\beta_3$, $c = (\beta_1\beta_2\beta_3)^5$ and given by the matrices

$$a = \frac{1}{2} \begin{pmatrix} 1 & -\tau & -\sigma \\ \tau & -\sigma & -1 \\ -\sigma & 0 & \tau \end{pmatrix}, \quad b = \frac{1}{2} \begin{pmatrix} -1 & -\tau & -\sigma \\ \tau & \sigma & 1 \\ -\sigma & 1 & \tau \end{pmatrix}, \quad c = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

Here, a and b generate the 3-dimensional irreducible representation of A_5 satisfying the relations

$$a^5 = b^3 = (ab)^2 = 1, \tag{35}$$

and c obviously commutes with a and b and $c^2 = 1$.

A remark at this point is in order. If we had constructed the E_8 roots in the form of (17), we could have used another pairing of the form $(A_1, A_3) + (A_3, A_2) + (A_2, A_1)$. This would lead to a different representation of the root system of E_8 and consequently an alternative description of the H_4 system by quaternions. These two versions of roots corresponds to two irreducible representations of the binary icosahedral group of degree 2. This can be reflected by the change $\sigma \leftrightarrow \tau$ in (9). A similar substitution in (24a-b) leads to an alternative representation of the roots of H_3 and D_6. With the new set of roots of H_3 the generators a' and b' will be obtained from (34) by the substitution $\sigma \leftrightarrow \tau$. The generator c' will remain as $c' = c$. This will account for the second irreducible representation of $2 \times A_5$ of degree 3. Using these data we can write down the character values of the group $2 \times A_5$ for the irreducible representations $\underline{3}$ and $\underline{3}'$ and the reducible representation of degree 6. They are tabulated in Table 2. It follows from Table 2 that 6-dimensional irreducible representation of the Weyl group $W(D_6)$ decomposes under Y_h as $6 = \underline{3} + \underline{3}'$. The same table is obtained in reference (16) from another consideration. This table indicates that the characters of the group Y_h are integers in 6-dimension, a necessary requirements to embed a noncrystallographic group in a crystallographic group in higher dimension.

Table 2. The character values of $2 \times A_5$

	I	AB	B	A	A^2	C	CAB	CB	CA	CA^2
$\chi(\underline{3})$	3	-1	0	τ	σ	-3	1	0	$-\tau$	$-\sigma$
$\chi(\underline{3}')$	3	-1	0	σ	τ	-3	1	0	$-\sigma$	$-\tau$
$\chi(6)$	6	-2	0	1	1	-6	2	0	-1	-1

The Weyl group $W(D_6)$ of order 23040 has 37 conjugacy classes which is equal to the number of its irreducible representations. The degree and the number of multiplicities

of the irreducible representations are as follows:
$\underline{1}(2), \underline{5}(4), \underline{6}(2), \underline{9}(2), \underline{10}(6), \underline{15}(4), \underline{16}(1), \underline{20}(1), \underline{24}(2), \underline{30}(4), \underline{36}(1), \underline{40}(4), \underline{45}(4)$.
This indicates that $W(D_6)$ has two 6-dimensional irreducible representations. We have checked that both irreducible representations of $W(D_6)$ of degree 6 decompose as $\underline{6} = \underline{3} + \underline{3}'$.

So far we have discussed the embedding of the icosahedral group $2 \times A_5$ in the Weyl group $W(D_6)$. Below we discuss the lattice structures associated with D_6. All the lattices generated by the root systems of the Lie algebras are well known [17]. There are three lattices associated with the Lie algebra D_6: (i) the root lattice which corresponds to the face centered cubic (FCC) lattice; (ii) the lattice Z^6 corresponding to the simple cubic (SC) lattice; (iii) the dual lattice D_6^* which corresponds to the body centered cubic (BCC) lattice. They are indeed the lattices obtained in the reference [4] and [16]. Below we discuss these lattices in our terminology of the quaternionic root system. Our approach is, in some sense, a new interpretation of Mermin et.al. [4] where they treat the unit vectors $\vec{i}, \vec{j}, \vec{k}, \tau\vec{i}, \tau\vec{j}, \tau\vec{k}$, as six integrally independent vectors. In the present work they are the quaternionic units $e_1, e_2, e_3, \sigma e_1, \sigma e_2, \sigma e_3$ which are orthonormal vectors under the scalar product (14). We use σ rather than τ.

The six vectors defined in Figure 1,

$$\alpha_1 = \ell_3 - \ell_4,\ \alpha_2 = \ell_2 - \ell_3,\ \alpha_3 = -\ell_1 - \ell_2,\ -\sigma\alpha_1 = \ell_1 - \ell_2,\ -\sigma\alpha_2 = \ell_4 - \ell_5,\ -\sigma\alpha_3 = \ell_5 - \ell_6 \tag{36}$$

generate the root lattice which correspond to the face centered cubic lattice in 6-dimensions.

The other two lattices Z^6 and D_6^* are obtained by adding the so called glue vectors [17]

$$\begin{aligned} p &= \frac{1}{2}(\ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 + \ell_6) = \frac{1}{\sqrt{2}}(e_1 + \tau e_3) \\ q &= \ell_6 = \frac{1}{\sqrt{2}}(e_3 - \sigma e_1) \\ r &= \frac{1}{2}(\ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 - \ell_6) = \frac{1}{\sqrt{2}}(\sigma^2 e_1 - \sigma e_3) = \frac{\sigma}{\sqrt{2}}(\sigma e_1 - e_3). \end{aligned} \tag{37}$$

For instance, Z^6 admits the root lattice as a sublattice of index 2 and can formally be written as

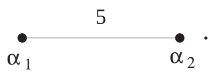
$$Z^6 = D_6^+ = D_6 \cup (p + D_6). \tag{38}$$

This lattice can be transformed into the simple cubic lattice generated by the vectors $\ell_i (i = 1, 2, \ldots, 6)$. The third lattice is the dual lattice D_6^* which can be written as

$$D_6^* = D_6 \cup (p + D_6) \cup (q + D_6) \cup (r + D_6). \tag{39}$$

This indicates that the dual lattice involves the root lattice as a sublattice with index 4 where $(0), p, q$ and r vectors are the coset representatives. An equivalent lattice to the lattice D_6^* can be generated by the vectors $\ell_i (i = 1, 2, 3, 4, 5)$ and the vector $\frac{1}{2}(\ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 + \ell_6)$. This is body centered cubic (BCC) lattice in 6-dimension. To sum up we have shown that the lattices obtained in 6-dimensions by different approaches are all related to the root system of D_6. Moreover, we have illustrated that it has an interesting relation with the root system of H_3 which is the symmetry of the icosahedron in 3-dimensions.

When we proceed in a similar manner we can associated the quasi crystals, if there is any, with the symmetry of the dihedral group of order 10 to the lattices generated by the root lattice of the Lie algebra of A_4. A subgraph of H_3 denoted by $H_2 = I_2(5)$ can be represented by the roots α_1, α_2 when α_3 is deleted in (16):

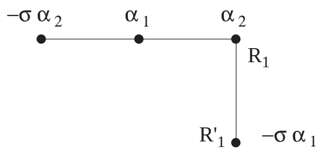


(40)

By reflections, one can get the 10 roots of H_2 as

$$H_2 : \quad \begin{array}{l} \pm e_2 \\ \pm\frac{1}{2}(e_1 + \tau e_2 + \sigma e_3) \\ \pm\frac{1}{2}(e_1 - \tau e_2 + \sigma e_3) \\ \pm\frac{1}{2}(-\tau e_1 + \sigma e_2 + e_3) \\ \pm\frac{1}{2}(-\tau e_1 - \sigma e_2 + e_3). \end{array} \tag{41}$$

They lie in the plane orthogonal to the vector $\ell_6 = e_3 - \sigma e_1$. After deleting α_3 and $-\sigma\alpha_3$ from the Coxeter-Dynkin diagram of D_6, one obtains the root diagrams of A_4 as



(42)

which yields the roots $H_2 + \sigma H_2$. The generators of H_2 can be chosen as

$$\beta_1 = R_1' R_1, \qquad \beta_2 = R_2' R_2, \tag{43}$$

satisfying the relations $\beta_1^2 = \beta_2^2 = 1$ and $(\beta_1\beta_2)^5 = 1$ and $\beta_1\beta_2 = R_1' R_1 R_2' R_2$ is the Coxeter element of H_2 and A_4 with period 5. The generators R_1', R_1, R_2', R_2 generate the Weyl group $W(A_4)$, which is isomorphic to the symmetric group S_5 of order 120 where β_1 and β_2 generate the dihedral subgroup D_5 of S_5. Again, here, the two 2-dimensional irreducible representations $\underline{2}, \underline{2}'$ of the dihedral group of order 10 can be associated with the replacement of $\tau \leftrightarrow \sigma$ in the root system of H_2. The two 4-dimensional representations of S_5 can be decomposed as $\underline{4} = \underline{2} + \underline{2}'$. We have two lattices in this case: (i) the root lattice of A_4 generated by the simple roots in (42); (ii) the dual lattice A_4^*. Their properties can be found in reference [17].

Before we conclude this section we would like to mention a number of other interesting aspects of the quaternionic root systems. One can show that the roots of H_4 can be generated from the roots of H_3 by quaternion multiplication. For this, let us define the quaternion products

$$\begin{aligned} A &= \alpha_1\alpha_2 = \frac{1}{2}(\tau - \sigma e_1 + e_3) \\ B &= \alpha_2\alpha_3 = \frac{1}{2}(1 - \sigma e_2 + \tau e_3), \end{aligned} \tag{44}$$

where α_1, α_2 and α_3 are the roots of H_3 given by (11). We can show that A and B satisfy the relations

$$A^5 = B^3 = (AB)^2 = -1, \tag{45}$$

which is the generation relation for the binary icosahedral group $\langle 5, 3, 2\rangle$. This shows that the products of the simple roots α_1, α_2 and α_3 of H_3 generate the root system of H_4 under multiplication. The automorphism group of the quaternionic set in (24a) preserving the quaternion algebra is the proper icosahedral group A_5 of order 60. Similarly, the automorphism group of the set of quaternions $\pm e_1, \pm e_2, \pm e_3$ preserving the quaternion multiplications is S_4. For the groups A_5 and S_4 are respectively, the automorphism groups of the icosahedron and the cube under rotations of the preceding arguments can be easily understood.

4. Conclusions

We have discussed a very interesting mathematical structure relating the root systems of the Lie algebras E_8, D_6, A_4 and their relationship to those of the noncrystallographic Coxeter graphs H_4, H_3, H_2. The magic square relating the two copies of the root systems of F_4 and $SP(3)$ respectively to the root system of E_8 and D_6 is impressive. The third mathematical structure associated with the above root systems is complemented by quaternions.

To our great surprise, the (H_3, D_6) system relating the symmetry of the icosahedron to the Weyl group $W(D_6)$ and the three lattices, simple cubic (SC), face centered cubic (FCC) and the body centered cubic (BCC) lattices has not been studied before by the crystallographers in connections with the quasi crystals with icosahedral point symmetry. The quasi crystallography has gained a great impetus in the last decade [18] and seems to possess enormous potentiality both from experiment and theoretical considerations.

Extension of the conformal field theory beyond the ADE classification seems to lead surprises such as the noncrystallographic H_3 symmetry and its implicit relation to the Coxeter-Dynkin diagram of D_6 [1] is related to the $SU(3)$ conformal field theory. If there is any further development in this direction there should necessarily be a link also with (H_4, E_8) and (H_2, A_4) systems and their quaternionic constructions.

We hope that we have introduced to the physics community a very rich mathematical structure which can serve as models in different fields of physics. Many of the mathematical properties of the systems discussed here are a new as far as the quaternionic connections are concerned.

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