Fibonaccization and Multiparameter q-oscillators

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Abstract

Starting from Coon-Baker-Yu (CBY) q-oscillator, a two parameter, multidimensional q-oscillator is formulated. It is shown that this procedure, called Fibonaccization, can be extended to any number of parameters. The relation between CBY states and Fibonacci CBY states are also investigated.

Generalization of integers into q-basic numbers and generalization of quantum oscillators into q-oscillators [1,2,3] are related deformations. These deformations have been investigated in recent studies [4,6,7,8]. For example, oscillators with a spectrum given by basic numbers defined in symmetric form [9]

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \tag{1}$$

where q is a real parameter, have been used in the construction of the $SU_q(2)$ Lie algebra [2,3], q correction to Planck radiation and q statistics [10]. On the other hand, the Jackson basic number [12], given by

$$[n] = \frac{q^n - 1}{q - 1}, \tag{2}$$

is used in the construction of $SL_q(n)$ and $SU_q(n)$ quantum [12,13]. This form also yields the energy spectrum of the multidimensional (nonquantum) U(n) invariant CBY q-oscillator [14]. n fact, the two parameter generalization of basic numbers,

$$[n] = \frac{q_1^n - q_2^n}{q_1 - q_2}, \tag{3}$$

with q_1, q_2 real numbers or a complex conjugate pair, solves the Fibonacci difference equation [15]. Among multidimensional q-oscillators, the multidimensional $U_q(n)$ invariant Pusz-Woronowicz (PW) oscillator [16] can be extended to a two parameter oscillator. This is the covariant Fibonacci oscillator whose spectrum is given by (3).

On the other hand, the CBY oscillator is simpler to construct and is related to PW oscillator through the oscillator states [12]. We will now propose a two parameter version of the CBY oscillator using a similar procedure as in [15]. We will call it the Fibonacci-CBY oscillator.

The defining commutation relation of the CBY q-oscillator is given by

$$[a_i, a_{j\dagger}]_q \equiv a_i a_{j\dagger} - q a_{j\dagger} a_i = \delta_{ij}, \tag{4}$$

where a and $a^\dagger$ are annihilation and creation operators, respectively. Using the fact that the right hand side of (4) is a scalar, we multiply (4) by $a_k^\dagger$ once from left, once from right and rearrange to obtain

$$a_i a_j^\dagger a_k^\dagger = q a_j^\dagger a_i a_k^\dagger - q a_k^\dagger a_j^\dagger a_i + a_k^\dagger a_i a_j^\dagger. \tag{5}$$

We now introduce a new parameter q_2 by defining a new operator b by

$$b_i \equiv q_2^{N/2} a_i, \tag{6}$$

where N is the number operator satisfying

$$a_i N = (N+1) a_i.$$

Using this fact it follows that

$$\begin{aligned} b_i f(N) &= f(N+1) b_i \\ b_j^\dagger f(N) &= f(N-1) b_j^\dagger \end{aligned} \tag{7}$$

for any analytic function $f(N)$.

It is possible to eliminate $q_2^{-3N/2}$ and find

$$b_i b_j^\dagger b_k^\dagger = q q_2 b_j^\dagger b_i b_k^\dagger - q q_2^2 b_k^\dagger b_j^\dagger b_i + q_2 b_k^\dagger b_i b_j^\dagger. \tag{8}$$

If we set $q = q_1/q_2$, we obtain a two parameter version of the CBY oscillator which we will call the fCBY oscillator with the "commutation" relation

$$\begin{aligned} b_i b_j^\dagger b_k^\dagger &= q_1 b_i b_j^\dagger b_i b_k^\dagger - q_1 q_2 b_k^\dagger b_j^\dagger b_i + q_2 b_k^\dagger b_i b_j^\dagger, \\ b_i |0> &= 0, \\ b_i b_m^\dagger |0> &= \delta_{im} |0> \end{aligned} \tag{9}$$

For the above equation (9), we calculate the inner products of oscillator states defined through:

$$|ijk\cdots>= b_i^\dagger b_j^\dagger b_k^\dagger \cdots | >,$$

where $|0>$ is the normalized ground state:

$$
\begin{aligned}
< i|j > &= \delta_j^i \\
< ij|km > &= q_2\delta_k^i < j|m > +q_1\delta_m^i < j|k > \\
< ijk|mnp > &= q_2^2\delta_m^i < jk|np > +q_1q_2\delta_n^i < jk|mp > +q_1^2\delta_p^i < jk|mn > .
\end{aligned}
\tag{10}
$$

The recursive property of the above definition permits us to define a tensor N from inner products of the fCBY q-oscillator states

$$< i_1\cdots i_n|k_1\cdots k_n >\equiv N_{k_1 k_2\cdots k_n}^{i_1 i_2\cdots i_n} \equiv N_k^i(n,q), \tag{11}$$

which has the recursion relation

$$
\begin{aligned}
N_{k_1 k_2 k_3\cdots k_n}^{i_1 i_2 i_3\cdots i_n} &= q_2^{n-1} N_{k_1}^{i_1} N_{k_2 k_3\cdots k_n}^{i_2 i_3\cdots i_n} \\
&+ q_1 q_2^{n-2} N_{k_2}^{i_1} N_{k_1 k_3\cdots k_n}^{i_2 i_3\cdots i_n} \\
&+ q_1^2 q_2^{n-3} N_{k_3}^{i_1} N_{k_1 k_2 k_4\cdots k_n}^{i_2 i_3\cdots i_n} \\
&+ \cdots \\
&+ q_1^{n-1} N_{k_2}^{i_1} N_{k_1 k_3\cdots k_{n-1}}^{i_2 i_3\cdots i_n}
\end{aligned}
\tag{12}
$$

If we consider the simplest form of the N tensor, N_j^i, we see that it is simply the Kronecker delta, δ_j^i. Since for the case $q_1 = -1$ or $q_2 = -1$, where we find the usual n-dimensional Kronecker delta, we name N the two parameter generalized Kronecker delta.

It is also possible to find a relation between the N tensor of the CBY q-oscillator and the N tensor of the new fCBY oscillator. To accomplish this, we expand the N tensor completely and we see that for two and three particle cases we have

$$
\begin{aligned}
N &= q_2 N_{CBY}|_{q=q_1/q_2} \\
N &= q_2^3 N_{CBY}|_{q=q_1/q_2},
\end{aligned}
\tag{13}
$$

where N denotes the sensor obtained from the inner products of CBY states [1,17]. For the generalization do a d-dimensional inner product case, one should note that in the above examples, N is each time divided by the highest power of q_2 when it is expanded

into its most simple form using (12). To find the expression for the highest power of q_2, we use the fact that the defining equation (9) is symmetric in powers of q_1 and q_2.

To obtain the highest power of q_1, in a d-dimensional inner product,

$$< i_1 i_2 \cdots i_d | j_1 j_2 \cdots j_d > = < |b_{i_d} \cdots b_{1_2} a_{i_1} b_{j1\dagger} b_{j2\dagger} \cdots b_{jd\dagger} | >,$$

we couple each b_{i_k} with the rightmost possible $b_{jm}^\dagger$ giving rise to a factor of q_1 each time it is permuted with a $b_j^\dagger$. For example, b_{i_1} will couple with $b_{jd}^\dagger$ picking up (d-1) factors of q_1. If we denote the total power of q_1 by $m(d)$, it can be expressed by a summation

$$m(d) = \sum_{i=1}^{d} (d - i) = \frac{d(d - 1)}{2}. \tag{14}$$

Using the symmetry of q_1 and q_2, we find that

$$N = q_2^{m(d)} N_{CBY}|_{q=q_1/q_2}. \tag{15}$$

The defining relation for this new oscillator (9) can also be written in the form of q-commutator notation in (4):

$$[[b_i, b_j^\dagger]_{q1}, b_k^\dagger]_{q2} = 0. \tag{16}$$

We can generalize this equation by writing

$$[[[b_i, b_j^\dagger]_{q1}, b_k^\dagger]_{q2}, b_l^\dagger]_{q3} = 0. \tag{17}$$

Note that (16) trivially solves (17). Further generalizing this equation by introducing new parmeters is straightforward. One can construct the n-parameter version of fCBY q-oscillator as

$$[\cdots [[b_i, b_j^\dagger]_{q1}, b_k^\dagger]_{q2}, \cdots b_z^\dagger]_{qn} = 0. \tag{18}$$

By setting $i = j = k = \cdots = z$, it is possible to obtain the one dimensional version of the above new q-oscillator. If this equation is written as a difference equation, the associated n-parameter basic number can be calculated. One should note that, for n parameters, the difference equation which determines the spectrum is of order n and therefore the number of initial conditions is also n. Different choices of initial condiitions lead to different results. For example, if one sets the first (n-1) basic numbers to be the integers up to (n-1) as the initial conditions, then it is possible to think of n, as the point above which the q effects are to be observed. Another interesting case can be an (n-1)-forld degenerate ground state and a unique first excited state.

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