Tr. J. of Physics
22 (1998) , 1005 – 1012.

The Space-Time Critical Dimension of an Open Parabosonic String*

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Received 13.05.1996

Abstract

Using analytical properties of a 1-loop open parabosonic M-point transition amplitude, we show that the space-time critical dimension depends on the order of the paraquantization.

1. Introduction

One of the main goals of quantum mechanics (QM) is to provide a consistent and unified description of the so-called wave-particle duality which is a direct consequence of the Heisenberg equations of motion. It turns out that the canonical commutation relations - which guarantee the Heisenberg equations - are not unique [1]. The general framework in which the canonical commutation relations are generalized is called paraquantization and characterized by an order parameter Q [2-9]. Although it is, in principle, possible to study the paraquantum observables within the usual Hilbert space, it is often convenient to use a larger Hilbert space in which the operators satisfy simple bilinear relations [2], [10-12]. Traditionally, for Focks-type irreducible representation of paraquantum theories

*This work was supported by the Algerian Ministry of Education and Research under contract No DD2501/01/17/93
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with a unique vacuum state, this is done by means of the Green decomposition [2], [10-12]:

$$a_n = \sum_{\beta=1}^{Q} a_n^{(\beta)} \tag{1}$$

where Q is the order of the paraquantization, β the Green index and $a_n^{(\beta)}$ is the bosonic annihilation operators with Green components satisfying the following bilinear but anomalous commutation relations:

$$\begin{aligned} \left[a_n^{(\beta)}, a_m^{+(\alpha)}\right]_+ &= 0 \quad \alpha \neq \beta \\ \left[a_n^{(\alpha)}, a_m^{+(\alpha)}\right]_- &= \delta_{mn}. \end{aligned} \tag{2}$$

The purpose of this paper is to derive the space-time critical dimension for an open parabosonic string by using the meromorphic properties of the M-point transition amplitude. In Section 2, we describe the formalism and in Section 3 we derive the critical dimension and finally in Section 4 we draw our conclusions.

2. Formalism

The Nambu-Goto classical action of a free relativistic open bosonic string is given by [13]:

$$S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma [(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2]^{1/2}, \tag{3}$$

where τ and σ are dimensionless word-sheet parameters and α' is the string tension (here, "' " as in x' and "·" as in $\dot{x}$ denotes $\frac{\partial}{\partial\sigma}$ and $\frac{\partial}{\partial\tau}$, respectively). The general solution of the equations of motion in the light cone gauge is [13]:

$$x^i(\sigma,\tau) = q^i + 2\alpha' p^i + 2\alpha' \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [a_n^i e^{-in\tau} + a_n^{+i} e^{in\tau}] \cos n\sigma, \tag{4}$$

where q^i and p^i are the string centre of mass coordinates and momentum, respectively.

After quantization the physical states $|\Psi\rangle_{phy}$ are subject to the Virasoro conditions:

$$\begin{gathered} L_n|\Psi\rangle_{phy} = 0 \qquad n > 1 \\ \text{and} \\ [L_0 - \alpha(0)]|\Psi\rangle_{phy} = 0, \end{gathered} \tag{5}$$

(here, $\alpha(0) = 1$) where the Virasoro generators L_n and L_0 are given by:

$$\begin{aligned} L_n &= \frac{1}{2\alpha'} \sum_{m=1}^{\infty} : \alpha_{n-m}^i \alpha_m^i : \\ L_0 &= \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i \end{aligned} \tag{6}$$

with

$$\alpha_0^i = 2\alpha' p^i, \quad \alpha_{-n}^i = \sqrt{2\alpha' n} a_n^{+i}, \quad \alpha_n^i = \sqrt{2\alpha' n} a_n^i.$$

It is to be noted that the string dynamical variables q^i, p^i, q^-, p^+, a^i and a^{+i} verify the following non vanishing canonical commutation relations:

$$\begin{aligned} [q^i, p^i] &= i\delta^{ij} \\ [q^-, p^+] &= -i \\ [a_n^i, a_m^{+j}] &= \delta_{mn}\delta^{ij}. \end{aligned} \tag{7}$$

Now, for the paraquantization the commutation relations (2.5) become:

$$\begin{aligned} [q^i, p^i] &= i\delta^{ij} \\ [q^-, p^+] &= -i \\ [a_n^{i(\beta)}, a_m^{+j(\alpha)}]_+ &= 0, \quad \alpha \neq \beta \\ [a_n^{i(\alpha)}, a_m^{+j(\alpha)}]_- &= \delta_{mn}\delta^{ij}, \end{aligned} \tag{8}$$

where we have used the Green decomposition (1.1) for a_n^i and a_m^{+j} and the fact that the observables, like q^i, p^i, q^-, p^+, which describe the center of mass coordinates and momentum of the string, should not be affected by the paraquantization [14-17]. In other words, the space-time properties of the string remain unchanged. This can be achieved by choosing a specific direction in the Green para-space-like relations [14-17]:

$$q^{i(\alpha)} = q^i \delta_{\alpha 1}, \quad p^{i(\alpha)} = p^i \delta_{\alpha 1}, \quad q^{-(\alpha)} = q^- \delta_{\alpha 1}, \quad p^{+(\alpha)} = p^+ \delta_{\alpha 1}. \tag{9}$$

3. M-Point Transition Amplitude

The 1-loop open parabosonic string M-point transition amplitude for a planar diagrams with M external tachyons, which is topologically equivalent to a disk with a hole quenched in the interior and external lines located on the exterior edge, can be written as:

$$A(1, 2, \ldots, M) = \int \prod_{\beta=1}^{Q} d^D p^{(\beta)} Tr[\Delta V(k_1, 1)\Delta V(k_2, 1) \cdots \Delta V(k_M, M)]. \tag{10}$$

(Here, $k_j = \overline{1, M}$ is the jth external tachyon momentum and propagator Δ has is expressed as

$$\Delta = (L_0 - \alpha(0))^{-1} \tag{11}$$

with L_0 as the paraquantum Virasoro operator [15-17] given by:

$$L_0 = -\sum_{\beta=1}^{Q} \sum_{m=1}^{\infty} : \alpha_{-m}^{i(\beta)} \alpha_m^{i(\beta)} : \tag{12}$$

(we take $2\alpha' = 1$) and

$$\alpha(0) = Q(D-2)/24.$$

(": :" means normal ordering). The paraquantum vertex operator $V(K_r, 1)$ has the expression

$$V(k_r, 1) = e^{iL_0} \ V(k_r, 0) e^{iL_0},$$

where

$$V(k_r, 0) = g : \exp\left[\frac{i}{2}\sum_{\gamma=1}^{Q}\sum_{i=1}^{D-2} K^{i(\gamma)} q^{i(\gamma)}\right], \tag{13}$$

where g is the coupling.

It is to be noted that the propagator Δ has the following useful integral representation:

$$\Delta = \int dx \ x^{L_0 - \alpha(0) - 1}. \tag{14}$$

Now, using the integral representation (3.5) and the fact that

$$x^{L_0} V(k_r, 1) = V(k_r, x) x^{L_0}, \tag{15}$$

where

$$V(k_r, x) = e^{i \times L_0} V_0(k_r, 0) e^{-i \times L_0}, \tag{16}$$

where

$$V(k_r, x) = e^{i \times L_0} V_0(k_r, 0) e^{-i \times L_0}, \tag{17}$$

the transition amplitude (3.1) can be rewritten as:

$$A(1, 2, \ldots, M) = \int \prod_{i=1}^{M} dx_i \int \prod_{\beta=1}^{Q} d^D p^{(\beta)} Tr\left[V_0(k_1, x_1) \cdots V_0(k_M, x_1 \cdots x_M) w^{L_0 - 1 - \alpha(0)}\right] \tag{18}$$

with

$$w = x_1 x_2 \cdots x_M. \tag{19}$$

Noticing that

$$\prod_{i=1}^{M} dx_i = dw \prod_{i=1}^{M-1} \frac{d\rho_i}{\rho_i}, \tag{20}$$

where

$$\rho_i = x_1 x_2 \cdots x_i, \tag{21}$$

Eq. (3.8) can be simplified to:

$$A(1, 2, \ldots, M) = \int \frac{dw}{w^{1+Q(D-2)/24}} \int \prod_{r=1}^{M-1} \frac{d\rho_r}{\rho_r} \vartheta(\rho_r - \rho_{r+1}) I(1, 2, \ldots, M) \tag{22}$$

with

$$I(1,2,\ldots,M)=\int\frac{dw}{\beta=1}d^{D}p^{(\beta)}Tr\left[V_0(k_1,\rho_1)V_0(k_2,\rho_2)\cdots V_0(k_M,\rho_M)w^{L_0}\right]. \tag{23}$$

The trace (3.13) can be easily calculated by using the paraquantum coherent state method [2]. In fact, using the identity

$$TrM=\sum_{\beta=1}^{Q}\frac{1}{\pi}\int d\lambda_n^{(\beta)}d\lambda_n^{(\beta)}e^{-|\lambda_n^{(\beta)}|^2}\langle\lambda_n^{(\beta)}|M|\lambda_n^{(\beta=}\rangle, \tag{24}$$

where

$$|\lambda_n^{(\beta)}\rangle=\exp\left[\lambda_n^{(\beta)}a_n^{+(\beta)}\right]|0\rangle \tag{25}$$

and

$$a_n^{(\alpha)}|\lambda_m^{(\beta)}\rangle=\delta_{\alpha\beta}\delta_{nm}\lambda_n^{(\beta)}|\lambda_m^{(\beta)}\rangle \tag{26}$$

$$\langle\mu_n^{(\alpha)}|\lambda_m^{(\beta)}\rangle=\exp\left[\mu_n^{*(\alpha)}\lambda_m^{(\beta)}\right]\delta_{\alpha\beta}\delta_{nm} \tag{27}$$

($\mu_n^{(\alpha)}$ and $\lambda_m^{(\beta)}$ are arbitrary complex numbers) and the fact that

$$x^{\sum_{i=1}^{D-2}a_n^{+i(\beta)}a_n^{i(\beta)}}|\lambda_m^{(\beta)}\rangle=\delta_{nm}|\lambda_m^{(\beta)}x\rangle \tag{28}$$

and

$$\langle 0|\exp\left(-K_I\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}a_n^{i(\beta)}\right)\quad +^{\sum_{k=1}^{D-2}\sum_{m=1}ma_m^{k+(\beta)}a_m^{k(\beta)}}\exp\left(K_J\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}a_n^{j(\beta)}\right)|0\rangle$$
$$=(1-x)^{K_IK_J}\delta_{ij}, \tag{29}$$

straightforward calculations give:

$$I(1,2,\ldots,M)=Q[f(w)]^{-Q(D-2)}\left(-\frac{2\pi}{Lnw}\right)^{Q(D-2)/2}\prod_{I<J}[\Psi_{IJ}]^{k_Ik_J}, \tag{30}$$

where

$$f(w)=\prod_{n=1}(1-w^n) \tag{31}$$

and

$$\Psi_{IJ}=-2\pi i\ \exp\left[\frac{\ln^2 C_{JI}}{2\ln w}\right]\vartheta_1\left(\frac{\ln C_{JI}}{2\pi i}|\frac{\ln w}{2\pi i}\right)/\vartheta_1'\left(0|\frac{\ln w}{2\pi i}\right) \tag{32}$$

with

$$C_{JI}=\rho_J/\rho_I \tag{33}$$

and ϑ_1 (resp ϑ'_1) being Jacobi function (resp. its derivative). Now, introducing new variables

$$\nu_r = \frac{\ln \rho_r}{\ln w} \tag{34}$$

and

$$q = \exp\left(\frac{2\pi^2}{\ln w}\right), \tag{35}$$

and using the identities

$$\frac{dw}{w}\prod_{r=1}^{M-1}\frac{d\rho_r}{\rho_r}\vartheta(\rho_r-\rho_{r+1}) = \frac{1}{2\pi^2}(-\ln w)^{M+1}\frac{dq}{q}\prod_{r=1}^{M-1}\vartheta(\nu_{r+1}-\nu_r)d\nu_r \tag{36}$$

and

$$\frac{1}{w^{Q(D-2)/24}}[f(w)]^{-Q(D-2)} = \left(-\frac{\pi}{\ln q}\right)^{Q(D-2)/2}\frac{1}{q^{Q(D-2)/12}}[f(q^2)]^{-Q(D-2)}, \tag{37}$$

the transition amplitude (3.12) takes the form:

$$\begin{aligned} A(1,2,\ldots,M) = {} & \frac{Q}{\pi}g^M\int_0^1\prod_{i=1}^{M-1}\vartheta(\nu_{i+1}-\nu_i)d\nu_i\int_0^1 dqq^{-1+Q(2-D)/12}W^{-1-Q(2-D)/24} \\ & \times\left(-\frac{2\pi^2}{\ln q}\right)^M[f(q^2)]^{-Q(D-2)}\prod_{I<J}[\Psi_{IJ}]^{k_Ik_J}. \end{aligned} \tag{38}$$

Now, by extracting the $\ln q$ factor from (3.22) and using the kinematical relation

$$\sum_{I<J}K_IK_J = -1/2\sum_I K_I^2 = -M, \tag{39}$$

and in order that the integrand in (3.28) can be a meromorphic function, i.e., the only existent singularities are a finite number of poles, the power of the W factor must vanish. Consequently, one deduces that the space time critical dimension must verify the relation

$$D = \frac{24}{Q}+2.$$

4. Conclusion

We conclude that the meromorphic property of the M-point transition amplitude with external tachyons and the generalization of the quantization procedure are strongly related to the critical space-time dimension of the parabosonic string $D=\frac{24}{Q}+2$. More details are being investigated [19].

Acknowledgements

We are very grateful to Professors M. Lagraa and M. Tahiri from ES SENIA University for fruitful discussions and one of us (N.M) would like to thank Professors C. Burgers and C.S. Lam from the Physics Department of McGill University for useful private communications, and Professor Provost for the hospitality which I had during my stay at the Institut Non Linéaire de Nice.

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