

Multiple Scales Method for a Matrix KdV Equation

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Abstract

In this paper the method of multiple scales is used to derive an integrable matrix non-linear Schrödinger equation (MNLS) as an amplitude equation from the Matrix Korteweg-de Vries equation (MKdV) associated with symmetric spaces. Then the integrability of the MNLS equation is deduced by the derivation of the spectral problem for the MNLS equation from the spectral problem for the MKdV equation.

1. Introduction

It is well known that a multiple scales analysis of the KdV equation (and, indeed a wide variety of equations) leads to the NLS equation for the modulated amplitude. In [11] Zakharov and Kuznetsov (ZK) considered the derivation of the NLS equation from the KdV equation so as to study both infinite line and periodic spectral properties of these equations and showed a much deeper correspondence between these integrable equations not only on the level of the equation but also on the level of the linear spectral problem.

In this paper the multiple scales expansion method proposed by ZK [11] is considered for deriving a matrix nonlinear Schrödinger equation (MNLS) from the matrix KdV (MKdV) equation associated with symmetric spaces. We introduce a multiscale (ϵ) expansion into the MKdV equation and then balance the resultant equation in leading order in ϵ for each of the end terms in the expansion. In particular, we consider $n = 0, 1, 2$ in the derivation of the MNLS equation from the MKdV equation. We finally give the spectral problem for the MNLS equation derived by the spectral problem for the MKdV equation considering $n = \pm 1, \pm 3$.

In Section 2 we present some background material on the Matrix KdV equation and its spectral problem. In Sections 3 and 4, following ZK [11], we construct the NLS matrix system [5] and the corresponding spectral problem associated with a given symmetric space, starting with both the matrix KdV system and the corresponding spectral problem

associated with the same symmetric space [1].

Throughout the paper we make extensive use of Reduce [6] to calculate and simplify our results.

2. Matrix KdV Equation

The matrix KdV equation is in the form

$$\mathbf{u}_t = 3(\mathbf{u}^2)_x + \mathbf{u}_{xxx}. \quad (1)$$

The spectral problem for (1) is given by the system of equations:

$$\mathbf{L}\Phi = (\partial^2 + \mathbf{u})\Phi = \lambda\Phi, \quad (2)$$

$$\Phi_t = \mathbf{P}\Phi = (4\partial^3 + 6\mathbf{u}\partial + 3\mathbf{u}_x)\Phi, \quad (3)$$

where \mathbf{u} is a $N \times N$ matrix of functions and Φ is an N -dimensional column vector (see [11, 10, 2, 4]).

In general, the matrix KdV equation (1) is a system of N^2 coupled (scalar) equations, but often only the symmetric case is considered, reducing the number of equations to $\frac{1}{2}N(N+1)$.

Athorne and Fordy [1] associated a matrix KdV equation with each of the Hermitian symmetric spaces giving rise to the significantly reduced sub systems of (1). This class of equations is the reductions of the third-order isospectral flow of the linear scattering problem presented in Fordy and Kulish [5] in the context of generalized NLS equations.

We now consider some special matrices for \mathbf{u} .

3. Derivation of Matrix NLS Equation Associated with Hermitian Symmetric Spaces

In this section we briefly remind the reader of the original Zakharov-Kuznetsov calculation to derive the Matrix NLS Equation associated with the Hermitian symmetric space, starting with the Matrix KdV Equation associated with the same Hermitian symmetric space [11].

3.1. The Case $N = 2$ and \mathbf{u} Hermitian

We consider the Hermitian matrix

$$\mathbf{u} = \begin{pmatrix} u_1 & z \\ z^* & u_2 \end{pmatrix}, \quad (4)$$

where u_1, u_2 are real and z^* is the complex conjugate of z [11].

Under these assumptions Eq. (1) is equivalent to the system:

$$\begin{aligned} u_{1t} &= u_{1xxx} + 3(u_1^2 + |z|^2)_x, \\ u_{2t} &= u_{2xxx} + 3(u_2^2 + |z|^2)_x, \\ z_t &= z_{xxx} + 3[(u_1 + u_2)z]_x. \end{aligned} \quad (5)$$

System (5) has the simple exact solution

$$u_1 = a_1, \quad u_2 = a_2, \quad z = e^{i(kx - \omega(k)t)}, \quad (6)$$

where the dispersion relation is defined by

$$\omega(k) = k^3 - 3k(a_1 + a_2)$$

and a_1, a_2 are arbitrary constants. We may perform a separation of scales for system (5) considering solutions locally close to solution (6). Introduce a scale parameter ϵ dependence assuming

$$\begin{aligned} u_1 &= a_1 + \epsilon^2 v_1(\xi, \tau), & u_2 &= a_2 + \epsilon^2 v_2(\xi, \tau), \\ z &= \epsilon e^{i(kx - \omega(k)t)} q(\xi, \tau), & \xi &= \epsilon x, \quad \tau = -3k\epsilon^2 t. \end{aligned} \quad (7)$$

Substituting these into the first two equations of (5), as $\epsilon \rightarrow 0$, we find

$$v_1 = -\frac{1}{2a_1}|q|^2 + f, \quad v_2 = -\frac{1}{2a_2}|q|^2 + g, \quad (8)$$

where f, g are constants of integration. Substituting (7) into the third equation of (5), as $\epsilon \rightarrow 0$, we find

$$\begin{aligned} \text{at order } \epsilon: & \quad k^2 = a_1 + a_2, \\ \text{at order } \epsilon^3: & \quad q_r - iq_{\xi\xi} - iq(v_1 + v_2) = 0. \end{aligned} \quad (9)$$

Using (8) we transform (9) into the NLS equation which is given by

$$iq_\tau + q_{\xi\xi} - \frac{k^2}{2a_1 a_2} q|q|^2 = 0. \quad (10)$$

3.1.1. The Linear Spectral Problem

Let us now consider the spectral problem given by (2) and (3), for which u_1 and u_2 are represented as before, and assume that $\Phi = (\Phi_1, \Phi_2)^T$ will be found in the form

$$\begin{aligned} \Phi_1(x, t) &= e^{i(p+k/2)x + i(r+\omega(k)/2)t} \Pi_1(\xi, \tau), \\ \Phi_2(x, t) &= e^{i(p-k/2)x + i(r-\omega(k)/2)t} \Pi_2(\xi, \tau). \end{aligned}$$

Here p and r are unknown constants. Equations (2) and (3) are considered in the λ -plane near the point $\lambda = \lambda_0^2$ which, together with constants p and r , are determined from the requirement of vanishing in (2) and (3) the terms of zero order in ϵ . From (2) we have

$$-\left(p + \frac{k}{2}\right)^2 + \lambda_0^2 + a_1 = 0, \quad -\left(p - \frac{k}{2}\right)^2 + \lambda_0^2 + a_2 = 0,$$

from which

$$p = \frac{(a_1 - a_2)}{2k} = \frac{(2a_1 - k^2)}{2k}, \quad \lambda_0 = \frac{a_1(a_1 - k^2)}{k^2} = \frac{-1}{2\alpha},$$

where

$$\alpha = \frac{k^2}{2a_1a_2} = \frac{k^2}{2a_1(k^2 - a_1)}.$$

Now, by setting $\lambda = \lambda_0 - \mu\epsilon/\lambda_0$ and taking the limit as $\epsilon \rightarrow 0$ in (2), we obtain the system

$$\begin{aligned} 2i \left(p + \frac{k}{2} \right) \frac{\partial \Pi_1}{\partial x} + \Phi \Pi_2 &= \mu \Pi_1, \\ 2i \left(p - \frac{k}{2} \right) \frac{\partial \Pi_2}{\partial x} + \Phi^* \Pi_1 &= \mu \Pi_2, \end{aligned}$$

which represents one of the possible versions of the operator L for NLS equation (10).

Similarly, one can find the operator P for the NLS equation (10) from the requirement of equating the leading zero order terms to zero in ϵ taking $r = 2p^3$ [11].

3.2. The Case When $N = n$ and \mathbf{u} are Hermitian

We separate the diagonal part in $\mathbf{u} = u_{ij}$ by taking

$$u_{ij} = u_i \delta_{ij} + z_{ij}, \quad z_{ii} = 0, \tag{11}$$

and introduce n real numbers a_i , ($i = 1, 2, \dots, n$) (see [11]). Moreover, introduce a scaling parameter ϵ dependence, assuming

$$\begin{aligned} u_i &= a_i^2 + \epsilon^2 v_i(\xi, \tau), \quad \xi = \epsilon x, \quad \tau = 3\epsilon t, \\ z_{ij} &= \epsilon k e^{i(a_i - a_j)x - 2i(a_i^3 - a_j^3)t} Z_{ij}(\xi, \tau). \end{aligned} \tag{12}$$

We now substitute (11) and (12) into (1) and take the limit as $\epsilon \rightarrow 0$. For $i \neq j$ there occurs the following closed set of equations for $Z_{ij} = Z_{ji}^*$:

$$\frac{\partial}{\partial \tau} Z_{ik} + 2a_i a_k \frac{\partial}{\partial \xi} Z_{ik} + 2i(a_i - a_k) \sum_{j=1}^n Z_{ij} Z_{jk} = 0. \tag{13}$$

System (13) is a special case of the hyperbolic nonlinear system for N -waves ($N = \frac{1}{2}n(n-1)$). In order to find operators L and P for this system, let us turn to system (2) and (3) in which Φ represents a column of n elements. Let us assume for them

$$\Phi_i = e^{i(a_i x - 2a_i^3 t)} \Psi_i(\xi, \tau), \quad \lambda^2 = \epsilon \mu. \tag{14}$$

Substituting (11), (12) and (14) into (2) and (3), and then setting $\epsilon \rightarrow 0$, we find

$$\mu \Psi_k + 2ia_k \frac{\partial}{\partial \xi} \Psi_k + \sum_{j=1}^n Z_{kj} \Psi_j = 0, \quad \frac{\partial}{\partial \tau} \Psi_k - 2a_k^2 \frac{\partial}{\partial \xi} \Psi_k + \sum_{j=1}^n (a_k - a_j) Z_{kj} \Psi_j = 0. \tag{15}$$

System (13) is the compatibility condition for system (15).

Now let us obtain a vector analogue of the NLS equation from the matrix KdV equation (see [7]). Set u in the matrix form defined according to (12) and then assume

$$\begin{aligned} u_1 &= a_1 + \epsilon^2 v(\xi, \tau), & u_{i+1} &= a_2 + \epsilon^2 v_i(\xi, \tau), \\ z_{1,i+1} &= \epsilon e^{i(kx-2k^3t)} Z_i(\xi, \tau), & z_{i+1,j+1} &= \epsilon^2 Z_{ij}(\xi, \tau), \\ \xi &= \epsilon x, & \tau &= -3\epsilon^2 t, & a_1 + a_2 &= k^2, & i, j &= 1, 2, \dots, n-1. \end{aligned}$$

Substituting these into (1) and taking the limit $\epsilon \rightarrow 0$, we obtain

$$Z_{ij} = -\frac{Z_i^* Z_j}{2a_2}, \quad v_i = -\frac{|Z|^2}{2a_2}, \quad v = -\frac{1}{2a_1} \sum_{i=1}^{n-1} |Z|^2.$$

We now have a system of equations for Z_i

$$i \frac{\partial Z_i}{\partial \tau} + \frac{\partial^2 Z_i}{\partial \xi^2} - \alpha \sum_{k=1}^{n-1} |Z_k|^2 Z_i = 0, \tag{16}$$

where $\alpha = \frac{k^2}{2a_2 a_1}$ as before. Operators L and P for the equation (16) can be easily deduced from (2) and (3) by analogy with the operators L, P for the system of N -waves [11].

4. Derivation of the Matrix NLS Equation Associated with Symmetric Spaces

We now derive the Matrix NLS systems [5] associated with the symmetric space, starting with the Matrix KdV systems associated with the same symmetric space [1].

4.1. The Case $N = 2$ with u as a 2×2 Symmetric Matrix

We now consider (following [1]) the symmetric matrix

$$u = \begin{pmatrix} u_1 & u_3 \\ u_3 & u_2 \end{pmatrix}, \tag{17}$$

where u_1, u_2, u_3 are functions of x and t . In terms of the symmetric space coordinates u (17) the associated matrix KdV equation (1) is equivalent to the system

$$\begin{aligned} u_{1t} &= u_{1xxx} + 3(u_1^2 + u_3^2)_x, \\ u_{2t} &= u_{2xxx} + 3(u_2^2 + u_3^2)_x, \\ u_{3t} &= u_{3xxx} + 3[(u_1 + u_2)u_3]_x. \end{aligned} \tag{18}$$

Let us seek the solution to system (18) in the form

$$u_j(x, t) = \sum_{n=-\infty}^{\infty} u_{jn}(x, t) e^{in\theta} \quad \text{for } j = 1, 2, 3,$$

$$\theta(x, t) = kx - \omega(k)t, \quad \omega(k) = k^3. \quad (19)$$

Then we introduce a scaling parameter ϵ dependence by assuming

$$\begin{aligned} u_{jn}(x, t) &= \epsilon^{\alpha(n)} v_{jn}(\xi, \tau), \quad \text{for } j = 1, 2, 3, \\ \alpha(0) &= 2, \quad \alpha(n) = |n|, \quad \text{for } n \neq 0, \\ \xi &= \epsilon \left(x - \frac{d\omega(k)}{dk} t \right) = \epsilon(x - 3k^2 t), \quad \tau = -\frac{1}{2} \epsilon^2 \frac{d^2\omega(k)}{dk^2} t = -3\epsilon^2 kt. \end{aligned} \quad (20)$$

Substituting (19) into (18) and equating coefficients at each $e^{in\theta}$ separately and then using (20), we obtain three infinite series in the powers of ϵ for each n :

$$\begin{aligned} \left(-3k^2 \epsilon \frac{\partial}{\partial \xi} - 3k\epsilon^2 \frac{\partial}{\partial \tau} - ink^3 \right) \epsilon^{\alpha(n)} v_{1n} &= \left(\epsilon \frac{\partial}{\partial \xi} + ink \right)^3 \epsilon^{\alpha(n)} v_{1n} + 3 \left(\epsilon \frac{\partial}{\partial \xi} + ink \right) \\ &\quad \sum_{m=-\infty}^{\infty} \epsilon^{\alpha(m)+\alpha(n-m)} (v_{1m} v_{1n-m} + v_{3m} v_{3n-m}), \\ \left(-3k^2 \epsilon \frac{\partial}{\partial \xi} - 3k\epsilon^2 \frac{\partial}{\partial \tau} - ink^3 \right) \epsilon^{\alpha(n)} v_{2n} &= \left(\epsilon \frac{\partial}{\partial \xi} + ink \right)^3 \epsilon^{\alpha(n)} v_{2n} + 3 \left(\epsilon \frac{\partial}{\partial \xi} + ink \right) \\ &\quad \sum_{m=-\infty}^{\infty} \epsilon^{\alpha(m)+\alpha(n-m)} (v_{2m} v_{2n-m} + v_{3m} v_{3n-m}), \\ \left(-3k^2 \epsilon \frac{\partial}{\partial \xi} - 3k\epsilon^2 \frac{\partial}{\partial \tau} - ink^3 \right) \epsilon^{\alpha(n)} v_{3n} &= \left(\epsilon \frac{\partial}{\partial \xi} + ink \right)^3 \epsilon^{\alpha(n)} v_{3n} + 3 \left(\epsilon \frac{\partial}{\partial \xi} + ink \right) \\ &\quad \sum_{m=-\infty}^{\infty} \epsilon^{\alpha(m)+\alpha(n-m)} (v_{1m} + v_{2m}) v_{3n-m}. \end{aligned} \quad (21)$$

Now let $\epsilon \rightarrow 0$ and consider the leading order terms at minimal powers of the parameter ϵ for each n . It is sufficient to consider the case $n \geq 0$. Except for case $n = 1$, the system (21) is significantly reduced as $\epsilon \rightarrow 0$ converting into explicit expressions for corresponding v_{jn} . In particular:

(i) For the case $n = 0$: At order ϵ^3 , we find

$$\begin{aligned} v_{10} &= \frac{-2}{k^2} (|v_{11}|^2 + |v_{31}|^2) + f_{10}, \\ v_{20} &= \frac{-2}{k^2} (|v_{21}|^2 + |v_{31}|^2) + f_{20}, \\ v_{30} &= \frac{-1}{k^2} ((v_{21} + v_{11})v_{31}^* + v_{31}(v_{11}^* + v_{21}^*)) + f_{30}, \end{aligned} \quad (22)$$

where f_{10}, f_{20}, f_{30} are constants of integration which will be selected as zero to get the matrix NLS equation.

(ii) For the case $n = 1$: At order ϵ^3 , we find

$$\begin{aligned} -\frac{\partial v_{11}}{\partial \tau} &= i\frac{\partial^2 v_{11}}{\partial \xi^2} + 2i(v_{11}v_{10} + v_{11}^*v_{12} + v_{31}^*v_{32} + v_{31}v_{30}), \\ -\frac{\partial v_{21}}{\partial \tau} &= i\frac{\partial^2 v_{21}}{\partial \xi^2} + 2i(v_{21}^*v_{22} + v_{32}v_{31}^* + v_{21}v_{20} + v_{30}v_{31}), \\ -\frac{\partial v_{31}}{\partial \tau} &= i\frac{\partial^2 v_{31}}{\partial \xi^2} + i(v_{31}^*(v_{12} + v_{22}) + v_{32}(v_{21}^* + v_{11}^*) + \\ &\quad v_{31}(v_{10} + v_{20}) + v_{30}(v_{11} + v_{21})). \end{aligned} \tag{23}$$

(iii) For the case $n = 2$: At order ϵ^3 , we find

$$v_{12} = \frac{1}{k^2}(v_{11}^2 + v_{31}^2), \quad v_{22} = \frac{1}{k^2}(v_{21}^2 + v_{31}^2), \quad v_{32} = \frac{1}{k^2}(v_{21} + v_{11})v_{31}. \tag{24}$$

Equations (22), (23) and (24) may be used to find the matrix NLS equation in the $\xi\tau$ -coordinates by assuming the following form of $q_1(\xi, \tau), q_2(\xi, \tau), q_3(\xi, \tau)$:

$$q_1 = \frac{v_{11}}{k}, \quad q_2 = \frac{v_{31}}{k}, \quad q_3 = \frac{v_{31}}{k}, \tag{25}$$

$$\begin{aligned} iq_{1\tau} &= q_{1\xi\xi} - 2q_1(|q_1|^2 + 2|q_3|^2) - 2q_3^2q_2^*, \\ iq_{2\tau} &= q_{2\xi\xi} - 2q_2(|q_2|^2 + 2|q_3|^2) - 2q_3^2q_1^*, \\ iq_{3\tau} &= q_{3\xi\xi} - 2q_3(|q_3|^2 + |q_1|^2 + |q_2|^2) - 2q_1q_2q_3^*. \end{aligned} \tag{26}$$

The matrix NLS equation (26) can be written in the following matrix form:

$$i \begin{pmatrix} q_1 & q_3 \\ q_3 & q_2 \end{pmatrix}_\tau = \begin{pmatrix} q_1 & q_3 \\ q_3 & q_2 \end{pmatrix}_{\xi\xi} - 2i \begin{pmatrix} q_1 & q_3 \\ q_3 & q_2 \end{pmatrix} \begin{pmatrix} q_1 & q_3 \\ q_3 & q_2 \end{pmatrix}^* \begin{pmatrix} q_1 & q_3 \\ q_3 & q_2 \end{pmatrix}.$$

We therefore get the matrix NLS equation (26) associated with the symmetric spaces [5] by starting the matrix KdV equation (18) associated with the symmetric coordinates. We now consider the associated spectral problem to symmetric spaces for the matrix KdV equation (18) and find it for the matrix NLS equation (26).

4.1.1. The Linear Spectral Problem

Inserting (17) and $\Phi = (\Phi_1, \Phi_2)^T$ into (2) and (3), we can write the spectral problem in the form:

$$\begin{aligned} \Phi_{1xx} + (u_1 - \lambda)\Phi_1 + u_3\Phi_2 &= 0, \\ \Phi_{2xx} + (u_2 - \lambda)\Phi_2 + u_3\Phi_1 &= 0. \end{aligned} \tag{27}$$

$$\begin{aligned} \Phi_{1t} &= 4\Phi_{1xxx} + 6(u_1\Phi_{1x} + u_3\Phi_{2x}) + 3(u_{1x}\Phi_1 + u_{3x}\Phi_2), \\ \Phi_{2t} &= 4\Phi_{2xxx} + 6(u_3\Phi_{1x} + u_2\Phi_{2x}) + 3(u_{3x}\Phi_1 + u_{2x}\Phi_2). \end{aligned} \tag{28}$$

As before, we represent u_j , ($j = 1, 2, 3$) in the form of (19) with (20) and assume

$$\Phi_j(x, t) = \sum_{n=-\infty}^{\infty} \Pi_{jn}(x, t) e^{jn\theta(x, t)/2}, \quad (29)$$

where the summing up is taken over the odd n and $j = 1, 2$. Let us introduce a scaling parameter ϵ dependence into (29) by assuming

$$\Pi_{jn}(x, t) = \epsilon^{(|n|-1)/2} \Phi_{jn}(\xi, \tau), \quad \lambda = -\left(\frac{k}{2} - \mu\epsilon\right)^2, \quad (30)$$

where ξ and τ are given by (20) and μ is an arbitrary constant.

Substituting (19) with (29) into (27) and equating coefficients at every $e^{in\theta/2}$ separately, and then using (20) and (30), we obtain two series in the powers of ϵ for each n :

$$\begin{aligned} & \left(\epsilon \frac{\partial}{\partial \xi} + \frac{ink}{2} - \lambda^2\right) \epsilon^{(|n|-1)/2} \Phi_{1n} + \sum_{m=-\infty}^{\infty} \epsilon^{\alpha(m)+(|n-2m|-1)/2} \\ & \quad (v_{1m} \Phi_{1n-2m} + v_{3m} \Phi_{2n-2m}) = 0, \\ & \left(\epsilon \frac{\partial}{\partial \xi} + \frac{ink}{2} - \lambda^2\right) \epsilon^{(|n|-1)/2} \Phi_{2n} + \sum_{m=-\infty}^{\infty} \epsilon^{\alpha(m)+(|n-2m|-1)/2} \\ & \quad (v_{2m} \Phi_{2n-2m} + v_{3m} \Phi_{1n-2m}) = 0. \end{aligned} \quad (31)$$

Taking limit $\epsilon \rightarrow 0$ we end up with the following results for $n = \pm 1, \pm 3$:

(i) For the case $n = \pm 1$: At order ϵ , we find:

$$\begin{aligned} i \frac{\partial}{\partial \xi} \Phi_{11} + \frac{1}{k} (v_{11} \Phi_{11}^* + v_{31} \Phi_{21}^*) &= \mu \Phi_{11}, \\ i \frac{\partial}{\partial \xi} \Phi_{21} + \frac{1}{k} (v_{21} \Phi_{21}^* + v_{31} \Phi_{11}^*) &= \mu \Phi_{21}, \end{aligned} \quad (32)$$

with their complex conjugates.

(ii) For the case $n = \pm 3$, we find:

$$\begin{aligned} \Phi_{13} &= \frac{1}{2k^2} (v_{11} \Phi_{11} + v_{31} \Phi_{21}) = \frac{1}{2k} (q_1 \Phi_{11} + q_3 \Phi_{21}), \\ \Phi_{23} &= \frac{1}{2k^2} (v_{31} \Phi_{11} + v_{21} \Phi_{21}) = \frac{1}{2k} (q_3 \Phi_{11} + q_2 \Phi_{21}), \end{aligned} \quad (33)$$

with their complex conjugates.

Similarly, substituting (19) with (29) into (28) and equating coefficients at every $e^{in\theta/2}$ separately and then using (20) and (30), we obtain two infinite series in the powers of ϵ for each n . Then if we let $\epsilon \rightarrow 0$ and consider the leading order terms for cases $n = \mp 1, \mp 3$, we have the following:

(i) For the case $n = 1$: At order ϵ^2 , we find

$$\begin{aligned} \frac{\partial \Phi_{11}}{\partial \tau} &= -i \frac{\partial^2 \Phi_{11}}{\partial \xi^2} - \frac{1}{k} \left(v_{11} \frac{\partial \Phi_{11}^*}{\partial \xi} + v_{31} \frac{\partial \Phi_{21}^*}{\partial \xi} \right) - \frac{1}{2k} \left(\frac{\partial v_{11}}{\partial \xi} \Phi_{11}^* + \frac{\partial v_{31}}{\partial \xi} \Phi_{21}^* \right) \\ &\quad - \frac{i}{2} (v_{10} \Phi_{11} + v_{30} \Phi_{21}) - i (v_{11}^* \Phi_{13} + v_{31}^* \Phi_{23}), \\ \frac{\partial \Phi_{21}}{\partial \tau} &= -i \frac{\partial^2 \Phi_{21}}{\partial \xi^2} - \frac{1}{k} \left(v_{31} \frac{\partial \Phi_{11}^*}{\partial \xi} + v_{21} \frac{\partial \Phi_{21}^*}{\partial \xi} \right) - \frac{1}{2k} \left(\frac{\partial v_{31}}{\partial \xi} \Phi_{11}^* + \frac{\partial v_{21}}{\partial \xi} \Phi_{21}^* \right) \\ &\quad - \frac{i}{2} (v_{30} \Phi_{11} + v_{20} \Phi_{21}) - i (v_{31}^* \Phi_{13} + v_{21}^* \Phi_{23}). \end{aligned} \quad (34)$$

(ii) For the case $n = -1$; At order ϵ^2 , we find the complex conjugate of (34).

(iii) For the cases $n = \pm 3$, we obtain (33) and its complex conjugate respectively.

At $n = \pm 1$ by using (25) we obtain the operator L and P for the matrix NLS equation (26), respectively:

$$\begin{aligned} \begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}_\xi &= i \begin{pmatrix} -\mu & Q \\ -Q^* & \mu \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}, \\ \begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}_\tau &= -2 \left[i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial^2}{\partial \xi^2} + \begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix} \frac{\partial}{\partial \xi} + \frac{1}{2} \right. \\ &\quad \left. \begin{pmatrix} i(QQ^* + v_0) & Q_\xi \\ Q_\xi^* & -i(v_0 + Q^*Q) \end{pmatrix} \right] \begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix} \end{aligned} \quad (35)$$

where $\Phi_1 = (\Phi_{11}, \Phi_{21})^T$, $\Phi_1^* = (\Phi_{11}^*, \Phi_{21}^*)^T$,

$$Q = \begin{pmatrix} q_1 & q_3 \\ q_3 & q_2 \end{pmatrix}, \quad \mu = \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v_0 = -(QQ^* + Q^*Q) = \begin{pmatrix} v_{10} & v_{30} \\ v_{30} & v_{20} \end{pmatrix},$$

and

$$\begin{aligned} v_{10} &= -2(|q_1|^2 + |q_3|^2), & v_{20} &= -2(|q_2|^2 + |q_3|^2), \\ v_{30} &= -(|q_1 + q_2|q_3^* + (q_1^* + q_2^*)q_3). \end{aligned}$$

4.2. The General $N \times N$ Symmetric Matrix Case

This derivation can be generalized to N -dimensions by starting the matrix KdV equation in order to get Matrix NLS equation, associated with symmetric matrix spaces. We now consider (following [1]) $N \times N$ symmetric matrix and separate diagonal part by taking

$$\mathbf{u} = u_{ij} = u_i \delta_{ij} + z_{ij}, \quad z_{ii} = 0, \quad (36)$$

with

$$z_{ij} = z_{ji} = u_k, \quad k = N + 1, \dots, \frac{1}{2}N(N + 1), \quad i, j = 1, \dots, N,$$

and u_j are functions of x and t .

We seek the solution to the associated matrix KdV equation (1) in terms of the symmetric space coordinates \mathbf{u} (36) in the form

$$u_j(x, t) = \sum_{n=-\infty}^{\infty} u_{jn}(x, t)e^{in\theta} \quad \text{for } j = 1, \dots, \frac{1}{2}N(N+1),$$

where $\theta(x, t)$ is given by (19). We then introduce a scaling parameter ϵ dependence by assuming (20) with $j = 1, \dots, \frac{1}{2}N(N+1)$.

Substituting these into (1) and equating coefficients at each $e^{in\theta}$ separately and then using (20), we obtain $\frac{1}{2}N(N+1)$ infinite series in the powers of ϵ for each n . We let $\epsilon \rightarrow 0$ and consider the leading order terms at minimal powers of the parameter ϵ for each n . It is sufficient to consider the case $n \geq 0$. Except for the case $n = 1$, the system (21) is significantly reduced as $\epsilon \rightarrow 0$ converting into explicit expressions for corresponding \mathbf{v}_n . In particular:

(i) For the case $n = 0$: At order ϵ^3 , we find

$$\mathbf{v}_0 = \frac{-1}{k^2}(\mathbf{v}_1\mathbf{v}_1^* + \mathbf{v}_1^*\mathbf{v}_1) + \mathbf{f}, \tag{37}$$

where \mathbf{f} , being $N \times N$ symmetric matrix of the form (36), is a constant of integration which will be selected as zero to get the matrix NLS equation (40).

(ii) For the case $n = 1$: At order ϵ^3 , we find

$$-\frac{\partial \mathbf{v}_1}{\partial \tau} = i\frac{\partial^2 \mathbf{v}_1}{\partial \xi^2} + 2i(\mathbf{v}_0\mathbf{v}_1 + \mathbf{v}_2\mathbf{v}_1^*), \tag{38}$$

with complex conjugates.

(iii) For the case $n = 2$: At order ϵ^3 , we find

$$\mathbf{v}_2 = \frac{1}{k^2}\mathbf{v}_1^2, \tag{39}$$

\mathbf{v}_j are $N \times N$ symmetric matrix of the form (36), given by

$$\mathbf{v}_0 = \begin{pmatrix} v_{10} & v_{30} \\ v_{30} & v_{20} \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} v_{11} & v_{31} \\ v_{31} & v_{21} \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} v_{12} & v_{32} \\ v_{32} & v_{22} \end{pmatrix}$$

for $N = 2$.

We now use equations (37), (38) and (39) to find the matrix NLS equation in the $\xi\tau$ -coordinates

$$i\mathbf{Q}_\tau = \mathbf{Q}_{\xi\xi} - 2\mathbf{Q}\mathbf{Q}^*\mathbf{Q}, \tag{40}$$

by assuming the following form of

$$\mathbf{Q} = \frac{\mathbf{v}_1}{k}, \tag{41}$$

with \mathbf{Q} being an $N \times N$ symmetric matrix of the form (36).

We therefore get the matrix NLS equation (40) associated with the symmetric spaces [5] by starting the matrix KdV equation (1) associated with the symmetric space coordinates.

4.2.1. The Linear Spectral Problem

We consider the associated spectral problem to symmetric spaces for the matrix KdV equation (1), defined by (2) and (3), with the matrix \mathbf{u} (36). As before, we represent u_j , ($j = 1, \dots, \frac{1}{2}N(N+1)$) in the form of (19) and assume

$$\Phi_m(x, t) = \sum_{n=-\infty}^{\infty} \Pi_{mn}(x, t) e^{mn\theta(x,t)/2}, \quad (42)$$

where the sum up is taken over the odd n and $m = 1, \dots, N$. We also introduce a scaling parameter ϵ dependence into (42) by assuming

$$\Pi_{mn}(x, t) = \epsilon^{(|n|-1)/2} \Phi_{mn}(\xi, \tau), \quad \lambda = -\left(\frac{k}{2} - \mu\epsilon\right)^2, \quad (43)$$

where ξ and τ are given by (20) and μ is an arbitrary constant.

In a similar way we obtain the operator L and P for the matrix NLS equation (40) which are given by (35) with $\Phi_1 = (\Phi_{11}, \dots, \Phi_{N1})^T$, \mathbf{Q} being an $N \times N$ symmetric matrix of the form (36),

$$\mu = \mu I, \quad \mathbf{v}_0 = -(\mathbf{Q}\mathbf{Q}^* + \mathbf{Q}^*\mathbf{Q}).$$

5. Conclusion

We have used the multiple scales method to derive the integrable system of matrix NLS equation, together with the corresponding spectral problem from the integrable system of matrix KdV equation. Starting with the KdV systems associated with symmetric spaces [1], we derive the NLS systems [5] associated with the same symmetric spaces. For instance, starting with the coupled KdV system, isospectral to an energy-dependent Schrödinger operator, we derive a new integrable system of NLS equations together with the corresponding spectral problem. The details are given in [3, 9].

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