Tr. J. of Physics
22 (1998) , 939 – 948.

Quantum Integrable Systems of the Group SU(1, 1)

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Received 12.11.1997

Abstract

Quantum integrable systems related with SU(1, 1) group manifold or hyperboloid $[x, x] = x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1$ in spherical, hyperbolic and parabolic coordinates systems are considered. The explicit expressions for waves functions, spectra and S-matrices are given.

1. Introduction

There exist many coordinate systems which reduce to separation of variables in Laplace-Beltrami operators given in [1]. But only for those which are geodesics, in other words, which relate to one parameter subgroup of symmetry group, does there exist a simple transformation of Laplace-Beltrami operators on symmetrical spaces (SS) to some Hamiltonian quantum systems. Hence only the distortion of the symmetry of the free particle motion on SS by the geodesic paths reduce to the dynamics of quantum systems.

The one dimensional integrable quantum systems related to free motions in symmetric spaces (SS) of the non compact groups SO(1,2), U(1,2), Sp(1,2) are considered in [2, 3]. As shown in [2, 3], the dynamics of a quantum system depend on the stabilizer (stationary) subgroup of the fixed point of the SS and coordinate systems on SS which is chosen. For the case of the SS with the compact stabilizer subgroup the quantum system has only continuous spectrum; but for the SS with the non compact one-quantum systems, it has discrete and continuous spectrum. This is because in the case of the SS with the compact stabilizer subgroup the distance between two points on SS is real but in the SS with the non compact case the distance has real and imaginary parts. The results of the case of the SS with the compact stabilizer subgroup are given in [4].

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The group SU(1,1) of the matrices $\begin{bmatrix} a & b \\ b & \overline{a} \end{bmatrix}$, $|a|^2 - |b|^2 = 1$ define the single sheet hyperboloid $[x, x] = x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1$. To our knowledge, no consistent and complete treatment for the single sheet hyperboloid exist up to now (see [5, 6]). Our solution of the hyperboloid for the parabolic coordinate is the original. Recently, there has been path integral treatments of the hyperboloid [6]. However, the author wrongly reported wave function on the hyperboloid $[x, x] = 1$ in the parabolic coordinate (see Eq. (46) in [6]). We shall show thatthe correct wave function for this case is expressed as Eq. (45) below.

We consider spherical, hyberbolic and parabolic (horispherical) coordinate systems on hyperboloid. In the Section 2, for a related quantum system, we give explicit expressions for wave functions, spectra and S-matrices.

2. Dynamics Related with SU(1,1) Group Manifold

Let us consider bispherical, hyperbolic and parabolic (horispherical) coordinates systems on the hyperboloid $[x, x] = x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1$, which define a SU(1,1) group manifold.

2.1. Bispherical Coordinate System

$$x = (\cosh\alpha\cos\varphi_1, \cosh\alpha\sin\varphi_1, \sinh\alpha\cos\varphi_2, \sinh\alpha\sin\varphi_2), \tag{1}$$

where $0 \leq \alpha < \infty$, $0 \leq \varphi_{1,2} \leq 2\pi$. From Eq. (1) follows the metric matrix

$$(g_{ab}) = diag(-1, \cosh^2\alpha, -\sinh^2\alpha)$$

and the Laplace-Beltrami operator

$$\Delta_{L,B} = \frac{-1}{J(\alpha)}\frac{\partial}{\partial\alpha}J(\alpha)\frac{\partial}{\partial\alpha} + \frac{1}{\cosh^2}\frac{\partial^2}{\partial\varphi_1^2} - \frac{1}{\sinh^2}\frac{\partial^2}{\partial\varphi_2^2}, \tag{2}$$

where $J(\alpha) = \sinh\alpha\cosh\alpha$.

Free motion on the hyperboloid $[x, x] = 1$ is defined by the equation

$$\Delta_{L,B}\Phi(\alpha, \varphi_1, \varphi_2) = -\sigma(\sigma+2)\Phi(\alpha, \varphi_1, \varphi_2). \tag{3}$$

After the substitution of

$$\Phi(\alpha, \varphi_1, \varphi_2) = \Lambda(\alpha)e^{im\varphi_1}e^{in\varphi_2}$$

into Eq.(3) we have

$$\frac{1}{J(\alpha)}\frac{\partial}{\partial\alpha}J(\alpha)\frac{\partial\Lambda}{\partial\alpha} + \left(\frac{m^2}{\cosh^2\alpha} - \frac{n^2}{\sinh^2\alpha}\right)\Lambda(\alpha) = \sigma(\sigma+2)\Lambda(\alpha), \tag{4}$$

with $\|\Lambda(\alpha)\| = \int_0^\infty |\Lambda(\alpha)|^2 J(\alpha)d\alpha$.

By the transformation

$$\Lambda(\alpha) = \frac{1}{\sqrt{J(\alpha)}}\Psi(\alpha), \tag{5}$$

Eq. (4) is reduced to the one dimensional Schrödinger equation with the potential

$$V(\alpha) = \frac{-\frac{1}{4} + n^2}{\sinh^2\alpha} + \frac{\frac{1}{4} - m^2}{\cosh^2\alpha} \tag{6}$$

and energy spectrum

$$E = -(\sigma + 1)^2. \tag{7}$$

By the substitution

$$\Lambda(\alpha) = \tanh^n \alpha \cosh^\sigma \alpha W(\alpha)$$

and the transformation $z = \tanh^2\alpha$, Eq. (4) is reduced to the hypergeometric equation and for the regular solution at $\alpha = 0$ we have [7, 8]:

$$\Lambda(\alpha) = \tanh^{|n|}\alpha \cosh^\sigma \alpha F\left(\frac{-\sigma + |m| + |n|}{2}, \frac{-\sigma - |m| + |n|}{2}; \quad |n| + 1; \quad \tanh^2\alpha\right). \tag{8}$$

It follows that the quantum system has bound and scattering states. The square integrable normalized wave function for $\sigma = 1$ with $l - |m| - |n| = 2k, \;\; k = 0, 1, 2, \ldots$ has the form:

$$\Psi_E(\alpha) = c_2 \sinh^{|n|+\frac{1}{2}}\alpha \cosh^{|m|+\frac{1}{2}}\alpha P^{(|n|,|m|)}_{\frac{l-|m|-|n|}{2}}(\cosh 2\alpha), \tag{9}$$

where

$$c_2 = \sqrt{\frac{(l+1)\Gamma\left(\frac{l-|m|+|n|+2}{2}\right)\Gamma\left(\frac{l+|m|-|n|+2}{2}\right)}{\Gamma^2(|n|+1)\Gamma\left(\frac{l-|m|-|n|+2}{2}\right)\Gamma\left(\frac{l+|m|-|n|+2}{2}\right)}}$$

For continuous spectrum $E = \rho^2 > 0, \;\; \sigma = -1 + i\rho$ we calculate the S-matrix using the analytical continuous formula 2.10 (1) for the hypergeometric function of [9]:

$$\begin{aligned} F(a, b; c; z) = A_1 F(a, b; a + b - c + 1; 1 - z) + A_2(1 - z)^{c-a-b} \\ F(c - a, c - b; c - a - b + 1; 1 - z), |\arg(1 - z)| < \pi \end{aligned} \tag{10}$$

$$A_1 = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \qquad A_2 = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}. \tag{11}$$

From the Eq. (10) and Eq. (8) we have the asymptotic expression

$$\Lambda(\alpha)_{\alpha\to\infty} = A(m, n, \rho)e^{(-1+i\rho)\alpha} + \overline{A}(m, n, \rho)e^{-(1+i\rho)\alpha}, \tag{12}$$

where

$$A(m, n, \rho) = \frac{\Gamma(|n| + 1)\Gamma(i\rho)}{\Gamma\left(\frac{i\rho-|m|+|n|+1}{2}\right)\Gamma\left(\frac{i\rho+|m|+|n|+1}{2}\right)}.$$

Solutions of the Schrödinger equation with potential Eq. (6 satisfying the condition

$$\int_0^\infty \Psi_E(\alpha)\overline{\Psi_E}(\alpha)d\alpha = \delta(\rho - \rho') \tag{13}$$

have the form:

$$\Psi_E(\alpha) = \tilde{c}_2 \sinh^{|n|+\frac{1}{2}} \alpha \cosh^{i\sqrt{E}-\frac{1}{2}} \alpha F \left(\frac{-i\sqrt{E} + |m| + |n| + 1}{2}, \right.$$
$$\left. \frac{-i\sqrt{E} - |m| + |n| + 1}{2}; \;\; |n| + 1; \;\; tanh^2\alpha \right), \tag{14}$$

where

$$|\tilde{c}_2| = \frac{1}{\sqrt{2\pi}|A|}.$$

The S-matrix is found to be

$$S = \frac{A}{\overline{A}} = \frac{\Gamma(i\sqrt{E})\Gamma\left(\frac{-i\sqrt{E}-|m|+|n|+1}{2}\right)\Gamma\left(\frac{-i\sqrt{E}+|m|+|n|+1}{2}\right)}{\Gamma(i\sqrt{E})\Gamma\left(\frac{i\sqrt{E}-|m|+|n|+1}{2}\right)\Gamma\left(\frac{i\sqrt{E}+|m|+|n|+1}{2}\right)}. \tag{15}$$

2.2. Hyperbolic Coordinate System

We have:

$$x = (\cosh\alpha\cosh\beta\cos\varphi, \; \cosh\alpha\cosh\beta\sin\varphi, \; \cosh\alpha\sinh\beta, \sinh\alpha), \tag{16}$$

where $-\infty < \alpha < \infty$, $-\infty < \beta < \infty$, $0 \le \varphi \le 2\pi$. From Eq. (16) follows the metric matrix

$$(g_{ab}) = diag(-1, -\cosh^2\alpha, \; \cosh^2\alpha\cosh^2\beta)$$

and the Laplace-Beltrami operator

$$\Delta^{(3)}_{L,B} = \frac{-1}{J(\alpha)}\frac{\partial}{\partial\alpha}J(\alpha)\frac{\partial}{\partial\alpha} + \frac{1}{\cosh^2\alpha}\Delta^{(2)}_{L,B}, \tag{17}$$

where $J(\alpha) = \cosh^2\alpha$ and

$$\Delta^{(2)}_{L,B} = \frac{-1}{J'(\beta)}\frac{\partial}{\partial\beta}J'(\beta)\frac{\partial}{\partial\beta} + \frac{1}{\cosh^2}\frac{\partial^2}{\partial\varphi^2}$$

with $J'(\beta) = \cosh\beta$.

Free motion on the hyperboloid is defined by the equation:

$$\Delta^{(3)}_{L,B}\Phi(\alpha,\beta,\varphi) = -\sigma(\sigma+2)\Phi(\alpha,\beta,\varphi). \tag{18}$$

Setting

$$\Phi(\alpha,\beta,\varphi) = A(\alpha)\Omega(\beta)e^{im\varphi}$$

and using

$$\Delta^{(2)}_{L,B}\Omega(\beta) = -\sigma_1(\sigma_1+1)\Omega(\beta) \tag{19}$$

we have

$$\frac{1}{J(\alpha)}\frac{\partial}{\partial\alpha}J(\alpha) + \frac{\sigma_1(\sigma_1+1)}{\cosh^2\alpha}A(\alpha) = \sigma(\sigma+2)A(\alpha), \tag{20}$$

with $\|A(\alpha)\| = \int_{-\infty}^{\infty}|A(\alpha)|^2 J(\alpha)d\alpha$.

Using the transformation

$$A(\alpha) = \frac{1}{\sqrt{J(\alpha)}}\Psi(\alpha),$$

Eq. (20) is reduced to the one dimensional Schrödinger equation with the potential

$$V(\alpha) = -\frac{\sigma_1(\sigma_1+1)}{\cosh^2\alpha} \tag{21}$$

and energy spectrum

$$E = -(\sigma+1)^2. \tag{22}$$

The substitution

$$A(\alpha) = \cosh^{\sigma_1}\alpha W(\alpha)$$

and the transformation $z = -\sinh^2\alpha$ reduces Eq.(20) to the hypergeometric equation. Regular solutions of Eq. (20) at $\alpha = 0$ have the form

$$A_1(\alpha) = \cosh^{\sigma_1}\alpha F\left(\frac{\sigma_1-\sigma}{2}, \frac{\sigma_1+\sigma+2}{2}; \frac{1}{2}; -\sinh^2\alpha\right)$$

and

$$A_2(\alpha) = \cosh^{\sigma_1}\alpha\sinh\alpha F\left(\frac{\sigma_1-\sigma+1}{2}, \frac{\sigma_1+\sigma+3}{2}; \frac{3}{2}; -\sinh^2\alpha\right). \tag{23}$$

It follows that the quantum system with the potential of Eq.(21) has bound and scattering states. The normalized wave function for the discrete spectrum $E = -(l+1)^2$ with

$$\sigma = 1, \quad \sigma_1 = l_1, \quad \frac{l-l_1}{2} = k, \quad \frac{l-l_1-1}{2} = k', \quad k, k' = 0, 1, 2\ldots$$

have the form:

$$\Psi^{(1)}_E(\alpha) = c_3\cosh^{l_1+1}\alpha\, P^{(-\frac{1}{2}, l_1+\frac{1}{2})}_{\frac{l-l_1}{2}}(\cosh 2\alpha)$$

and

$$\Psi_E^{(2)}(\alpha) = \tilde{c}_3 \cosh^{l_1+1}\alpha\ \sinh\alpha\ P^{(+\frac{1}{2}, l_1+\frac{1}{2})}_{\frac{l-l_1-1}{2}}(\cosh 2\alpha), \tag{24}$$

where

$$c_3 = \sqrt{\frac{(l+1)\Gamma\left(\frac{l-l_1+1}{2}\right)\Gamma\left(\frac{l+l_1+2}{2}\right)}{\pi\Gamma\left(\frac{l-l_1+2}{2}\right)\Gamma\left(\frac{l+l_1+3}{2}\right)}}, \quad \tilde{c}_3 = \sqrt{\frac{(l+1)\Gamma\left(\frac{l-l_1+2}{2}\right)\Gamma\left(\frac{l+l_1+3}{2}\right)}{\pi\Gamma^2\left(\frac{3}{2}\right)\Gamma\left(\frac{l-l_1+1}{2}\right)\Gamma\left(\frac{l+l_1+2}{2}\right)}}.$$

To calculate S-matrices for the continuous spectrum $\sqrt{E} = \rho,\ \sigma = -1 + i\rho,\ \ 0 < \rho < \infty$ we use the formula 2.10 (2) of [9]:

$$F(a, b; c; z) = B_1(-z)^a\ F\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right) + B_2(-z)^{-b} F\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right),\ |\arg(-z)| < \pi, \tag{25}$$

where

$$B_1 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}, \quad B_2 = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}.$$

From the Eq. (23) and Eq. (25) we have asymptotic expressions:

$$A_1\alpha \to \infty = B(\sigma_1, \rho)e^{(-1+i\rho)\alpha} + \overline{B}(\sigma_1, \rho)e^{-(1+i\rho)\alpha}$$

and

$$A_2\alpha \to \infty = \tilde{B}(\sigma_1, \rho)e^{(-1+i\rho)\alpha} + \overline{\tilde{B}}(\sigma_1, \rho)e^{-(1+i\rho)\alpha}, \tag{26}$$

where

$$B(\sigma_1, \rho) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(i\rho)}{\Gamma\left(\frac{i\rho+\sigma_1+1}{2}\right)\Gamma\left(\frac{i\rho-\sigma_1}{2}\right)}, \quad \tilde{B}(\sigma_1, \rho) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(i\rho)}{\Gamma\left(\frac{i\rho+\sigma_1+2}{2}\right)\Gamma\left(\frac{i\rho-\sigma_1+1}{2}\right)}.$$

Thus the wave functions $\Psi_E(\alpha)$ with the condition

$$\int_{-\infty}^{\infty} \Psi_E(\alpha)\overline{\Psi_E}(\alpha)d\alpha = \delta(\rho - \rho') \tag{27}$$

has the form

$$\Psi_E^{(1)}(\alpha) = c_4 \cosh^{\sigma_1+1}\alpha F\left(\frac{-i\sqrt{E}+\sigma_1+1}{2}, \frac{i\sqrt{E}+\sigma_1+1}{2}; \frac{1}{2}; -\sinh^2\alpha\right)$$

and

$$\Psi_E^{(2)}(\alpha) = \tilde{c}_4 \cosh^{\sigma_1+1}\alpha \sinh\alpha F\left(\frac{-i\sqrt{E}+\sigma_1+2}{2}, \frac{i\sqrt{E}+\sigma_1+2}{2}; \frac{3}{2}; -\sinh^2\alpha\right), \tag{28}$$

where

$$|c_4| = \frac{1}{2\sqrt{\pi}|B|}, \qquad \tilde{c}_4| = \frac{1}{2\sqrt{\pi}|\tilde{B}|}.$$

The S-matrices to be found are given by

$$S^{(1)} = \frac{B}{\overline{B}} = \frac{\Gamma(i\sqrt{E})\Gamma\left(\frac{-i\sqrt{E}+\sigma_1+1}{2}\right)\Gamma\left(\frac{-i\sqrt{E}-\sigma_1}{2}\right)}{\Gamma(-i\sqrt{E})\Gamma\left(\frac{i\sqrt{E}+\sigma_1+1}{2}\right)\Gamma\left(\frac{i\sqrt{E}-\sigma_1}{2}\right)}$$

and

$$S^{(2)} = \frac{\tilde{B}}{\overline{\tilde{B}}} = \frac{\Gamma(i\sqrt{E})\Gamma\left(\frac{-i\sqrt{E}+\sigma_1+2}{2}\right)\Gamma\left(\frac{-i\sqrt{E}-\sigma_1+1}{2}\right)}{\Gamma(-i\sqrt{E})\Gamma\left(\frac{i\sqrt{E}+\sigma_1+2}{2}\right)\Gamma\left(\frac{i\sqrt{E}-\sigma_1+1}{2}\right)}. \tag{29}$$

2.3. Parabolic (Horispherical) Coordinate System

We have

$$x = \left(\cosh\frac{t}{2} - \frac{1}{2}e^{\frac{t}{2}}q^2,\ \ e^{\frac{t}{2}}q_1, e^{\frac{t}{2}}q_2,\ \ \sinh\frac{t}{2} + \frac{1}{2}e^{\frac{t}{2}}q^2\right), \tag{30}$$

where $q^2 = q_1^2 - q_2^2$, $-\infty < t < \infty$, $-\infty < q_{1,2} < \infty$. From Eq. (30) follows the metric matrix

$$(g_{ab}) = diag\left(-\frac{1}{4},\ e^t, -e^t\right)$$

and the Laplace-Beltrami operator

$$\Delta_{L,B} = \frac{-4}{J(t)}\frac{\partial}{\partial t}J(t)\frac{\partial}{\partial t} + \frac{1}{e^t}\left(\frac{\partial^2}{\partial q_1^2} - \frac{\partial^2}{\partial q_2^2}\right), \tag{31}$$

where $J(t) = e^t$.

The free motion on the hyperboloid is defined by the equation:

$$\Delta_{L,B}\Phi(t, q_1, q_2) = -\sigma(\sigma+2)\Phi(t, q_1, q_2). \tag{32}$$

By the separation of the variables,

$$\Phi(t, q_1, q_2) = T(t)e^{ivq_1}e^{i\mu q_2}$$

from Eq. (32) and we have:

$$4\frac{d^2T}{dt^2} + 4\frac{dT}{dt} + \frac{(v^2-\mu^2)}{e^t}T(t) = \sigma(\sigma+2)T(t) \tag{33}$$

with $\|T(t)\| = \int_{-\infty}^{\infty} |T(\alpha)|^2 e^t dt.$

By the transformation

$$T(t) = \frac{1}{\sqrt{J(t)}}\Psi(t),$$

Eq. (33) is reduced to the one dimensional Schrödinger equation with the potential

$$V(t) = \frac{\mu^2 - v^2}{4e^t} \tag{34}$$

and energy spectrum

$$E = -\frac{(\sigma+1)^2}{4}. \tag{35}$$

After the transformation $z = e^{-\frac{t}{2}}$ we have Bessel's equation 7.2.1 (1) of [9]:

$$z^2\frac{d^2\Psi}{dz^2} + z\frac{d\Psi}{dz} + ((v^2-\mu^2)z^2 - (\sigma+1)^2)\Psi(z) = 0. \tag{36}$$

In the case when $v^2 - \mu^2 < 0$ the quantum system has only scattering states with energy $E = \frac{\rho^2}{4}$, $\sigma = -1 + i\rho$, $0 < \rho < \infty$ and the wave functions which tend to zero for $t \to -\infty$ are given by the Mcdonald K-function:

$$\Psi_E(t) = c_5 K_{i\sqrt{E}}(\sqrt{\mu^2 - v^2}e^{-t/2}), \qquad \rho = \sqrt{E}. \tag{37}$$

In order to calculate the S-matrix and c_5-factor we use the definition of the K-function from Eqs. 7.2.2 (12) and (13) of [9]:

$$K_v(z) = \frac{\pi}{2\sin v\pi}[I_{-v}(z) - I_v(z)], \tag{38}$$

where

$$I_v(z) = \frac{\left(\frac{z}{2}\right)^v}{\Gamma(v+1)^0}F_1\left(v+1; \frac{z^2}{4}\right).$$

It follows

$$I_v(z)_{z\to 0} \to \frac{z^v}{2^v\Gamma(1+v)}. \tag{39}$$

Thus we have:

$$\Psi_E(t)_{t\to\infty} = De^{i\rho t/2} + \overline{D}e^{-i\rho t/2}, \tag{40}$$

where $D = \dfrac{\pi(\sqrt{\mu^2-v^2})^{i\rho}}{2i\sinh\rho\pi 2^{i\rho}\Gamma(-i\rho+1)}$. Thus the wave function $\Psi_E(t)$ with condition Eq. (27) has the form

$$\Psi_E(t) = \frac{1}{2\sqrt{\pi}|D|}K_{i\sqrt{E}}(\sqrt{\mu^2-v^2}e^{-t/2}). \tag{41}$$

The S-matrix to be found is given by

$$S = (\sqrt{\mu^2-v^2})^{-2i\sqrt{E}}2^{2i\sqrt{E}}\frac{\Gamma(i\sqrt{E}+1)}{\Gamma(-i\sqrt{E}+1)}. \tag{42}$$

In the case when $v^2 - \mu^2 > 0$ the quantum system has bound and scattering states. The orthonormalized wave functions $\Psi_E(t)$ for the discrete spectrum $\sigma = \ell$, $E = -\dfrac{(l+1)^2}{4}$ with $l = 2n$, $n = 0, 1, 2, \ldots$ has the form

$$\Psi_E(t) = \tilde{c}_5 \; J_{l+1}(\sqrt{v^2 - \mu^2} e^{-t/2}), \tag{43}$$

where $\tilde{c}_5 = \sqrt{2(l+1)}$. Here, we used the orthogonality condition for the Bessel function 7.14 (32) of [9]:

$$\int_0^\infty x^{-1} J_{v+2n+1}(x) J_{v+2m+1}(x) dx = \begin{cases} 0, & m \neq n \\ (4n+2v+2)^{-1}, & m = n, v > -1. \end{cases} \tag{44}$$

The wave function $\Psi_E(t)$ for the continuous spectrum $E = \frac{\rho^2}{4}$ are given by the analytical continuation from Eq. (37):

$$\Psi_E(t) = \frac{\pi}{4i\sqrt{\pi} \sinh \sqrt{E}\pi |D|} \left[J_{-i\sqrt{E}}(\sqrt{v^2 - \mu^2} e^{-t/2}) - J_{i\sqrt{E}}(\sqrt{v^2 - \mu^2} e^{-t/2}) \right]. \tag{45}$$

Finally, we give the result

$$\Psi_E(t) = \left[J_{i\sqrt{E}}(\sqrt{v^2 - \mu^2} e^{-t/2}) + J_{-i\sqrt{E}}(\sqrt{v^2 - \mu^2} e^{-t/2}) \right] \tag{46}$$

from [6] (formulae 8.2, where $\sqrt{E} \equiv p$, $t \equiv 2\rho$, $|k| \equiv \sqrt{v^2 - \mu^2}$, $v^2 - \mu^2 > 0$). This solution defined for $v^2 - \mu^2 > 0$ is not continuously related with solution (37) defined for $v^2 - \mu^2 < 0$, hence it is not correct.

Acknowledgements

The authors would like to thank I. Hakkı Duru and G.A. Kerimov for helpful discussions.

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