Contractions of quantum algebraic structures *

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Abstract

A general framework for obtaining certain types of contracted and centrally extended algebras is presented. The whole process relies on the existence of quadratic algebras, which appear in the context of boundary integrable models.

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1 Introduction

We show that the symmetry breaking mechanism due to the presence of appropriate boundary conditions may be exploited in order to obtain centrally extended algebras via suitable contraction procedures. We use the boundary algebra to obtain the relevant Casimir operator. We explicitly demonstrate that this is perhaps the simplest and most straightforward way to obtain the Casimir operator of usual and deformed Lie algebras.

One main point of this investigation is that we are able to show that the associated open transfer matrix commutes with the elements of the emerging contracted algebra. We study here the simplest case, that is the E_2^c algebra, in order to illustrate the procedure followed, however this description may be generalized to more complicated algebraic structures.

This brief note is based on [1] where the interested reader can find all details of the construction.

2 Quadratic algebras

We give first a short review of the fundamental quadratic algebraic relations, ruling quantum integrable models, that is the Yang–Baxter and reflection equations. The Yang–Baxter equation [2] is defined as

$$R_{12}(\lambda_1-\lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2)\ =\ R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1-\lambda_2)\ , \tag{2.1}$$

acting on $\mathbb{V}^{\otimes 3}$ and $R \in \mathrm{End}(\mathbb{V}^{\otimes 2})$ $R_{12} = R\otimes\mathbb{I}$, $R_{23} = \mathbb{I}\otimes R$, where R physically describes the scattering among particle-like excitations displayed in integrable models. Given an R matrix we introduce the following fundamental algebraic relations [3], which define the algebra $\mathcal{A}$ (see e.g. [3, 4])

$$R_{12}(\lambda_1-\lambda_2)\ L_1(\lambda_1)\ L_2(\lambda_2)\ =\ L_2(\lambda_2)\ L_1(\lambda_1)\ R_{12}(\lambda_1-\lambda_2)\ , \tag{2.2}$$

where $L \in \mathrm{End}(\mathbb{V})\otimes\mathcal{A}$. This allows the construction of tensorial representations of the later algebra as [3, 4]

$$T_a(\lambda) = L_{aN}(\lambda-\theta_N)\ L_{aN-1}(\lambda-\theta_{N-1})\dots L_{a2}(\lambda-\theta_2)\ L_{a1}(\lambda-\theta_1)\ , \tag{2.3}$$

where $T(\lambda) \in \mathrm{End}(\mathbb{V}) \otimes \mathcal{A}^{\otimes N}$. Traditionally, the space a is called "auxiliary", whereas the spaces $1,\dots,N$ are called "quantum". For simplicity we usually suppress all quantum spaces when writing down the monodromy matrix. The free to choose complex parameters θ_i are called inhomogeneities. Using the fundamental algebra (2.2) one may show that

$$\Big[trT(\lambda),\ trT(\mu)\Big] = 0\ , \tag{2.4}$$

where $trT(\lambda) \in \mathcal{A}^{\otimes N}$ and the trace is taken over the auxiliary space a. The latter relation guarantees the integrability of the system.

We next introduce the reflection equation associated to the reflection algebra $\mathcal{R}$. It is given by [5, 6]

$$R_{12}(\lambda_1-\lambda_2)\ \mathbb{K}_1(\lambda_1)\ R_{21}(\lambda_1+\lambda_2)\ \mathbb{K}_2(\lambda_2)=\mathbb{K}_2(\lambda_2)\ R_{12}(\lambda_1+\lambda_2)\ \mathbb{K}_1(\lambda_1)\ R_{21}(\lambda_1-\lambda_2)\ , \tag{2.5}$$

acting on $\mathbb{V}^{\otimes 2}$ and as customary, we follow the notation $\mathbb{K}_1=\mathbb{K}\otimes\mathbb{I}$ and $\mathbb{K}_2=\mathbb{I}\otimes\mathbb{K}$. Also $R_{21}=\mathcal{P}\ R_{12}\ \mathcal{P}$, where $\mathcal{P}$ is the permutation operator, acting as $\mathcal{P}(a\otimes b)=b\otimes a$ and in addition $\mathbb{K}\in \text{End}(\mathbb{V})\otimes\mathcal{R}$. In general, the representations of the later algebra may be expressed as [6]

$$\mathbb{K}(\lambda)=L(\lambda)\ K(\lambda)\otimes\mathbb{I}\ L^{-1}(-\lambda)\ , \tag{2.6}$$

where the matrix K is a c-number representation of the aforementioned algebra, called also the reflection matrix and physically describes the interaction of a particle-like excitation with the boundary. Tensor representations of these algebra are given by (2.6) after replacing L by T defined in (2.3). We define the open transfer matrix as [6]

$$t(\lambda)=tr\{K^{+}(\lambda)\mathbb{K}(\lambda)\}\ , \tag{2.7}$$

where K^{+} matrix is a c-number solution of the reflection algebras. With the help of the quadratic exchange relations one may show that (see e.g. [6])

$$[t(\lambda),\ t(\mu)]=0\ , \tag{2.8}$$

which again guarantees the integrability of the system under consideration.

3 The E_2^c extended algebra

We aim at constructing the centrally extended E_2^c algebra. To achieve this we start from the gl_3 spin chain and break the symmetry down to $sl_2\otimes u(1)$ by using the results and techniques developed in [7]. Indeed, by implementing appropriate boundary conditions one can break the gl_n symmetry of a spin chain model to $gl_l\otimes gl_{n-l}$, where l is an integer depending on the choice of boundary. We will exploit this phenomenon in order to perform a contraction of the boundary algebra to E_2^c.

The gl_n algebra is generated by $J^{+(k)},\ J^{-(k)}$ and $e^{(i)}$, with $i=1,2,\dots,n$. Defining $s^{(k)}=e^{(k)}-e^{(k+1)}$ they satisfy the commutation relations

$$[J^{+(k)},\ J^{-(l)}]=\delta_{kl}s^{(k)}\ ,\qquad [s^{(k)},\ J^{\pm(l)}]=\pm(2\delta_{kl}-\delta_{k\ l+1}-\delta_{k\ l-1})J^{\pm(l)} \tag{3.1}$$

and $\sum_{i=1}^{n}e^{(i)}$ belongs to the center of the algebra. We will focus here to the gl_3 algebra. The L matrix is expressed as $L(\lambda)=\lambda+i\mathbb{P}$, where $\mathbb{P}$ in terms of the gl_3 elements takes the form

$$\mathbb{P}=\begin{pmatrix} e^{(1)} & J^{-(1)} & \Lambda^{+} \\ J^{+(1)} & e^{(2)} & J^{-(2)} \\ \Lambda^{-} & J^{+(2)} & e^{(3)} \end{pmatrix}\ ,\qquad \text{where}\qquad \Lambda^{\pm}=\pm[J^{\pm(1)},\ J^{\pm(2)}]\ . \tag{3.2}$$

We choose as K in (2.6) the following diagonal matrix

$$K(\lambda) = k = \text{diag}(1,\ 1,\ -1) \tag{3.3}$$

and expand $\mathbb{K}(\lambda)$ as

$$\mathbb{K}_0(\lambda) = L_{01}(\lambda)\ k_0\ L_{01}^{-1}(-\lambda) = k + \sum_j \frac{\mathbb{K}^{(j-1)}}{\lambda^j}\ . \tag{3.4}$$

The first two coefficients are

$$\mathbb{K}^{(0)} = i\Big(\mathbb{P}_{01}k_0 + k_0\mathbb{P}_{01}\Big)\ , \qquad \mathbb{K}^{(1)} = -\mathbb{P}_{01}\ k_0\ \mathbb{P}_{01} - k_0\mathbb{P}_{01}^2\ . \tag{3.5}$$

Clearly, what remains are the generators of the $sl_2 \otimes u(1)$ algebra. Specifically, $\big(J^{\pm(1)},\ s^{(1)}\big)$ satisfy the sl_2 commutation relations, whereas $c = \sum_i e_i^{(3)}$ commutes with everything. The first two conserved quantities are given by taking the trace over the auxiliary space in $\mathbb{K}^{(0)}$, $\mathbb{K}^{(1)}$. If for notational convenience, we set

$$s^{(1)} \equiv 2J\ , \qquad e^{(3)} \equiv 2\tilde{J}\ , \qquad J^{-(1)} \equiv -J^-\ , \qquad J^{+(1)} \equiv J^+\ . \tag{3.6}$$

and also consider the transfer matrix expansion as

$$t(\lambda) = \sum_k \frac{t^{(k-1)}}{\lambda^k}\ , \tag{3.7}$$

we obtain the integrals of motion

$$t^{(0)} = c - 4\tilde{J}, \qquad t^{(1)} \propto J^2 - \frac{1}{2}\{J^+,\ J^-\} - \tilde{J}^2 - c\tilde{J} + \frac{c^2}{4}\ . \tag{3.8}$$

From the first integral of motion it is clear that $e^{(3)} = 2\tilde{J}$ is also a conserved quantity, so that in the second charge we may drop the last two terms. In conclusion, the conserved changes can be taken to be

$$I^{(0)} = c - 4\tilde{J}\ , \qquad I^{(1)} = J^2 - \frac{1}{2}\{J^+,\ J^-\} - \tilde{J}^2\ . \tag{3.9}$$

Similarly, the N-tensor representations are obtained in a straightforward manner (see [1] for more details).

We will exploit the breaking of the symmetry due to the presence of non-trivial integrable boundaries to obtain the centrally extended E_2^c algebra. We consider a contraction known in the mathematics literature as a Saletan contraction and is distinct from the more common Inonü–Winger contraction. It is closely related to the so called Penrose limit in gravity, that constructs a plane wave starting from any gravitational background by magnifying the region around a null geodesic. This was first used in the string literature in WZW models [8], more recently in various supersymmetric brane solutions of string and M-theory [9] and has been instrumental in understanding issues within the AdS/CFT correspondence involving sectors of large quantum numbers [10]. According to this contraction

$$J^{\pm} = \frac{1}{\sqrt{2\epsilon}} P^{\pm} \ , \qquad J = \frac{1}{2}\left(T + \frac{F}{\epsilon}\right) \ , \qquad \tilde{J} = -\frac{F}{2\epsilon} \ , \qquad \epsilon \to 0 \ . \tag{3.10}$$

Then one obtains the following commutation relations that define the E_2^c algebra

$$[P^+,\ P^-] = -2F \ , \qquad [T,\ P^{\pm}] = \pm P^{\pm} \ , \tag{3.11}$$

where F is an *exact* central element of the algebra. It is obvious that the conserved quantities, after contracting and keeping the leading order contribution are

$$I^{(0)} = F \ , \qquad I^{(1)} = TF - \frac{1}{2}\{P^+,\ P^-\} \ . \tag{3.12}$$

Recall the representation of the reflection algebra

$$\begin{aligned} \mathbb{K}(\lambda) &= L(\lambda)\ k\ L^{-1}(-\lambda) = (1 + \frac{i}{\lambda}\mathbb{P})\ k\ (1 + \frac{i}{\lambda}\mathbb{P} - \frac{1}{\lambda^2}\mathbb{P}^2 - \frac{i}{\lambda^3}\mathbb{P}^3 + \frac{1}{\lambda^4}\mathbb{P}^4 \ldots) \\ &= k + \frac{i}{\lambda}(\mathbb{P}k + k\mathbb{P}) - \frac{1}{\lambda^2}(\mathbb{P}k\mathbb{P} + k\mathbb{P}^2) - \frac{1}{\lambda^2}(\mathbb{P}k\mathbb{P}^2 + k\mathbb{P}^3) \ldots \ , \end{aligned} \tag{3.13}$$

where k is given in (3.3). Then, after taking the trace over the auxiliary space and recalling (3.7), we obtain

$$t^{(k-1)} \propto \sum_{a,b} (\mathbb{P}_{ab}\ k_{bb}\ \mathbb{P}_{ba}^{k-1} + k_{aa}\ \mathbb{P}_{aa}^{k}) \ . \tag{3.14}$$

This is a quite general result that holds irrespectively of the particular application based on the $sl_2 \times u(1)$ algebra we have just employed. Indeed, based on this example we may infer that, before the contraction, $t^{(k)}$ are the higher Casimir operators of $sl_n \otimes u(2)$. After the contraction is taken one has to consistently keep only the highest order contribution in the $\frac{1}{\epsilon}$ expansion of each $t^{(k)}$. Then, each one of $t^{(k)}$ commutes by construction with the contracted algebra (E_2^c in our elementary example) and clearly, the same does the transfer matrix. This logic may be generalized to contractions of any higher rank algebras and similarly for generic N-particle representations (see [1]). In fact, this argument holds independently of the context one realizes the contraction (see e.g. [8]). More precisely, having in general a set of Casimir operators of say the gl_n algebra, after the contraction is taken one consistently should keep the highest order contribution in order to obtain the contracted Casimir quantities. Depending on the rank of the considered algebra the expansion of $t(\lambda)$ should truncate at some point, or in other words the higher Casimir quantities should be trivial combinations of the lower ones.

To simplify the analysis and clearly demonstrate the main ideas we restricted our discussion to examples involving originally gl_3 symmetry. A similar construction for the q deformed case has been also analyzed in [1]. A natural extension of the present work is to consider generic symmetry breaking of the type $\mathrm{G} \otimes \mathrm{H}$ where G and H are generic algebras ($\mathrm{H} \subset \mathrm{G}$), and then follow a contraction procedure similar to the ones for ordinary Lie and current algebras [8].

Finally, we note that our systematic approach for taking limits in general algebraic structures has been instrumental in resolving, in a natural way, a quite old misunderstanding concerning the opposite to contraction procedure, commonly known as expansion. For the details the interested reader is referred to [1].

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