

Canonical and grand canonical partition functions of Dyson gases as tau-functions of integrable hierarchies and their fermionic realization

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Abstract

The partition function for a canonical ensemble of 2D Coulomb charges in a background potential (the Dyson gas) is realized as a vacuum expectation value of a group-like element constructed in terms of free fermionic operators. This representation provides an explicit identification of the partition function with a tau-function of the 2D Toda lattice hierarchy. Its dispersionless (quasiclassical) limit yields the tau-function for analytic curves encoding the integrable structure of the inverse potential problem and parametric conformal maps. A similar fermionic realization of partition functions for grand canonical ensembles of 2D Coulomb charges in the presence of an ideal conductor is also suggested. Their representation as Fredholm determinants is given and their relation to integrable hierarchies, growth problems and conformal maps is discussed.

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1 Introduction

Statistical ensembles of 2D Coulomb particles (referred to also as logarithmic gases, β -ensembles or Dyson gases) at a particular value of inverse temperature, $\beta = 2$, are known to have remarkable integrable properties. Their partition functions, regarded as functions of properly chosen parameters of the background trapping potential (coupling constants), can be identified with tau-functions for hierarchies of nonlinear integrable equations such as 2D Toda lattice (2DTL) or Kadomtsev-Petviashvili (KP) hierarchies. Equivalently, the tau-function provides a generating series for correlation functions of the Dyson gas.

The aim of this paper is to give an explicit representation of the partition functions as vacuum expectation values of certain operators constructed from free fermions, in the spirit of the Kyoto school [1, 2]. Similar constructions in the context of random matrix models of different types were given in [3, 4, 5, 6]. However, most of the previous studies were devoted to canonical ensembles with a fixed number of particles, N , which after F. Dyson [7] are customarily viewed as eigenvalues of a $N \times N$ random matrix. The canonical partition function, Z_N , is then tau-function of the 2DTL hierarchy with the discrete variable N .

A natural question is whether this construction can be extended to grand canonical ensembles. Taking a weighted sum of the Z_N 's, $\mathcal{Z} = \sum_N e^{\mu N} Z_N$, one obtains a quantity whose interpretation in terms of integrable hierarchies is presently not clear. A way to introduce a grand canonical ensemble of 2D Coulomb charges with transparent integrable properties was first suggested in [8]. The idea is to consider the logarithmic gas with varying number of particles in the presence of an ideal conductor, so that each particle interacts not only with other particles of the gas but also with their “mirror images” of opposite charge. The grand canonical partition functions of such systems appear to be tau-functions of the 2DTL or KP hierarchies (depending on whether the conductor fills a disk or a half-plane), whose “times” again serve as parameters of the background potential but in a different manner. The fermionic operator construction of these tau-functions is technically even simpler than that for canonical ensembles.

Our motivation comes from growth problems of Laplacian type such as viscous flows in the Hele-Shaw cell [9, 10]. At zero surface tension, the model has an integrable structure of the 2DTL hierarchy in the zero dispersion limit [11]. From mathematical point of view, the same integrable hierarchy stays behind some classical problems of complex analysis: parametric deformations of conformal maps [12], boundary value problems for Laplace operator [13, 14] and the inverse potential problem in two dimensions [15]. Switching on the dispersion thus coming back to the original 2DTL hierarchy, one obtains a non-trivial deformation of this integrable structure. However, the corresponding deformation of Laplacian growth and, what would be even more intriguing, of the above mentioned problems of complex analysis, is not yet formulated explicitly in proper terms.

The theory of logarithmic gases provides another view on these matters [16, 17]. In the thermodynamic limit, when the number of particles is very large, the logarithmic gas in the leading approximation macroscopically looks like a charged fluid with continuous density. The equilibrium state of the system is a result of competition between the mutual repelling of particles, which tends to remove them to infinity, and the external force which attracts each particle to local minima of the background potential. Typically,

in the equilibrium the fluid occupies some compact domains in the plane around local minima of the background potential, which we call droplets. The shape of the droplets is determined by their total charge and by the profile of the trapping potential. When one increases the total charge keeping the potential fixed, the droplet changes its shape according to the growth law specific for the Laplacian growth processes (the Darcy's law). At the same time, the integrable deformation of the $N = \infty$ picture is naturally build in the 2D Coulomb gas. We believe that the reformulation of the model in terms of free fermions may help to clarify the geometrical meaning of this deformation.

Another interesting question is what kind of growth problems emerge in the thermodynamic limit of the grand canonical ensembles. We give a partial answer restricting ourselves to the grand canonical ensemble in the upper half-plane, in which case the growth problem is equivalent to Laplacian growth of "fat slits" considered in [18].

2 Free fermions and tau-functions

In this section we recall the free fermionic construction [1, 2] of KP and 2DTL tau-functions.

Let ψ_n, ψ_n^* , $n \in \mathbb{Z}$, be free fermionic operators with usual anticommutation relations $[\psi_n, \psi_m]_+ = [\psi_n^*, \psi_m^*]_+ = 0$, $[\psi_n, \psi_m^*]_+ = \delta_{mn}$. They generate an infinite dimensional Clifford algebra. We also use their Fourier transforms

$$\psi(z) = \sum_{k \in \mathbb{Z}} \psi_k z^k, \quad \psi^*(z) = \sum_{k \in \mathbb{Z}} \psi_k^* z^{-k} \quad (2.1)$$

which are regarded as free fermionic fields in the complex plane of the variable z .

Neutral bilinear combinations $\sum A_{mn} \psi_m \psi_n^*$ of the fermions generate the Lie algebra $gl(\infty)$. Normal ordering of the generators (see below) allows one to consider certain infinite sums as well. Exponentiating these expressions, one obtains an infinite dimensional group which is a central extension of GL_∞ . Let us call elements of the Clifford algebra of the form $g = \exp(\sum_{mn} A_{mn} \psi_m \psi_n^*)$ group-like elements. A characteristic property of the group-like elements g is that $g \psi_n g^{-1}$ is a linear combination of ψ_j 's and similarly for ψ_n^* : $g \psi_n g^{-1} = \sum_l \psi_l R_{ln}$, $g \psi_n^* g^{-1} = \sum_l \psi_l^* R_{nl}^{-1}$, where the matrix R is determined by the matrix A . Of particular importance are the group-like elements obtained by exponentiating the operators

$$J_+ = \sum_{k \geq 1} t_k J_k, \quad J_- = \sum_{k \geq 1} t_{-k} J_{-k} \quad (2.2)$$

where

$$J_k = \sum_{j \in \mathbb{Z}} \psi_j \psi_{j+k}^*$$

are Fourier modes of the "current operator" and t_k are parameters (called times). In general, they are complex numbers. It is convenient to denote the collection of times with positive (negative) indices by $t_+ = \{t_1, t_2, \dots\}$ and $t_- = \{t_{-1}, t_{-2}, \dots\}$ respectively and to introduce their generating function

$$\xi(t_\pm, z) = \sum_{k \geq 1} t_{\pm k} z^k.$$

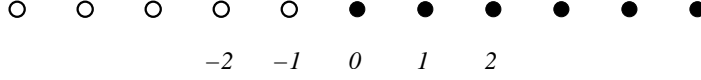


Figure 1: *The Dirac sea* $|0\rangle$. Filled states are shown in black, empty ones in white.

It is easy to check that the fields $\psi(z)$, $\psi^*(z)$ transform diagonally under the adjoint action of the group-like elements e^{J_+} , e^{J_-} :

$$\begin{aligned} e^{J_\pm} \psi(z) e^{-J_\pm} &= e^{\xi(t_\pm, z^{\pm 1})} \psi(z) \\ e^{J_\pm} \psi^*(z) e^{-J_\pm} &= e^{-\xi(t_\pm, z^{\pm 1})} \psi^*(z). \end{aligned} \quad (2.3)$$

In terms of polynomials $p_k(t_\pm)$ defined by

$$e^{\xi(t_\pm, z)} = \sum_{k \geq 0} p_k(t_\pm) z^k \quad (2.4)$$

the corresponding formulas for ψ_n, ψ_n^* can be written as

$$\begin{aligned} e^{J_\pm} \psi_n e^{-J_\pm} &= \sum_{k \geq 0} \psi_{n \mp k} p_k(t_\pm) \\ e^{J_\pm} \psi_n^* e^{-J_\pm} &= \sum_{k \geq 0} \psi_{n \pm k}^* p_k(-t_\pm). \end{aligned} \quad (2.5)$$

Next, we introduce a vacuum state $|0\rangle$ which is a “Dirac sea” where all negative mode states are empty and all positive ones are occupied (Fig. 1):

$$\psi_n |0\rangle = 0, \quad n < 0; \quad \psi_n^* |0\rangle = 0, \quad n \geq 0.$$

(For brevity, we call indices $n \geq 0$ *positive*.) With respect to this vacuum, the operators ψ_n with $n < 0$ and ψ_n^* with $n \geq 0$ are annihilation operators while the operators ψ_n^* with $n < 0$ and ψ_n with $n \geq 0$ are creation operators. Similarly, the dual vacuum state has the properties

$$\langle 0 | \psi_n^* = 0, \quad n < 0; \quad \langle 0 | \psi_n = 0, \quad n \geq 0.$$

We also need “shifted” Dirac vacua $|n\rangle$ and $\langle n|$ defined as

$$\begin{aligned} |n\rangle &= \begin{cases} \psi_{n-1} \dots \psi_1 \psi_0 |0\rangle, & n > 0 \\ \psi_n^* \dots \psi_{-2}^* \psi_{-1}^* |0\rangle, & n < 0 \end{cases} \\ \langle n| &= \begin{cases} \langle 0 | \psi_0^* \psi_1^* \dots \psi_{n-1}^*, & n > 0 \\ \langle 0 | \psi_{-1} \psi_{-2} \dots \psi_n, & n < 0 \end{cases} \end{aligned}$$

The vacuum expectation value $\langle 0 | \dots | 0 \rangle$ is a hermitian linear form on the Clifford algebra defined on bilinear combinations of fermions by the properties $\langle 0 | 0 \rangle = 1$, $\langle 0 | \psi_n \psi_m | 0 \rangle = \langle 0 | \psi_n^* \psi_m^* | 0 \rangle = 0$ for all m, n and

$$\langle 0 | \psi_n \psi_m^* | 0 \rangle = \delta_{mn} \quad \text{for } m < 0, \quad \langle 0 | \psi_n \psi_m^* | 0 \rangle = 0 \quad \text{for } m \geq 0.$$

Its extension to the whole algebra is given by the Wick's theorem: Let w_i be arbitrary linear combinations of ψ_j and ψ_j^* , then $\langle 0 | w_1 \dots w_{2n+1} | 0 \rangle = 0$ and

$$\langle 0 | w_1 \dots w_{2n} | 0 \rangle = \sum_{\sigma} (-1)^{P(\sigma)} \langle 0 | w_{\sigma(1)} w_{\sigma(2)} | 0 \rangle \dots \langle 0 | w_{\sigma(2n-1)} w_{\sigma(2n)} | 0 \rangle.$$

Here the sum is over permutations σ of $1, \dots, 2n$ such that $\sigma(1) < \sigma(2)$, $\sigma(3) < \sigma(4)$, \dots , $\sigma(2n-1) < \sigma(2n)$ and $\sigma(1) < \sigma(3) < \dots < \sigma(2n-1)$ and $P(\sigma)$ is the parity of σ . The Wick's theorem is often used in the following form. Let w_i be linear combinations of ψ_j 's only and w_i^* be linear combinations of ψ_j^* 's only, then

$$\langle 0 | w_1 \dots w_n w_n^* \dots w_1^* | 0 \rangle = \det_{i,j=1,\dots,n} \langle 0 | w_i w_j^* | 0 \rangle.$$

Let us also give an explicit formula for the expectation value of products of the fields $\psi(z)$, $\psi^*(\zeta)$:

$$\begin{aligned} \langle N | \psi(z_1) \dots \psi(z_n) \psi^*(\zeta_n) \dots \psi^*(\zeta_1) | N \rangle &= \prod_{l=1}^n (z_l / \zeta_l)^N \cdot \det_{i,j} \frac{\zeta_i}{z_i - \zeta_j} \\ &= \prod_l \frac{z_l^N \zeta_l^{1-N}}{(z_l - \zeta_l)} \prod_{i < j} \frac{(z_i - z_j)(\zeta_j - \zeta_i)}{(z_i - \zeta_j)(z_j - \zeta_i)}. \end{aligned} \quad (2.6)$$

One may define the normal ordering $\bullet(\dots)\bullet$ with respect to the Dirac vacuum $|0\rangle$ when all annihilation operators are moved to the right and all creation operators are moved to the left taking into account their anticommutativity under mutual permutations. For example, $\bullet\psi_1^* \psi_1 \bullet = -\psi_1 \psi_1^* = \psi_1^* \psi_1 - 1$, and, more generally, $\bullet\psi_m^* \psi_n \bullet = \psi_m^* \psi_n - \langle 0 | \psi_m^* \psi_n | 0 \rangle$. We also note the identities

$$\begin{aligned} e^{\alpha \psi_k \psi_k^*} &= 1 + (e^\alpha - 1) \psi_k \psi_k^* = \bullet e^{(e^\alpha - 1) \psi_k \psi_k^*} \bullet \quad \text{for } k \geq 0, \\ e^{\alpha \psi_k^* \psi_k} &= 1 + (e^\alpha - 1) \psi_k^* \psi_k = \bullet e^{(e^\alpha - 1) \psi_k^* \psi_k} \bullet \quad \text{for } k < 0. \end{aligned} \quad (2.7)$$

As it has been established in the works of the Kyoto school, the expectation values of group-like elements are tau-functions of integrable hierarchies of nonlinear differential equations. In particular,

$$\tau_n(t_+, t_-) = \langle n | e^{J_+} g e^{-J_-} | n \rangle \quad (2.8)$$

is the tau-function of the 2DTL hierarchy meaning that it obeys the infinite set of Hirota bilinear equations for the 2DTL hierarchy the simplest of which is

$$\partial_{t_1} \tau_n \partial_{t_{-1}} \tau_n - \tau_n \partial_{t_1} \partial_{t_{-1}} \tau_n = \tau_{n+1} \tau_{n-1}.$$

In a similar manner,

$$\tau(t_+) = \langle 0 | e^{J_+} g | 0 \rangle \quad (2.9)$$

is the tau-function of the KP hierarchy.

It is worthwhile to note that the group-like elements of the Clifford algebra can be also written as normal ordered exponents of the form

$$g = \bullet \exp \left(\sum_{mn} \tilde{A}_{mn} \psi_m \psi_n^* \right) \bullet \quad (2.10)$$

For regular (invertible) elements g , this normal ordered expression is just equal, modulo a constant factor, to an exponent of a fermionic bilinear form without normal ordering but with some other matrix \tilde{A} (an example is provided by (2.7), see also [19] for a more general case). If g is invertible, then the tau-function at any n is not identically zero. However, there is an important class of tau-functions which vanish identically at some values of n . They correspond to *singular* elements g which are not invertible (and thus do not belong to a group in a strict sense) but still can be represented in the normal ordered form (2.10).

Here are two important examples of such singular elements:

$$\begin{aligned} P_+ &= \text{:} \exp \left(\sum_{i < 0} \psi_i \psi_i^* \right) \text{:} = \prod_{i < 0} (1 - \psi_i^* \psi_i) = \prod_{i < 0} \psi_i \psi_i^*, \\ P_- &= \text{:} \exp \left(- \sum_{i \geq 0} \psi_i \psi_i^* \right) \text{:} = \prod_{i \geq 0} (1 - \psi_i \psi_i^*) = \prod_{i \geq 0} \psi_i^* \psi_i. \end{aligned} \tag{2.11}$$

In a sense, these operators are projectors to positive and negative modes respectively. Their properties (extensively used in what follows) can be easily seen from the definition. The both operators obey the projector property: $P_{\pm}^2 = P_{\pm}$. The operator P_+ kills negative creation modes standing to the right and negative annihilation modes standing to the left and commutes with all positive modes:

$$\begin{aligned} P_+ \psi_k^* &= \psi_k P_+ = 0, & k < 0, \\ [P_+, \psi_k^*] &= [P_+, \psi_k] = 0, & k \geq 0. \end{aligned} \tag{2.12}$$

The operator P_- kills positive creation modes standing to the right and positive annihilation modes standing to the left and commutes with all negative modes:

$$\begin{aligned} P_- \psi_k &= \psi_k^* P_- = 0, & k \geq 0, \\ [P_-, \psi_k^*] &= [P_-, \psi_k] = 0, & k < 0. \end{aligned} \tag{2.13}$$

From this it is obvious that $P_+|n\rangle = 0$ at $n < 0$ and $P_+|n\rangle = |n\rangle$ at $n \geq 0$. Similarly, $P_-|n\rangle = 0$ at $n \geq 0$ and $P_-|n\rangle = |n\rangle$ at $n < 0$. Somewhat less obvious properties (also used in what follows) are $P_+ e^{-J_-} |0\rangle = |0\rangle$, $\langle 0| e^{J_+} P_+ = \langle 0|$.

As an example, let us calculate the tau-function corresponding to the singular element P_+ : $\tau_N = \langle N| e^{J_+} P_+ e^{-J_-} |N\rangle$. Here we follow [3]. First of all, it is not difficult to see that $\tau_N = 0$ at $N < 0$ and $\tau_0 = 1$. For $N > 1$ we have:

$$\tau_N = \langle 0| \psi_{N-1}^* \dots \psi_0^* e^{J_+} P_+ e^{-J_-} \psi_0 \dots \psi_{N-1} |0\rangle.$$

To proceed, it is convenient to use the short hand notation $\psi_n(H) = e^H \psi_n e^{-H}$, $\psi_n^*(H) = e^H \psi_n^* e^{-H}$, where H is any operator from the Clifford algebra, then we can write

$$\tau_N = \langle 0| e^{J_+} \psi_{N-1}^*(-J_+) \dots \psi_0^*(-J_+) P_+ P_+ \psi_0(-J_-) \dots \psi_{N-1}(-J_-) e^{-J_-} |0\rangle.$$

Equations (2.5) imply

$$\begin{aligned} \psi_n(-J_-) &= \sum_{k \geq 0} \psi_{n+k} p_k(-t_-) \\ \psi_n^*(-J_+) &= \sum_{k \geq 0} \psi_{n+k}^* p_k(t_+). \end{aligned} \tag{2.14}$$

so the operators $\psi_n(-J_-)$, $\psi_n^*(-J_+)$ in the formula for τ_N contain only positive modes and, therefore, commute with P_+ . Moving one P_+ to the right and another one to the left, and using the properties mentioned above, we obtain

$$\begin{aligned}\tau_N &= \langle 0 | \psi_{N-1}^*(-J_+) \dots \psi_0^*(-J_+) \psi_0(-J_-) \dots \psi_{N-1}(-J_-) | 0 \rangle \\ &= \det_{1 \leq j, k \leq N} \langle 0 | \psi_{j-1}^*(-J_+) \psi_{k-1}(-J_-) | 0 \rangle\end{aligned}$$

by the Wick's theorem. The expectation value under the determinant can be represented as a contour integral as follows:

$$\begin{aligned}\langle 0 | \psi_j^*(-J_+) \psi_k(-J_-) | 0 \rangle &= \sum_{a, b \geq 0} p_a(t_+) p_b(-t_-) \langle 0 | \psi_{j+a}^* \psi_{k+b} | 0 \rangle \\ &= \sum_{a, b \geq 0} p_a(t_+) p_b(-t_-) \delta_{j+a, k+b} = \oint_{|z|=1} z^{j-k} e^{\xi(t_+, z) - \xi(t_-, 1/z)} \frac{dz}{2\pi i z}.\end{aligned}$$

The whole determinant can then be written as an N -fold contour integral:

$$\tau_N = \frac{1}{N!} \oint \dots \oint \prod_{j < k} (z_j - z_k) (z_j^{-1} - z_k^{-1}) \prod_{l=1}^N e^{\xi(t_+, z_l) - \xi(t_-, 1/z_l)} \frac{dz_l}{2\pi i z_l}. \quad (2.15)$$

When $t_{-k} = -\bar{t}_k$, the expression $\xi(t_+, z) - \xi(t_-, 1/z)$ is purely real for z on the unit circle and τ_N coincides with the partition function of the unitary random matrix model written in terms of the eigenvalues. In this form, it can be treated also as the partition function of a canonical ensemble of N 2D Coulomb particles confined on a circle.

3 Partition function of the 2D Coulomb gas as a tau-function: canonical ensemble

Let us fix an arbitrary measure $d\mu(z)$ in the complex plane and consider the following group-like element:

$$g_0 = \bullet \exp \left(\int_{\mathbb{C}} \psi_+(z) \psi_+^*(1/\bar{z}) d\mu(z) - \sum_{j \geq 0} \psi_j \psi_j^* \right) \bullet \quad (3.1)$$

Here $\psi_+(z) = \sum_{n \geq 0} \psi_n z^n$, $\psi_+^*(z) = \sum_{n \geq 0} \psi_n^* z^{-n}$ are truncated Fourier series containing only positive modes. Obviously, g_0 commutes with P_+ . Expanding the exponent into a series, one can represent g_0 in a more explicit form:

$$g_0 = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{C}^m} \psi_+(z_1) \dots \psi_+(z_m) P_- \psi_+^*(1/\bar{z}_m) \dots \psi_+^*(1/\bar{z}_1) d\mu_1 \dots d\mu_m, \quad (3.2)$$

where $d\mu_j \equiv d\mu(z_j)$.

Let us consider the expectation value

$$\tau_N(t_+, t_-) = \langle N | e^{J_+} g_0 P_+ e^{-J_-} | N \rangle \quad (3.3)$$

and apply to it a chain of transformations similar to the ones used in the simpler case $g_0 = 1$. Again, $\tau_N = 0$ at $N < 0$ and $\tau_0 = 1$. For $N > 1$ we have:

$$\begin{aligned}\tau_N &= \langle 0 | \psi_{N-1}^* \dots \psi_0^* e^{J_+} g_0 P_+ e^{-J_-} \psi_0 \dots \psi_{N-1} | 0 \rangle \\ &= \langle 0 | e^{J_+} \psi_{N-1}^*(-J_+) \dots \psi_0^*(-J_+) P_+ g_0 P_+ \psi_0(-J_-) \dots \psi_{N-1}(-J_-) e^{-J_-} | 0 \rangle \\ &= \langle 0 | \psi_{N-1}^*(-J_+) \dots \psi_0^*(-J_+) g_0 \psi_0(-J_-) \dots \psi_{N-1}(-J_-) | 0 \rangle.\end{aligned}$$

Substituting the explicit form of g_0 , we get:

$$\begin{aligned}\tau_N &= \sum_{m \geq 0} \frac{1}{m!} \int_{\mathbb{C}^m} d\mu_1 \dots d\mu_m \langle 0 | \psi_{N-1}^*(-J_+) \dots \psi_0^*(-J_+) \psi_+(z_1) \dots \psi_+(z_m) \\ &\quad \times P_- \psi_+^*(1/\bar{z}_m) \dots \psi_+^*(1/\bar{z}_1) \psi_0(-J_-) \dots \psi_{N-1}(-J_-) | 0 \rangle.\end{aligned}$$

The next step is to notice that only the term with $m = N$ contributes to the sum and all other terms vanish. Indeed, at $m > N$ the state

$$\psi_+^*(1/\bar{z}_m) \dots \psi_+^*(1/\bar{z}_1) \psi_0(-J_-) \dots \psi_{N-1}(-J_-) | 0 \rangle$$

is in fact the null state because the number of annihilation operators exceeds the number of creation operators while at $m < N$ the operator

$$P_- \psi_+^*(1/\bar{z}_m) \dots \psi_+^*(1/\bar{z}_1) \psi_0(-J_-) \dots \psi_{N-1}(-J_-)$$

is in fact the null operator because P_- multiplied by the uncompensated positive ψ -modes from the right gives 0 (see (2.13)). Therefore, the expression simplifies to

$$\begin{aligned}\tau_N &= \frac{1}{N!} \int_{\mathbb{C}^N} d\mu_1 \dots d\mu_N \langle 0 | \psi_{N-1}^*(-J_+) \dots \psi_0^*(-J_+) \psi_+(z_1) \dots \psi_+(z_N) \\ &\quad \times P_- \psi_+^*(1/\bar{z}_N) \dots \psi_+^*(1/\bar{z}_1) \psi_0(-J_-) \dots \psi_{N-1}(-J_-) | 0 \rangle.\end{aligned}$$

Since there are as many annihilation operators to the right of P_- as creation ones, the state that they produce from the vacuum is proportional to the vacuum state itself, i.e.,

$$\psi_+^*(1/\bar{z}_N) \dots \psi_+^*(1/\bar{z}_1) \psi_0(-J_-) \dots \psi_{N-1}(-J_-) | 0 \rangle = | 0 \rangle C_N,$$

where the constant C_N is

$$\begin{aligned}C_N &= \langle 0 | \psi_+^*(1/\bar{z}_N) \dots \psi_+^*(1/\bar{z}_1) \psi_0(-J_-) \dots \psi_{N-1}(-J_-) | 0 \rangle \\ &= \det_{1 \leq j, k \leq N} \langle 0 | \psi_+^*(1/\bar{z}_j) \psi_{k-1}(-J_-) | 0 \rangle.\end{aligned}$$

Because

$$\begin{aligned}\langle 0 | \psi_+^*(1/\bar{z}_j) \psi_{k-1}(-J_-) | 0 \rangle &= \sum_{a, l \geq 0} \bar{z}_j^l p_a(-t_-) \langle 0 | \psi_l^* \psi_{k+a-1} | 0 \rangle \\ &= \sum_{a \geq 0} \bar{z}_j^{k+a-1} p_a(-t_-) = \bar{z}_j^{k-1} e^{-\xi(t_-, \bar{z}_j)},\end{aligned}$$

the constant C_N is explicitly given by

$$C_N = \Delta_N(\bar{z}_i) \prod_{l=1}^N e^{-\xi(t_-, \bar{z}_l)}, \quad (3.4)$$

where we use the convenient short-hand notation for the Vandermonde determinant:

$$\Delta_N(z_i) = \det_{1 \leq j, k \leq N} (z_k^{j-1}) = \prod_{i>j} (z_i - z_j).$$

Now, it remains to calculate

$$\begin{aligned} \langle 0 | \psi_{N-1}^*(-J_+) \dots \psi_0^*(-J_+) \psi_+(z_1) \dots \psi_+(z_N) | 0 \rangle &= \det_{1 \leq j, k \leq N} \langle 0 | \psi_{j-1}^*(-J_+) \psi_+(z_k) | 0 \rangle \\ &= \det_{1 \leq j, k \leq N} z_k^{j-1} e^{\xi(t_+, z_k)} = \Delta_N(z_i) \prod_{l=1}^N e^{\xi(t_+, z_l)}, \end{aligned}$$

which can be done in a completely similar manner. Collecting everything together, we obtain the result:

$$\tau_N(t_+, t_-) = \frac{1}{N!} \int_{\mathbb{C}^N} |\Delta_N(z_i)|^2 \prod_{l=1}^N e^{\xi(t_+, z_l) - \xi(t_-, \bar{z}_l)} d\mu(z_l). \quad (3.5)$$

Assume that $d\mu(z) = e^{-U(z, \bar{z})} d^2z$ is a smooth measure on the plane and $t_{-k} = -\bar{t}_k$, then the expression $\xi(t_+, z) - \xi(t_-, \bar{z})$ is purely real and the integral (3.5) has a physical interpretation as the partition function of a canonical ensemble of N identical Coulomb particles in the plane in the background potential $W(z, \bar{z}) = -U(z, \bar{z}) + 2\mathcal{R}e \sum_k t_k z^k$:

$$Z_N = \frac{1}{N!} \int_{\mathbb{C}^N} |\Delta_N(z_i)|^2 \prod_{l=1}^N e^{W(z_l, \bar{z}_l)} d^2z_l. \quad (3.6)$$

It is proportional to the partition function of the ensemble of normal random $N \times N$ matrices Φ [20]: $Z_N \propto \int D\Phi e^{\text{tr} W(\Phi, \Phi^\dagger)}$, with z_i being their eigenvalues.

If the measure $d\mu$ is concentrated on a curve $\Gamma \subset \mathbb{C}$, then the 2D integrals $\int_{\mathbb{C}} (\dots) d^2z$ are reduced to 1D integrals $\int_{\Gamma} (\dots) |dz|$ along Γ . This means that the 2D Coulomb particles are confined to the curve Γ . For particular choices of Γ the integral (3.5) yields the partition functions of random matrix models of certain types in terms of eigenvalues. For example, if Γ is the real line, one obtains the partition function of hermitian random matrices and if Γ is the unit circle, the integral (3.5) becomes identical to (2.15) which is the partition function of unitary random matrices.

Let us show that for axially symmetric measures $d\mu$ the operator representation (3.3) is equivalent to the one suggested by A. Orlov et al [21]. For an axially symmetric measure, the bilinear form in the fermion operators in (3.1) becomes diagonal:

$$\int_{\mathbb{C}} \psi_+(z) \psi_+^*(1/\bar{z}) d\mu(z) = \sum_{m, n \geq 0} \psi_n \psi_m^* \int_{\mathbb{C}} z^n \bar{z}^m e^{-U(|z|)} d^2z = \sum_{n \geq 0} h_n \psi_n \psi_n^*,$$

where

$$h_n = \int_{\mathbb{C}} |z|^{2n} e^{-U(|z|)} d^2z,$$

(we assume that the measure is smooth with $U(z, \bar{z}) = U(|z|)$), so in this case

$$g_0 = \text{:} \exp \left(\sum_{n \geq 0} (h_n - 1) \psi_n \psi_n^* \right) \text{:} = \exp \left(\sum_{n \geq 0} \log h_n \psi_n \psi_n^* \right) \quad (3.7)$$

and the tau-function (3.3) does have the form $\langle N | e^{J_+} e^X e^{-J_-} | N \rangle$ with $X = \sum_{j \in \mathbb{Z}} X_j \text{:} \psi_j \psi_j^* \text{:}$ studied in [21]. More precisely, it corresponds to a singular limit of the latter with $X_j = \log h_j$ for $j \geq 0$ and $X_j \rightarrow +\infty$ for all $j < 0$. Indeed, writing

$$e^X = \prod_{j \geq 0} \left(1 + (e^{X_j} - 1) \psi_j \psi_j^* \right) \cdot \prod_{j < 0} \left(1 + (e^{-X_j} - 1) \psi_j^* \psi_j \right),$$

we see that the first product is equal to $\prod_{j \geq 0} (1 + (h_j - 1) \psi_j \psi_j^*) = g_0$ while the limit of the second one is the singular operator P_+ .

An important example is $U(z, \bar{z}) = c|z|^2$, then $h_n = \pi c^{-n-1} n! = \pi c^{-n-1} \Gamma(n+1)$ and $X_n = -(n+1) \log c + \log \Gamma(n+1)$ (the common constant $\log \pi$ is irrelevant because the operator $\sum_{j \in \mathbb{Z}} \text{:} \psi_j \psi_j^* \text{:}$ commutes with all elements of the Clifford algebra). Note that the analytic continuation of this formula to negative values of n with the help of the gamma-function automatically implies the required singular limit $X_n = +\infty$ at $n < 0$. The tau-function (3.5) for this case has the following expansion in Schur functions [21, 22]:

$$\tau_N(t_+, t_-) = \pi^N c^{-N(N+1)/2} \prod_{k=1}^N \Gamma(k) \cdot \sum_{\lambda} c^{-|\lambda|} (N)_{\lambda} s_{\lambda}(t_+) s_{\lambda}(-t_-). \quad (3.8)$$

Here λ denotes the Young diagram with $\ell(\lambda)$ rows of lengths $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)} > 0$, $|\lambda|$ is the total number of boxes in λ ,

$$(N)_{\lambda} := \prod_{i=1}^{\ell(\lambda)} (N+1-i)(N+2-i) \dots (N+\lambda_i-i)$$

and the sum runs over all Young diagrams including the empty one (its contribution is 1). The Schur functions s_{λ} are defined in the standard way as determinants of the polynomials $p_k(t)$ (2.4):

$$s_{\lambda}(t) = \det_{1 \leq i, j \leq \ell(\lambda)} p_{\lambda_i - i + j}(t).$$

The dispersionless limit of the tau-function (3.5) is achieved via rescaling of times $t_k = T_k/\hbar$, $N = T_0/\hbar$ and tending $\hbar \rightarrow 0$. Clearly, this implies $N \rightarrow \infty$. However, in order to get a meaningful limit the measure $d\mu$ should be chosen appropriately. In the case of a smooth measure one should set $d\mu(z) = e^{-\frac{1}{\hbar} U(z, \bar{z})} d^2 z$, then

$$\tau_N(t_+, t_-) = \exp \left(\frac{F_0}{\hbar^2} + O(\hbar^{-1}) \right), \quad \hbar \rightarrow 0. \quad (3.9)$$

The function $F_0 = F_0(\dots, T_{-1}, T_0, T_1, \dots)$ is what is called ‘‘dispersionless tau-function’’ [23, 24]. When the reality condition $T_{-k} = -\bar{T}_k$ is imposed, this function encodes a formal solution to the inverse potential problem in 2D and admits a nice geometric/electrostatic description [15]. Set $\sigma(z, \bar{z}) = \frac{1}{\pi} \partial_z \partial_{\bar{z}} U(z, \bar{z})$ to be density of the background charges. We

assume that $\sigma > 0$. Given a real positive T_0 and complex T_k 's, $k \geq 1$, it is a subject of the inverse potential problem to find a domain D in the complex plane such that T_0 is the total charge contained in D and T_k 's are harmonic moments of its exterior with respect to the density σ :

$$T_0 = \int_D \sigma(z, \bar{z}) d^2 z, \quad T_k = -\frac{1}{k} \int_{\mathbb{C} \setminus D} z^{-k} \sigma(z, \bar{z}) d^2 z, \quad k \geq 1. \quad (3.10)$$

Leaving aside the very difficult questions about existence and uniqueness of the solution, we simply assume, just for an illustrative purpose, that we are in a situation when D is a compact connected domain (containing the origin). Then F_0 is given by

$$F_0 = - \int_D \int_D \sigma(z, \bar{z}) \log |z^{-1} - \zeta^{-1}| \sigma(\zeta, \bar{\zeta}) d^2 z d^2 \zeta, \quad (3.11)$$

which is basically the electrostatic energy of D filled by electric charge with density σ (and with a point-like charge at the origin). In the case $U(z, \bar{z}) = z\bar{z}$, $\sigma = 1/\pi$ one obtains the ‘‘tau-function of analytic curves’’ [25] which encodes the integrable structure of parametric families of conformal maps [12] and the Dirichlet boundary value problem in $\mathbb{C} \setminus D$ [13, 14]. The analytic continuation of the function F_0 to general values of T_k (not necessarily constrained by the condition $T_{-k} = -\bar{T}_k$) also has a geometric meaning in terms of pairs of conformal maps [26]. As the results of [27] suggest, higher terms in the \hbar -expansion of the tau-function (3.9) may be related to spectral invariants of the domain D .

For tau-functions of the form (3.5) with a singular measure $d\mu$ concentrated on a contour, the dispersionless limit does not require introducing any \hbar -dependence of the measure. The leading $\hbar \rightarrow 0$ behavior has the same form (3.9) and there is an integral representation for F_0 similar to (3.11) with the domain D being replaced by a segment of the curve. However, the parameters T_k in this case do not admit a direct geometric interpretation.

4 Partition function of the 2D Coulomb gas as a tau-function: grand canonical ensemble

A direct attempt of passing to a grand canonical ensemble via $\mathcal{Z} = \sum_{N \geq 0} e^{\mu N} Z_N$ with Z_N given by (3.6) leads to a quantity whose interpretation as a tau-function of an integrable hierarchy is not known. A way to introduce a grand canonical ensemble with good integrable properties, first suggested by I. Loutsenko et al [8], is to consider the logarithmic gas in the presence of an ideal conductor, so that each particle interacts not only with other particles of the gas but also with their ‘‘mirror images’’ of opposite charge as well as with its one image. For the reflection to be globally well-defined the conductor should be a disk.

Let \mathbb{D} be the unit disk and \mathbb{D}^* its exterior. We assume that particles of the gas with complex coordinates z_i occupy \mathbb{D}^* and \mathbb{D} is an ideal conductor. Then the mirror images are inside \mathbb{D} at the points $1/\bar{z}_i$ and the Coulomb energy of the system is

$$E_N = \sum_{i < j}^N \left(\log |z_i - z_j| + \log |\bar{z}_i^{-1} - \bar{z}_j^{-1}| \right) - \sum_{i,j}^N \log |z_i - \bar{z}_j^{-1}|.$$

The grand canonical partition function of the system of identical particles in a background trapping potential W (which is supposed to include the chemical potential) is defined as

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\mathbb{D}^*} \cdots \int_{\mathbb{D}^*} e^{E_N + W_N} d^2 z_1 \cdots d^2 z_N, \quad W_N = \sum_{l=1}^N W(z_l, \bar{z}_l).$$

In a more explicit form, it reads

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\mathbb{D}^*} \cdots \int_{\mathbb{D}^*} \prod_{j < k}^N \left| \frac{z_j - z_k}{z_j \bar{z}_k - 1} \right|^2 \prod_{l=1}^N \frac{|z_l| e^{W(z_l, \bar{z}_l)}}{|z_l|^2 - 1} d^2 z_l. \quad (4.1)$$

In order to ensure convergency of the integrals, the function $e^{W(z, \bar{z})}$ should vanish at $|z| = 1$ as $(|z|^2 - 1)^\alpha$ with $\alpha > 0$.

Let us fix an arbitrary measure $d\mu(z)$ in \mathbb{D}^* and consider the following group-like element:

$$G = \underset{\times}{\times} \exp \left(\int_{\mathbb{D}^*} \psi(z) \psi^*(1/\bar{z}) d\mu(z) \right) \underset{\times}{\times}. \quad (4.2)$$

Here we use another normal ordering $\underset{\times}{\times}(\dots)\underset{\times}{\times}$, the one with respect to the completely filled vacuum $|\!-\infty\rangle$, which means that the ψ -modes are moved to the left while ψ^* -modes are moved to the right. Expanding the exponent into a series, one can represent G in a more explicit form:

$$G = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\mathbb{D}^*} \psi(z_1) \cdots \psi(z_N) \psi^*(1/\bar{z}_N) \cdots \psi^*(1/\bar{z}_1) d\mu_1 \cdots d\mu_m, \quad (4.3)$$

where $d\mu_j \equiv d\mu(z_j)$. Using (2.6) it is straightforward to see that the expectation value

$$\tau_n^{(G)}(t_+, t_-) = \langle n | e^{J_+} G e^{-J_-} | n \rangle \quad (4.4)$$

has the structure of partition function of a grand canonical ensemble:

$$\tau_n^{(G)}(t_+, t_-) = e^{-\sum_{k \geq 1} k t_k t_{-k}} \sum_{N \geq 0} \frac{1}{N!} \int_{\mathbb{D}^*} \cdots \int_{\mathbb{D}^*} \prod_{j < k}^N \left| \frac{z_j - z_k}{z_j \bar{z}_k - 1} \right|^2 \prod_{l=1}^N e^{\omega(t; z_l, \bar{z}_l)} \frac{d\mu(z_l)}{|z_l|^2 - 1}, \quad (4.5)$$

where

$$\begin{aligned} \omega(t; z, \bar{z}) &= t_0 \log |z|^2 + \xi(t_+, z) + \xi(t_-, 1/z) - \xi(t_+, 1/\bar{z}) - \xi(t_-, \bar{z}) \\ &= t_0 \log |z|^2 + \sum_{k \geq 1} (t_k (z^k - \bar{z}^{-k}) - t_{-k} (\bar{z}^k - z^{-k})) \end{aligned} \quad (4.6)$$

and $n \equiv t_0$. Clearly, at $t_{-k} = -\bar{t}_k$ $\omega(t; z, \bar{z})$ is real and can be interpreted as a harmonic part of the background potential in \mathbb{D}^* . More precisely, the identification with (4.1) goes as follows: set $W(z, \bar{z}) = -U(z, \bar{z}) + \omega(t; z, \bar{z}) - \log |z|$, then $d\mu(z) = e^{-U(z, \bar{z})} d^2 z$. It is convenient to redefine the tau-function by extracting the simple factor in front of the sum in (4.5): $\tilde{\tau}_n^{(G)}(t_+, t_-) = e^{\sum_{k \geq 1} k t_k t_{-k}} \tau_n^{(G)}(t_+, t_-)$, then $\mathcal{Z} = \tilde{\tau}_n^{(G)}(t_+, t_-)$.

Note that the tau-function $\tilde{\tau}^{(G)}$ is formally a ∞ -soliton tau-function with momenta of solitons distributed with the measure $d\mu$. It is the Fredholm determinant of an integral operator:

$$\tilde{\tau}_n^{(G)}(t_+, t_-) = \det(\mathbf{1} + \hat{K}). \quad (4.7)$$

Here $\mathbf{1}$ is the identity operator and the operator \hat{K} acts to functions on \mathbb{D}^* as follows:

$$\hat{K}f(z) = \int_{\mathbb{D}^*} \frac{f(\zeta)e^{\omega(t;\zeta,\bar{\zeta})}}{z\bar{\zeta} - 1} d\mu(\zeta).$$

To see this, we again calculate the expectation value of (4.3) with the help of (2.6) but now using the determinant representation of each term. This gives an expansion of the Fredholm determinant.

As an example, consider the case when the measure $d\mu$ is concentrated on a circle of radius e^ϵ , $\epsilon > 0$. Set $z = e^{\epsilon+i\phi}$, then $d\mu(z) = e^\epsilon d\phi$ and the sum in the r.h.s. of (4.5) becomes the grand canonical partition function of the charged particles on the circle $|z| = e^\epsilon$ in the presence of the ideal conductor:

$$\mathcal{Z} = \sum_{N \geq 0} \frac{e^{\mu N}}{N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{j < k}^N \left| \frac{\sin \frac{\phi_j - \phi_k}{2}}{\sin \frac{\phi_j - \phi_k + 2i\epsilon}{2}} \right|^2 \prod_{l=1}^N e^{V(\phi_l)} d\phi_l, \quad (4.8)$$

where the chemical potential μ is expressed through t_0 as $\mu = 2\epsilon t_0 - \log(e^\epsilon - e^{-\epsilon})$ and

$$V(\phi) = 2 \sum_{k \in \mathbb{Z}} \sinh(\epsilon k) t_k e^{ik\phi}.$$

This partition function is a compactified version of the one considered by V. Kazakov et al in [28]. As a function of the ‘‘times’’ t_k , it is tau-function of the 2DTL hierarchy.

A slightly more general setting is to take the disk-like conductor of an arbitrary radius R and to consider the grand canonical ensemble of charged particles in its exterior. In fact for any finite R this hardly brings anything new because the systems at different R can be transformed into each other by a simple rescaling of z and by redefining the background potential. However, in the singular limit $R \rightarrow \infty$ a new grand canonical ensemble emerges, which can be introduced independently without a reference to any limiting process. It is defined in the upper half plane \mathbb{H} , with the lower half plane being an ideal conductor. Its partition function is tau-function of the KP hierarchy. The limiting procedure is rather sophisticated and will not be described here. We only remark that it is similar to the one developed by E. Antonov et al [29] for the transition 2DTL \rightarrow KP in the continuum limit of the 2DTL hierarchy.

Let us give the main formulas related to the grand canonical ensemble in \mathbb{H} and its operator realization. Fix an arbitrary measure $d\mu_{\mathbb{H}}(z)$ in \mathbb{H} and consider the following group-like element:

$$G_{\mathbb{H}} = \underset{\times}{\times} \exp \left(i \int_{\mathbb{H}} \psi(z) \psi^*(\bar{z}) \bar{z}^{-1} d\mu_{\mathbb{H}}(z) \right) \underset{\times}{\times}. \quad (4.9)$$

The expectation value

$$\tau(t) = \langle 0 | e^{J^+} G_{\mathbb{H}} | 0 \rangle \quad (4.10)$$

is KP tau-function depending on the ‘‘times’’ $t = \{t_1, t_2, \dots\}$. (For the abuse of notation we use the same letters as in (4.4) but one should remember that their meaning is different, see below.) A calculation similar to the one for \mathbb{D}^* yields

$$\tau(t) = \sum_{N \geq 0} \frac{1}{N!} \int_{\mathbb{H}} \cdots \int_{\mathbb{H}} \prod_{j < k}^N \left| \frac{z_j - z_k}{z_j - \bar{z}_k} \right|^2 \prod_{l=1}^N e^{\xi(t; z_l) - \xi(t; \bar{z}_l)} \frac{d\mu_{\mathbb{H}}(z_l)}{2\mathcal{I}m z_l}, \quad (4.11)$$

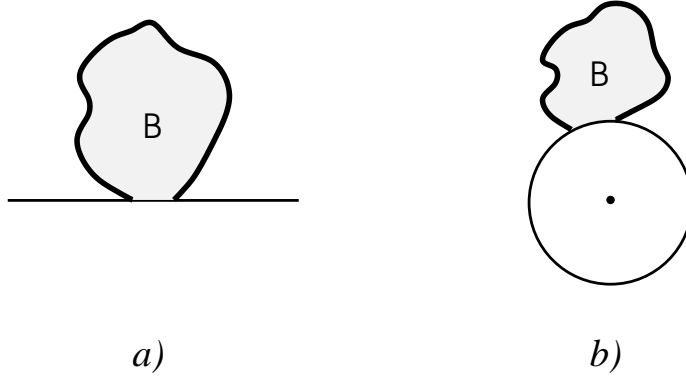


Figure 2: A fat slit B a) in the upper half plane, b) in the exterior of the unit disk.

which looks like a ∞ -soliton tau-function with continuously distributed momenta. (The analogy with soliton solutions of the KP hierarchy was first noticed in [8].) In order to ensure convergency on the real line, the measure should vanish there as $d\mu_{\mathbb{H}}(z) \propto (\mathcal{I}m z)^\alpha$ with $\alpha > 0$. Assuming that all the t_k 's are purely imaginary and $d\mu_{\mathbb{H}}(z) = e^{-U(z, \bar{z})} d^2 z$, this tau-function is identical to the partition function of the grand canonical ensemble of 2D Coulomb particles in \mathbb{H} with the background potential $W(z, \bar{z}) = -U(z, \bar{z}) + \xi(t, z) - \xi(t, \bar{z})$. Choosing the measure $d\mu_{\mathbb{H}}$ concentrated on the line $x + i\epsilon$, one obtains from (4.11) a non-compact analog of (4.8), which is precisely the partition function considered in [28].

The dispersionless limit of the tau-function (4.11) is again achieved via rescaling of times $t_k = T_k/\hbar$, setting the measure to be $d\mu_{\mathbb{H}}(z) = e^{-\frac{1}{\hbar}U(z, \bar{z})} d^2 z$ and tending $\hbar \rightarrow 0$. In contrast to the canonical case, the mean total charge of the system can not be taken arbitrary but is fixed by the equilibrium condition. In other words, at $\hbar \rightarrow 0$ the leading contribution to the sum (4.11) comes from its maximal term. One has:

$$\tau(t) = \exp\left(\frac{\tilde{F}_0}{\hbar^2} + O(\hbar^{-1})\right), \quad \hbar \rightarrow 0. \quad (4.12)$$

The function $\tilde{F}_0 = \tilde{F}_0(T_1, T_2, \dots)$ is the tau-function of the dispersionless KP hierarchy [23, 24]. When the reality condition $\mathcal{R}e T_k = 0$ is imposed, this function admits a nice geometric/electrostatic description given in [18] for the particular case $\sigma = 1/\pi$. Let $\sigma(z, \bar{z}) = \frac{1}{\pi} \partial_z \partial_{\bar{z}} U(z, \bar{z})$ be density of the background charges, as before. Given purely imaginary T_k 's, $k \geq 1$, a version of the inverse potential problem in the upper half plane allows one to find a domain B (a “fat slit” [18], see Fig. 2a) in \mathbb{H} such that

$$T_1 = -2i \mathcal{I}m \int_B \sigma(z, \bar{z}) d^2 z, \quad T_k = \frac{2i}{k} \mathcal{I}m \int_{\mathbb{H} \setminus B} z^{-k} \sigma(z, \bar{z}) d^2 z, \quad k \geq 2. \quad (4.13)$$

Then \tilde{F}_0 is given by

$$\tilde{F}_0 = - \int_B \int_B \sigma(z, \bar{z}) \log \left| \frac{z - \zeta}{z - \bar{\zeta}} \right| \sigma(\zeta, \bar{\zeta}) d^2 z d^2 \zeta, \quad (4.14)$$

which is basically the electrostatic energy of B filled by electric charge with density σ in the presence of an ideal conductor filling the lower half-plane. As a function of the T_k 's it

obeys the infinite set of dispersionless Hirota relations. The corresponding Lax function performs the conformal map from the upper half plane \mathbb{H} to the complement of the “fat slit” \mathcal{B} in \mathbb{H} . Increasing T_1 and keeping all T_k ’s with $k \geq 2$ fixed, one obtains a growth problem of Laplacian type in the upper half plane, with a specific boundary condition on the real line, which is associated with the dispersionless KP hierarchy in the same way as the problem in the whole plane is associated with the dispersionless 2DTL hierarchy.

The tau-function (4.5) regarded as a function of “slow times” $T_k = \hbar t_k$, $k \in \mathbb{Z}$, has a similar $\hbar \rightarrow 0$ limit yielding tau-function of the dispersionless 2DTL hierarchy. It generates a growth problem for “fat slits” in the exterior of the unit disk (Fig. 2b), which will be discussed elsewhere.

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