Invariant formulation of surfaces associated with $\mathbb{C}P^{N-1}$ models

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Abstract.   In this paper, we provide an invariant formulation of completely integrable $\mathbb{C}P^{N-1}$ Euclidean sigma models in two dimensions defined on the Riemann sphere S^2. The scaling invariance is explicitly taken into account by expressing all the equations in terms of projection operators. Properties of the projectors mapping onto one-dimensional subspaces are discussed in detail. The paper includes a discussion of surfaces connected with the $\mathbb{C}P^{N-1}$ models and the wave functions of their linear spectral problem.

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1. Introduction

The simple Dirichlet Lagrangian

$$\mathcal{L} = \frac{1}{4}(D_\mu z)^\dagger \cdot (D_\mu z), \tag{1.1}$$

where † denotes the Hermitian conjugate, may represent e.g. a free quantum particle described by a wave function z. It is trivial when z is a scalar function of its variables and D_μ is just a partial derivative. However the model becomes nontrivial and has found many applications if the target space is a complex Grassmanian manifold and the partial derivatives $(D_\mu z)$ turn into the appropriate covariant derivatives. The most popular are $\mathbb{C}P^{N-1}$ models, whose target spaces are complex $G(1, N)$ Grassmanians, equivalent to projective spaces ($\mathbb{C}P$ stands for complex projective). The target space is a set of lines intersecting at the origin or, equivalently, the $N-1$ dimensional Riemann sphere immersed in an N-dimensional vector space. The two-dimensional space of independent variables is the simplest nontrivial one. In that case the variables z are subject to the constraint

$$z^\dagger \cdot z = 1, \tag{1.2}$$

and the covariant derivative has the form

$$D_\mu z = \partial_\mu z - (z^\dagger \cdot \partial_\mu z)z, \qquad \partial_\mu = \partial_{\xi^\mu}, \qquad \mu = 1, 2. \tag{1.3}$$

These systems are the subject of our investigation in this paper.

The main objective of the paper is to formulate differential projective geometry in terms of projective operators, which makes it explicitly invariant under scaling by any scalar function. Some applications of the projectors have been introduced earlier in [9, 16]. In this paper we construct a basis of projectors which map onto orthogonal one-dimensional subspaces and use it to express all other quantities. As the model is exactly solvable [9, 15], the formulation encompasses the spectral problem and the surfaces whose immersion conditions are the dynamics equation of the system.

Instead of the Cartesian variables $(\xi^1, \xi^2) \in \mathbb{R}^2$, we use more convenient complex variables $(\xi, \bar{\xi}) \in \mathbb{C}$, where $\xi = \xi^1 + i\,\xi^2$ (complex conjugates are marked by a bar over a symbol). The complex plane is usually compactified to the Riemann sphere.

To avoid the inconvenient non-analytic condition (1.2) the model dynamics is usually expressed in terms of

$$z = f/|f|, \quad |f| = \left(f^\dagger \cdot f\right)^{1/2}, \tag{1.4}$$

without any constraints on the new variable f. The Euler-Lagrange (E-L) equations in the new variables read

$$\left(\mathbb{I} - \frac{f \otimes f^\dagger}{f^\dagger \cdot f}\right) \cdot \left[\partial\bar{\partial}f - \frac{1}{f^\dagger \cdot f}\left((f^\dagger \cdot \bar{\partial}f)\partial f + (f^\dagger \cdot \partial f)\bar{\partial}f\right)\right] = 0, \tag{1.5}$$

where ∂ and $\bar{\partial}$ are derivatives with respect to the complex independent variables ξ and $\bar{\xi}$ respectively, $\mathbb{I}$ is the $N \times N$ unit matrix.

On the other hand, equation (1.5) does not have the simplicity of the original Lagrangian. Moreover the solutions in terms of the new variables are not unique: the same Grassmanian solution corresponds to infinitely many f's. To achieve uniqueness, a constraint has to be imposed on f; most often it relies on putting its first nonzero component equal to 1.

It seems that more natural variables in the $\mathbb{C}P^{N-1}$ and all Grassmanian models are projection operators, more precisely orthogonal projectors mapping onto individual directions in the $\mathbb{C}P^{N-1}$ models or on the appropriate subspaces in Grassmanians of higher order. The orthogonal projector which maps onto a one-dimensional subspace in the direction f may be written as

$$P = [1/(f^{\dagger} \cdot f)] f \otimes f^{\dagger}. \tag{1.6}$$

It is evident that such projectors (as well as other orthogonal projectors) are Hermitian $P^{\dagger} = P$. They are also subject to a constraint, but the constraint is analytic and simple

$$P^2 = P, \tag{1.7}$$

while the Lagrangian is as simple as the one for z [16, 8]:

$$\mathcal{L} = \text{tr}(\partial P \cdot \bar{\partial} P). \tag{1.8}$$

Similarly to the case of the z variables, the appropriate constraint, (1.7) in this case, is multiplied by a Lagrange multiplier and subtracted from the Lagrangian before taking the variation of the action integral. The variation yields the E-L equations for the projectors, which can be expressed in the well-known form of a conservation law [15, 16, 8], namely

$$\partial \left[\bar{\partial} P, P \right] + \bar{\partial} \left[\partial P, P \right] = 0. \tag{1.9}$$

More details on the $\mathbb{C}P^{N-1}$ sigma models may be found in [16, 8]. In the present paper we concentrate on the consequences of expressing their theory in terms of the projectors.

This paper is organized as follows. In section 2 we list the basic algebraic and analytical properties of orthogonal projectors which map onto one-dimensional subspaces. Invariant recurrence relations, which follow from those properties, are summarized without much detail (the detailed discussion is given in our other paper [3]). In section 3 we discuss the mutual connection between the projectors and the surfaces whose conditions for immersion (the Gauss-Codazzi-Ricci equations) are equivalent to (1.9). Section 4 contains a discussion of the linear problem and the corresponding wave function. Finally, we list the conclusions and possible directions for further work.

2. Properties of projectors mapping onto 1^D subspaces

Here we list the properties of the orthogonal projection matrices P which map onto one-dimensional subspaces. All of the discussed properties follow from the defining property (1.7) and from the fact that the target is one-dimensional.

(i) From the definition of the projective property (1.7) it follows that the operators P are diagonalizable and their eigenvalues are 0 or 1.

(ii) If a projector maps onto a one-dimensional subspace, its rank is one and thus only one of the eigenvalues is one, the rest being zero. Hence for such P

$$\mathrm{tr}(P) = 1. \tag{2.1}$$

The diagonalisation may always include placing the only nonzero eigenvalue in the first row and first column.

(iii) By differentiating the defining property (1.7) we obtain after straightforward computation

$$\partial P \cdot P = (\mathbb{I} - P) \cdot \partial P, \quad P \cdot \partial P = \partial P \cdot (\mathbb{I} - P) \tag{2.2}$$

and the same holds for the "barred" derivative $\bar{\partial}$. In other words: an exchange with ∂P or $\bar{\partial} P$ turns P into $\mathbb{I} - P$ and vice versa.

Induction yields a more general property about the exchange of P with an arbitrary number of ∂P and $\bar{\partial} P$ in arbitrary order

$$P \cdot \partial P \cdot \bar{\partial} P \cdot ... \cdot \partial P = \partial P \cdot \bar{\partial} P \cdot ... \cdot \partial P \cdot P \tag{2.3}$$

if the total number of the derivatives ∂P, $\bar{\partial} P$ is even, or

$$P \cdot \partial P \cdot \bar{\partial} P \cdot ... \cdot \partial P = \partial P \cdot \bar{\partial} P \cdot ... \cdot \partial P \cdot (\mathbb{I} - P) \tag{2.4}$$

if the total number of the derivatives ∂P, $\bar{\partial} P$ is odd.

The above properties hold for all projection operators, regardless of the dimension of their target subspace and the projection angle.

(iv) If an orthogonal projector P maps onto a one-dimensional subspace then, for any square matrix A having the same dimension as the space, we have

$$P \cdot A \cdot P = \mathrm{tr}(P \cdot A)\, P. \tag{2.5}$$

The proof by diagonalisation follows directly from the property that only one eigenvalue of P is one, while the others are zero. A consequence of this property is the necessary and sufficient condition that a projection of any projector Q onto the projector P is a zero matrix, that is

$$P \cdot Q \cdot P = 0 \quad \mathrm{iff} \quad \mathrm{tr}(P \cdot Q) = 0, \tag{2.6}$$

which is compatible with the definition of the scalar product [3]

$$(A, B) = -(1/2)\mathrm{tr}(A \cdot B). \tag{2.7}$$

(v) The following traces vanish:

$$\mathrm{tr}(P \cdot \partial P \cdot P \cdot \bar{\partial} P \cdot ... \cdot \partial P) = 0, \tag{2.8}$$

where the matrix product of derivatives contains any odd number of the ∂ and $\bar{\partial}$ derivatives in arbitrary order, while the number of the projectors P and their positions in the product are also arbitrary.

Proof: If the product contains at least one operator P: Write any of the operators P in the product as $P \cdot P$ and exchange the right P with ∂P and $\bar{\partial} P$, one by one, up to the rightmost position, while moving the left P in the same way to the leftmost position. On each exchange, P turns into $\mathbb{I} - P$ and vice versa (property 2.4). If we do not encounter the product $P \cdot (\mathbb{I} - P) = 0$ or $(\mathbb{I} - P) \cdot P = 0$ (which ends the procedure), we end up with

$$\mathrm{tr}\left(P \cdot \partial P \cdot P \cdot \bar{\partial} P \cdot ... \cdot \partial P \cdot (\mathbb{I} - P)\right) \tag{2.9}$$

if the number of the derivatives to the left of the P is even while that to the right is odd, or

$$\mathrm{tr}\left((\mathbb{I} - P) \cdot \partial P \cdot P \cdot \bar{\partial} P \cdot ... \cdot \partial P\right) \tag{2.10}$$

if the number of the derivatives to the left of the P is odd while that to the right is even. In either case the trace is zero because a cyclic permutation of the factors yields an expression containing the product $P \cdot (\mathbb{I} - P) = 0$ or $(\mathbb{I} - P) \cdot P = 0$.
If the product contains only derivatives, without P operators, the unit matrix may be put as the first factor and represented as $P + (\mathbb{I} - P)$. The same procedure as before yields zero for each of the components. *Q.E.D.*

(vi) Properties involving 2nd derivatives of the projectors P are interesting as such derivatives occur in the E-L equations (1.9). It follows from $\mathrm{tr}(P \cdot \partial P) = 0$ and $\mathrm{tr}(P \cdot \bar{\partial} P) = 0$ (a special case of (2.8)) that

$$\mathrm{tr}(P \cdot \partial^2 P) = -\mathrm{tr}(\partial P \cdot \partial P), \tag{2.11}$$

with analogous formulae for the $\partial\bar{\partial}$ and $\bar{\partial}^2$ derivatives.

(vii) If the P operator also satisfies the E-L equations (1.9), then we have

$$\mathrm{tr}(P \cdot \bar{\partial} P \cdot \partial\bar{\partial} P) = 0 \tag{2.12}$$

The proof is straightforward if we use property (2.2) and the invariance of traces on cyclic permutations.

(viii) While traces of products of an odd number of derivatives vanish for the P projectors, the traces of an even number can significantly be simplified by the following property: For any square matrix A (of the proper dimension) we have the factorisation property

$$\mathrm{tr}(A \cdot \partial P \cdot \partial P \cdot P) = \mathrm{tr}(A \cdot P)\mathrm{tr}(\partial P \cdot \partial P \cdot P), \tag{2.13}$$

with analogous formulae in which one or both ∂ derivatives are replaced by $\bar{\partial}$.

Proof: From equation (2.3) we have

$$\text{tr}(A \cdot \partial P \cdot \partial P \cdot P) = \text{tr}(A \cdot P \cdot \partial P \cdot \partial P \cdot P), \qquad (2.14)$$

which yields equation (2.13) by property (2.5).

(ix) **Gram-Schmidt orthogonalization**

Given a set of linearly independent vectors in a vector space, we can always construct an orthonormal basis by using the well-known Gram-Schmidt orthogonalization procedure. An orthogonal basis has its counterpart in the corresponding set of projectors. If we represent one-dimensional projectors as matrices in the new orthonormal basis, the i-th projector P_i is represented by a matrix having one nonzero diagonal element, $z_{ii} = 1$, while all other elements of the matrix are zero.

This straightforward procedure is not so trivial if the vectors and projectors are functions of the mainfold parameters e.g. $\xi, \bar{\xi}$ and we want the basis vectors to satisfy the E-L equations (1.5) or, equivalently, the corresponding projectors P_i to satisfy equation (1.9). The Gram-Schmidt orthogonalization operators which map solutions of (1.5) to consecutive orthogonal solutions are [2, 16]

$$P_+(f) = (\mathbb{I} - P) \cdot \partial f, \qquad (2.15)$$

which we refer to as a "creation operator" and

$$P_-(f) = (\mathbb{I} - P) \cdot \bar{\partial} f, \qquad (2.16)$$

which is called an "annihilation operator". The corresponding "creation" and "annihilation" operators for one-dimensional projectors were found in [3]. They are defined by

$$\mathbf{\Pi}_-(P_k) = P_{k-1}, \qquad \mathbf{\Pi}_+(P_k) = P_{k+1}. \qquad (2.17)$$

while their explicit form reads

$$\mathbf{\Pi}_-(P) = \frac{\bar{\partial}P \cdot P \cdot \partial P}{\text{tr}(\bar{\partial}P \cdot P \cdot \partial P)} = \frac{(\mathbb{I} - P) \cdot \bar{\partial}P \cdot \partial P}{\text{tr}(\bar{\partial}P \cdot P \cdot \partial P)} = \frac{\bar{\partial}P \cdot \partial P \cdot (\mathbb{I} - P)}{\text{tr}(\bar{\partial}P \cdot P \cdot \partial P)}, \qquad (2.18)$$

and

$$\mathbf{\Pi}_+(P) = \frac{\partial P \cdot P \cdot \bar{\partial}P}{\text{tr}(\partial P \cdot P \cdot \bar{\partial}P)} = \frac{(\mathbb{I} - P) \cdot \partial P \cdot \bar{\partial}P}{\text{tr}(\partial P \cdot P \cdot \bar{\partial}P)} = \frac{\partial P \cdot \bar{\partial}P \cdot (\mathbb{I} - P)}{\text{tr}(\partial P \cdot P \cdot \bar{\partial}P)}, \qquad (2.19)$$

The complete basis is obtained if we apply the creation operator $0, 1, \ldots, N - 1$ times to any holomorphic solution of (1.5) or its projector counterpart (2.17). The construction may also be performed in the opposite direction by means of the annihilation operator, starting from an antiholomorphic solution.

It immediately follows from (2.18, 2.19), that the result of the "creation" or "annihilation" is always orthogonal to the original projector

$$\mathbf{\Pi}_+(P) \cdot P = P \cdot \mathbf{\Pi}_+(P) = 0 \quad \text{and} \quad \mathbf{\Pi}_-(P) \cdot P = P \cdot \mathbf{\Pi}_-(P) = 0 \qquad (2.20)$$

(x) If the basis is built by the above Gram-Schmidt orthogonalisation, starting from a vector which is a holomorphic function of ξ, then the projectors whose target subspaces are vectors of the basis, satisfy

$$\mathrm{tr}(\partial P \cdot \partial P) = 0, \quad \mathrm{tr}(\bar{\partial} P \cdot \bar{\partial} P) = 0. \tag{2.21}$$

Other projectors mapping onto one-dimensional subspaces do not have to satisfy this equation. The proof will be given in Appendix A.

Using properties (i)–(x) we can prove all the properties required for the model to be consistent.

(i) If P is an orthogonal projector and $P_{+1} = \mathbf{\Pi}_+(P)$ exists, then P_{+1} is also an orthogonal projector: it has the projective property $P_{+1}^2 = P_{+1}$ and its kernel is orthogonal to its target subspace. The same is true of $\mathbf{\Pi}_-(P)$. Moreover the trace of $\partial P \cdot P \cdot \bar{\partial} P$ vanishes iff the whole matrix vanishes (the same holds for $\bar{\partial} P \cdot P \cdot \partial P$), which ensures the possibility of constructing $P_{\pm 1}$ whenever the matrix is nonzero.

(ii) The operators $\mathbf{\Pi}_+$ and $\mathbf{\Pi}_-$ are inverses of each other, i.e. $\mathbf{\Pi}_+(\mathbf{\Pi}_-(P)) = \mathbf{\Pi}_-(\mathbf{\Pi}_+(P)) = P$, provided that the inner operation on P is possible.

(iii) If P satisfies the E-L equations (1.9) and $P_{+1} = \mathbf{\Pi}_+(P)$ exists, then P_{+1} also satisfies those equations.

The proofs are in Appendix A.

3. Projectors and soliton surfaces

We first recall some of the previously known results. It has been shown in [4] that the conservation law (1.9) may be interpreted as a condition for the contour integral

$$X(\xi, \bar{\xi}) = i \int_\gamma \left(-[\partial P, P] d\xi + [\bar{\partial} P, P] d\bar{\xi} \right), \tag{3.1}$$

to be independent of the path of integration γ. This defines a mapping of an area on a Riemann sphere into a set of $su(N)$ matrices $\Omega \ni (\xi, \bar{\xi}) \mapsto X(\xi, \bar{\xi}) \in su(N) \simeq \mathbb{R}^{N^2-1}$. This generalised Weierstrass formula for immersion of 2D surfaces in $\mathbb{R}^{N^2-1}$ [6, 7, 10] defines surfaces in terms of the projectors P. The compatibility conditions of the immersion constitute the conservation law (1.9). The integration may be performed explicitly for the surfaces corresponding to the projectors P_k obtained recursively from the holomorphic solution. It yields ([5], see also the proof in Appendix A)

$$X_k = -i \left(P_k + 2 \sum_{j=0}^{k-1} P_j \right) + \frac{i(1+2k)}{N} \mathbb{I}, \qquad k = 0, \ldots, N-2. \tag{3.2}$$

For $k = N - 1$ equation (3.2) gives an equation equivalent to that for $k = 0$, which reduces the number of surfaces (or algebraically independent immersion functions).

Inversely, we can obtain the projectors P_k from the surfaces X_k either as a linear combination of the surfaces $X_0, \ldots, X_k$

$$P_k = i \sum_{j=1}^{k} (-1)^{k-j} (X_j - X_{j-1}) + (-1)^k i X_0 + \frac{1}{N} \mathbb{I}, \tag{3.3}$$

or by a nonlinear formula which depends on X_k only [3]

$$P_k = X_k{}^2 - 2i \left(\frac{2k+1}{N} - 1 \right) X_k - \frac{2k+1}{N} \left(\frac{2k+1}{N} - 2 \right) \mathbb{I}. \tag{3.4}$$

The projective property $P_k^2 = P_k$ apparently imposes a constraint on the surfaces X_k. Does it constitute an equation defining those surfaces or is it identically satisfied by the surfaces constructed from (3.2)? To verify this, we examine the projective property for P_k, where we substitute the P_k with (3.4). Direct substitution of (3.4) into the projective property yields a 4$^{\text{th}}$ degree equation. However a simpler, 3$^{\text{rd}}$ degree condition may be obtained by multiplying (3.4) by X_k and making use of the fact that P_k is orthogonal to all the lower-index projectors. The 3$^{\text{rd}}$ degree condition obtained in this way may be factorised to the form

$$\left[X_k - i \left(\frac{1+2k}{N} - 2 \right) \mathbb{I} \right] \left[X_k - i \left(\frac{1+2k}{N} - 1 \right) \mathbb{I} \right] \left[X_k - i \frac{1+2k}{N} \mathbb{I} \right] = 0. \tag{3.5}$$

This condition has a simple interpretation if we diagonalise it, which is always possible as the X_k matrices are antihermitian. The diagonalised form (3.5) consists of a product of matrices containing merely eigenvalues of X_k minus a number equal to $i[(1+2k)/N-2]$, $i[(1+2k)/N-1]$ or $i(1+2k)/N$. We find that equation (3.5) is always satisfied if the surfaces have been constructed according to (3.2). That is

- The component $i(1+2k)/N$ has been added to each diagonal element of the sum of projectors (3.2) to make X_k traceless. Therefore it occurs as a component of every eigenvalue of X_k.
- The component $2i$ subtracted from $i(1+2k)/N$ is a contribution due to $2i \sum_{j=0}^{k-1} P_j$ as each of the P_j has one eigenvalue equal to 1 and the other eigenvalues equal to 0. It occurs at the indices of the dimensions onto which $P_1 \ldots P_{k-1}$ map.
- The component i subtracted from $i(1+2k)/N$ is a contribution due to $-i P_k$. It is a component of the eigenvalue at the index pointing at the dimension onto which P_k maps.
- Nothing is subtracted from $i(1+2k)/N$ at the indices $k+1 \ldots N$ pointing at the dimensions onto which none of $P_0, \ldots, P_k$ map.

Thus equation (3.5) is the lowest degree constraint on the immersion functions X_k of the surfaces. If we directly substituted (3.4) into the projective property, we would get an equivalent condition: the equation would differ from (3.5) by the middle factor: in the 4$^{\text{th}}$ degree condition the factor $[X_k - i((1+2k)/N - 1) \mathbb{I}]$ is squared.

Although equation (3.5) is obvious when we look at the source of X_k (3.2), it is nevertheless a nontrivial constraint on the surfaces. Since all the eigenvalues are independent of the coordinates $(\xi, \bar{\xi})$, the whole kinematics of a moving frame (vielbein) may only be due to variation of the diagonalising (unitary) matrix.

Differential geometry of the surfaces. Once we have the immersion functions of the surfaces, we can describe their metric and curvature properties.

(i) The diagonal elements of the metric tensor are zero. This property, proven in [3], directly follows from property (2.21). Let g_k be the metric tensor corresponding to the surface X_k. Its components will be marked with indices outside the parentheses to distinguish them from the number of the surface. We have

$$(g_k)_{11} = -\frac{1}{2}\text{tr}(\partial X_k \cdot \partial X_k) = \frac{1}{2}\text{tr}([\partial P_k, P_k] \cdot [\partial P_k, P_k]) = -\frac{1}{2}\text{tr}(\partial P_k \cdot \partial P_k) = 0, \quad (3.6)$$

where we have successively applied the definition of the metric tensor, the definition of X_k given in (3.1), property (2.2) and property (2.21). The vanishing of $(g_k)_{22}$ follows from the Hermitian conjugate of (3.6).

(ii) The nonzero off-diagonal element $(g_k)_{12} = (g_k)_{21}$ is equal to

$$(g_k)_{12} = -\frac{1}{2}\text{tr}(\partial X_k \cdot \bar{\partial} X_k) = -\frac{1}{2}\text{tr}([\partial P_k, P_k] \cdot [\bar{\partial} P_k, P_k]) = \frac{1}{2}\text{tr}(\partial P_k \cdot \bar{\partial} P_k). \quad (3.7)$$

Thus the 1$^{\text{st}}$ fundamental form reduces to

$$I_k = \text{tr}(\partial P_k \cdot \bar{\partial} P_k)\, d\xi d\bar{\xi}. \quad (3.8)$$

The second form

$$II_k = (\partial^2 X_k - (\Gamma_k)^1_{11}\partial X_k)d\xi^2 + 2\partial\bar{\partial}X_k d\xi d\bar{\xi} + (\bar{\partial}^2 X_k - (\Gamma_k)^2_{22}\bar{\partial}X_k)d\bar{\xi}^2, \quad (3.9)$$

is easy to find when we determine the Christoffel symbols $(\Gamma_k)^1_{11}$ and $(\Gamma_k)^2_{22}$. These are the only nonzero components of the Γ. We have from (3.7)

$$(\Gamma_k)^1_{11} = \partial \ln (g_k)_{12}, \quad (\Gamma_k)^2_{22} = \bar{\partial} \ln (g_k)_{12}. \quad (3.10)$$

Using (3.1) and the E-L equations (1.9) together with (3.10), we can write (3.9) as

$$\begin{aligned} II_k = {} & -\text{tr}(\partial P_k \cdot \bar{\partial} P_k)\,\partial\frac{[\partial P, P]}{\text{tr}(\partial P_k \cdot \bar{\partial} P_k)}d\xi^2 + 2[\bar{\partial}P, \partial P]d\xi d\bar{\xi} \\ & + \text{tr}(\partial P_k \cdot \bar{\partial} P_k)\,\bar{\partial}\frac{[\bar{\partial}P, P]}{\text{tr}(\partial P_k \cdot \bar{\partial} P_k)}d\bar{\xi}^2 \end{aligned} \quad (3.11)$$

Examples of the metric for surfaces induced by Veronese solutions of the E-L equations (1.5) are given in [3].

4. Projectors and the spectral problem

The spectral problem is closely related to the immersion functions of the surfaces. The relation between the wave functions and the immersion functions is given by the Sym-Tafel formula [11, 12, 13, 14], and they are also related by their asymptotic properties. These aspects of the theory were discussed in [3]. In this section we concentrate on the consequences of their representation in terms of projectors.

Similarly to the surfaces, the wave functions of the spectral problem can also be expressed in terms of the projectors. The spectral problem found by Zakharov and Mikhailov [15] reads

$$\partial\Phi_k = \frac{2}{1+\lambda}[\partial P_k, P_k]\Phi_k, \qquad \bar{\partial}\Phi_k = \frac{2}{1-\lambda}[\bar{\partial}P_k, P_k]\Phi_k, \qquad k = 0, 1, \ldots, N-1, \quad (4.1)$$

where $\lambda \in \mathbb{C}$ is the spectral parameter and the wave functions are given by [1]

$$\Phi_k = \mathbb{I} + \frac{4\lambda}{(1-\lambda)^2}\sum_{j=0}^{k-1} P_j - \frac{2}{1-\lambda}P_k, \qquad (4.2)$$

$$\Phi_k^{-1} = \mathbb{I} - \frac{4\lambda}{(1+\lambda)^2}\sum_{j=0}^{k-1} P_j - \frac{2}{1+\lambda}P_k. \qquad (4.3)$$

This in turn yields the projectors P_k in terms of the wave functions [3]

$$P_k = (1/4)\left[2(1+\lambda^2)\mathbb{I} - (1-\lambda)^2\Phi_k - (1+\lambda)^2\Phi_k^{-1}\right]. \qquad (4.4)$$

The projective property may be represented in terms of Φ_k as a factorisable 4th degree expression with one double (squared) factor, resembling the corresponding equation for the surfaces X_k, namely

$$P_k^2 - P_k = (1/16)\Phi_k^{-2}(\mathbb{I} - \Phi_k)\left[(1+\lambda)^2 - (1-\lambda)^2\Phi_k\right]\left[(1+\lambda) - (1-\lambda)\Phi_k\right]^2 = 0. \quad (4.5)$$

It may be interpreted in the same way as the equivalent relation for the surfaces (3.5). We can also obtain a 3rd degree equation in which all of the linear factors are of the 1st degree. This may be performed in a way similar to the derivation of (3.5), i.e. by multiplying (4.4) by Φ_k and applying the orthogonality of P_k to $P_0 \ldots P_{k-1}$.

5. Concluding remarks

The description of $\mathbb{C}P^{N-1}$ models in terms of orthogonal projection operators has a few advantages compared with their description in terms of vectors. It is natural, and the picture which it provides is clear. At the same time it need not be more difficult than that in terms of vectors, provided that we know a few identities of the projector algebra and analysis (such as those listed in Section 2). The construction of an orthogonal basis in terms of projectors is straightforward. Also the principal conditions required for the consistency of the model are easy to prove.

The technique presented above for constructing an increasing number of surfaces associated with CP^{N-1} sigma models on Euclidean spaces can lead to a detailed analytical description of the surfaces in question. This description provides us with effective tools for finding surfaces without invoking any additional considerations, proceeding directly from the given CP^{N-1} sigma model equations (1.9).

In the next stage of this research, it would be worthwhile to extend the presented approach to more general sigma models based on Grassmannian manifolds, i.e. the homogeneous spaces

$$G(m,n) = \frac{SU(N)}{S\left(U(m) \times U(n)\right)}, \quad N = m + n. \tag{5.1}$$

Grassmannian sigma models are a generalization of CP^{N-1} sigma models. Their important common property is that the Euler-Lagrange equations can best be written in terms of projectors. They share a lot of properties like an infinite number of local and/or nonlocal conserved quantities, infinite-dimensional symmetry algebras, Hamiltonian structures, complete integrability, the existence of multisoliton solutions, etc. The investigation of surfaces for this case can lead to different classes and much more diverse types of surfaces than the ones discussed in this paper. The geometrical aspects of such surfaces will be described in more detail in a future work.

Acknowledgments

Appendix A. Proofs of consistency properties

Here we use the properties (i–x) of the projection operators mapping onto one-dimensional subspaces in order to prove several properties required for the consistency of the description of the $\mathbb{C}P^{N-1}$ model in terms of projection operators.

(i) The projective property of the P "promoted" by the creation operator (2.17) has been proven in [3]. However we may obtain it immediately by using the property (2.5)

$$\mathbf{\Pi}_{+}(P) \cdot \mathbf{\Pi}_{+}(P) = \frac{\partial P \cdot P \cdot \bar{\partial} P \cdot \partial P \cdot P \cdot \bar{\partial} P}{[\mathrm{tr}(\partial P \cdot P \cdot \bar{\partial} P)]^2} = \frac{\partial P \cdot P \cdot \bar{\partial} P}{[\mathrm{tr}(\partial P \cdot P \cdot \bar{\partial} P)]} = \mathbf{\Pi}_{+}(P), \tag{A.1}$$

where the property (2.5) has been applied to transform the numerator in (A.1) according to

$$(\partial P \cdot P \cdot \bar{\partial} P) \cdot (\partial P \cdot P \cdot \bar{\partial} P) = \mathrm{tr}(\partial P \cdot P \cdot \bar{\partial} P)\, \partial P \cdot P \cdot \bar{\partial} P. \tag{A.2}$$

A by-product of the proof is a demonstration that the trace of $\partial P \cdot P \cdot \bar{\partial} P$ vanishes iff the whole matrix vanishes. Namely, if the trace vanishes then the r.h.s. of (A.2) is zero, but it is a square of a Hermitian matrix $\partial P \cdot P \cdot \bar{\partial} P$. Hence it vanishes iff the matrix vanishes.

This means that the construction of $\mathbf{\Pi}_+(P)$ from P (2.19) is correct and always possible, except for the cases in which $\partial P \cdot P \cdot \bar{\partial} P$ vanishes. In this case $\mathbf{\Pi}_+(P) = 0$ is indeterminate.

Mutatis mutandis we may prove the projective property and correctness of the construction for $\mathbf{\Pi}_-(P)$.

The orthogonality of the projectors $\mathbf{\Pi}_+(P)$ and $\mathbf{\Pi}_-(P)$ follows from the fact that they are Hermitian (which may be checked in a straightforward way).

(ii) In order to make the model consistent, we should have

$$\mathbf{\Pi}_- \left(\mathbf{\Pi}_+(P) \right) = P \quad \text{and} \quad \mathbf{\Pi}_+ \left(\mathbf{\Pi}_-(P) \right) = P, \tag{A.3}$$

provided that the action of the creation operator (first case) or the annihilation operator (second case) can be executed. For shorthand notation we introduce

$$P_{+1} = \mathbf{\Pi}_+(P); \quad P_{-1} = \mathbf{\Pi}_-(P) \tag{A.4}$$

According to the definition of $\mathbf{\Pi}_\pm$, we have to prove that $\bar{\partial} P_{+1} \cdot P_{+1} \cdot \partial P_{+1} / \mathrm{tr}(\bar{\partial} P_{+1} \cdot P_{+1} \cdot \partial P_{+1}) = P$, provided that $P_{+1} \neq 0$.

To assess $\bar{\partial} P_{+1} \cdot P_{+1} \cdot \partial P_{+1}$ and its trace, note that we may double P_{+1} in this expression, due to its previously proven projective property.

$$\bar{\partial} P_{+1} \cdot P_{+1} \cdot \partial P_{+1} = \bar{\partial} P_{+1} \cdot P_{+1} \cdot P_{+1} \cdot \partial P_{+1} \tag{A.5}$$

Note also that the second half of the r.h.s. in equation (A.5) is the Hermitian conjugate of its first half. Consider the first half. We have

$$\begin{aligned}
\bar{\partial} P_{+1} \cdot P_{+1} &= \bar{\partial} \frac{\partial P \cdot P \cdot \bar{\partial} P}{\mathrm{tr}(\partial P \cdot P \cdot \bar{\partial} P)} \cdot \frac{\partial P \cdot P \cdot \bar{\partial} P}{\mathrm{tr}(\partial P \cdot P \cdot \bar{\partial} P)} \\
&= \frac{(P \cdot \bar{\partial}\partial P \cdot \bar{\partial} P + \partial P \cdot \bar{\partial} P \cdot \bar{\partial} P)\partial P \cdot P \cdot \bar{\partial} P}{\left[\mathrm{tr}(\partial P \cdot P \cdot \bar{\partial} P)\right]^2}.
\end{aligned} \tag{A.6}$$

We apply properties (2.3) and (2.5) in order to replace some of the factors in the numerator by the appropriate traces, then we use the property (2.13) to factor those traces. The invariance of traces under cyclic permutations of factors allows us to obtain (after cancellation of the common factor)

$$\bar{\partial} P_{+1} \cdot P_{+1} = \frac{\left[\mathrm{tr}(\bar{\partial}\partial P \cdot P) + \mathrm{tr}(\partial P \cdot \bar{\partial} P \cdot P)\right] P \cdot \bar{\partial} P}{\mathrm{tr}(\partial P \cdot P \cdot \bar{\partial} P)} \tag{A.7}$$

Using the property (2.11) and then the exchange property (2.2), we further get

$$\bar{\partial} P_{+1} \cdot P_{+1} = \frac{\left[-\mathrm{tr}(\partial P \cdot \bar{\partial} P) + \mathrm{tr}(\partial P \cdot \bar{\partial} P \cdot P)\right] P \cdot \bar{\partial} P}{\mathrm{tr}(\partial P \cdot P \cdot \bar{\partial} P)} = -P \cdot \bar{\partial} P \tag{A.8}$$

Hence, combining the above result with its Hermitian conjugate, we finally get

$$\mathbf{\Pi}_- \left(\mathbf{\Pi}_+(P)\right) = \frac{P \cdot \bar{\partial}P \cdot \partial P \cdot P}{\text{tr}(P \cdot \bar{\partial}P \cdot \partial P \cdot P)} = P \tag{A.9}$$

due to property (2.5). *Q.E.D.*

(iii) We now prove, by a different method than that of [5], the equation (3.2), expressing X_k as a sum of projectors. An equation obtained in the proof will also be used to demonstrate another property necessary for the consistency of the model: if P_m satisfies the E-L equation (1.9), then $P_{m+1} = \mathbf{\Pi}_+(P_m)$ also does.

Proof: The surfaces X_k are defined by (3.1) up to a constant matrix (whose diagonal elements are uniquely determined by the condition that the traces of X_k vanish). Hence it is sufficient to prove that

$$\partial X_k = -i\partial P_k - 2i\sum_{j=0}^{k-1} \partial P_j, \tag{A.10}$$

$$\bar{\partial} X_k = -i\bar{\partial} P_k - 2i\sum_{j=0}^{k-1} \bar{\partial} P_j, \tag{A.11}$$

or equivalently

$$[\partial P_k, P_k] = \partial P_k + 2\sum_{j=0}^{k-1} \partial P_j, \tag{A.12}$$

$$[\bar{\partial} P_k, P_k] = -\bar{\partial} P_k - 2\sum_{j=0}^{k-1} \bar{\partial} P_j. \tag{A.13}$$

This thesis will be proven by induction. For $k = 0$, equations (A.12) and (A.13) reduce to

$$\partial P_0 P_0 - P_0 \partial P_0 = \partial P_0 \quad \text{and} \quad \bar{\partial} P_0 P_0 - P_0 \bar{\partial} P_0 = -\bar{\partial} P_0 \tag{A.14}$$

which by property (2.2) are equivalent to

$$P_0 \cdot \partial P_0 = 0 \quad \text{and} \quad \bar{\partial} P_0 \cdot P_0 = 0 \tag{A.15}$$

where P_0 maps onto a direction of a holomorphic function. Let z be a unit vector in that direction. Then $P_0 = z \otimes z$ and z depends only on ξ, while $z^\dagger$ depends only on $\bar{\xi}$. Hence

$$P_0 \cdot \partial P_0 = z \otimes z^\dagger \cdot \partial z \otimes z^\dagger = (z^\dagger \cdot \partial z)z \otimes z^\dagger = 0 \tag{A.16}$$

since $z^\dagger \cdot \partial z = \partial(z^\dagger \cdot z) = 0$ for any holomorphic unit vector z. The second half of (A.14) is the Hermitian conjugate of the first half.

Let (A.12 and A.13) now hold for $k = m \geq 0$. By property (2.2) we have

$$\bar{\partial} P_{m+1} \cdot P_{m+1} - P_{m+1} \cdot \bar{\partial} P_{m+1} = 2\partial P_{m+1} \cdot P_{m+1} - \bar{\partial} P_{m+1} \tag{A.17}$$

We may replace $2\,\bar{\partial}P_{m+1}\cdot P_{m+1}$ by $-2\,P_m\bar{\partial}P_m$ using (A.8). Applying property (2.2) to one of these two $P_m\bar{\partial}P_m$ we get

$$[\bar{\partial}P_{m+1}, P_{m+1}] = -\bar{\partial}P_{m+1} + [\bar{\partial}P_m, P_m] - \bar{\partial}P_m \tag{A.18}$$

On the basis of the induction hypothesis, (A.18) turns into

$$[\bar{\partial}P_{m+1}, P_{m+1}] = -\bar{\partial}P_{m+1} - 2\sum_{j=0}^{m}\bar{\partial}P_j, \tag{A.19}$$

which is exactly the second part of the thesis, i.e. (A.13). The first part of the thesis: (A.12) is its Hermitian conjugate. *Q.E.D.*

From the intermediate result (A.18) we obtain the connection between the E-L equations (1.9) for P_m and those for P_{m+1}. Applying the ∂ derivative to (A.18), the $\bar{\partial}$ derivative to its Hermitian conjugate, and subtracting the results of the differentiation from each other we obtain

$$\partial\,[\bar{\partial}P_{m+1}, P_{m+1}] + \bar{\partial}\,[\partial P_{m+1}, P_{m+1}] = \partial\,[\bar{\partial}P_m, P_m] + \bar{\partial}\,[\partial P_m, P_m], \tag{A.20}$$

whence P_{m+1} satisfies the E-L equations (1.9) iff P_m does, provided that the construction of P_{m+1} from P_m is possible. This result may also be used in the opposite direction: when the construction of P_{m-1} from P_m is possible, then P_{m-1} satisfies the E-L equations iff P_m does. This is another criterion of consistency of the model.

Another by-product of the proof is the demonstration of property (x) (2.21) (from which it follows that the metric tensor g_k of the surfaces X_k has zeros on the diagonal $(g_k)_{11} = (g_k)_{22} = 0$). Indeed (2.21) obviously holds for $k = 0$, when $P = z\otimes z^\dagger$. If it holds for $k = m$ then, writing (A.18) as $[\bar{\partial}P_{m+1}, P_{m+1}] + \bar{\partial}P_{m+1} = [\bar{\partial}P_m, P_m] - \bar{\partial}P_m$ and squaring both sides, we obtain (using property (2.2) and the invariance of traces under cyclic permutations)

$$\mathrm{tr}(\bar{\partial}P_{m+1}\cdot\bar{\partial}P_{m+1}) = \mathrm{tr}(\bar{\partial}P_m\cdot\bar{\partial}P_m), \tag{A.21}$$

which yields the thesis for all k by induction.

References

[1] Din A M, Horvath Z and Zakrzewski W J 1984 *The Riemann-Hilbert problem and finite action* $\mathbb{C}P^{N-1}$ *solutions* Nucl. Phys. **B 233**, 269

[2] Din A M and Zakrzewski W J 1980 General classical solutions of the $\mathbb{C}P^{N-1}$ model *Nucl. Phys.* B **174** 397–403.

[3] Goldstein P and Grundland A M Invariant recurrence relations for $\mathbb{C}P^{N-1}$ models J. Phys. A submitted 2009, arXiv math-phys. 09100624.

[4] Grundland A M, Strasburger A and Zakrzewski W J 2005 Surfaces immersed in $\mathfrak{su}(N+1)$ Lie algebras obtained from the $\mathbb{C}P^N$ sigma models, *J. Phys. A: Math. Gen.* **39** 9187–9213.

[5] Grundland A M and Yurdusen I 2009 On analytic descriptions of two-dimensional surfaces associated with the CP^{N-1} sigma models *J. Phys. A: Math. Gen.* **42** 172001 (5pp).

[6] Konopelchenko B 1996 Induced surfaces and their integrable dynamics *Stud. Appl. Math.* **96** 9–51.

[7] Konopelchenko B and Taimanov I 1996 Constant mean curvature surfaces via an integrable dynamical system *J. Phys. A: Math. Gen.* **29** 1261–1265

[8] Manton N and Sutcliffe P 2004 *Topological Solitons* (Cambridge: Cambridge University Press).

[9] Mikhailov A V 1986 Integrable magnetic models in: Solitons (Modern Problems in Condensed Matter vol. 17) ed S E Trullinger, V E Zakharov and V L Pokrovsky (Amsterdam: North-Holland) pp 623–690

[10] Nomizu K and Sasaki T 1994 *Affine differential geometry* (Cambridge: Cambridge University Press).

[11] Sym A 1982 *Soliton surfaces* Lett. Nuovo Cimento **33** 394–400

[12] Sym A 1983 *Soliton surfaces II: Geometric unification of solvable nonlinearities* Lett. Nuovo Cimento **36** 307–312

[13] Sym A 1985 *Soliton surfaces and their applications (Soliton geometry from spectral problems)*, Geometric Aspects of the Einstein Equations and Integrable systems (Lecture Notes in Physics vol 239) ed R Martini (Berlin: Springer), 154–231

[14] Tafel J 1995 *Surfaces in R^3 with prescribed curvature*, J. Geom. Phys. **17** 381–390

[15] Zakharov V E and Mikhailov A V 1979 Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method Sov. Phys.–JETP **47** 1017.

[16] Zakrzewski W J 1989 *Low Dimensional Sigma Models* (Bristol: Adam Hilger 1989), Chapter 3 (46–74).