

Homotopy perturbation method for fractional-order Burgers-Poisson equation

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Abstract

In this paper, the fractional-order Burgers-Poisson equation is introduced by replacing the first-order time derivative by fractional derivative of order α . Both exact and approximate explicit solutions are obtained by employing homotopy perturbation method. The numerical results reveal that the proposed method is very effective and simple for handling fractional-order differential equations.

Keywords: Fractional Burgers-Poisson equation; Homotopy perturbation method; Fractional derivative; Symbolic computation

1. Introduction

In 2004, the Burgers-Poisson (BP) equation has firstly been proposed to describe the unidirectional propagation of long waves in dispersive media [1], denoted by

$$u_t + uu_x = \varphi_x, \quad (1)$$

$$\varphi_{xx} = \varphi + u, \quad (2)$$

where φ and u depend on $(t, x) \in (0, \infty) \times R$, and subscripts denote partial derivatives. In order to well study BP equation, by applying $1 - \partial_x^2$ to (1) and using (2) on the resulting right-hand side, we rewrite the BP system as a single

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differential equation for u :

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx}. \quad (3)$$

Because BP equation is a shallow water equation modeling unidirectional water wave subject to weaker dispersive effects than the KdV equation. This means that BP equation is very important in the field of mathematical physics. In Ref.[1], the authors turned out that BP equation features wave breaking in finite time, a local existence result for smooth solutions and a global existence result for weak entropy solutions were further proved. Later on, BP equation was also proposed by Fetecau and Levy [2] by Padé (2,2) approximation of the phase velocity that arises in the linear water wave theory. In Ref.[3], the authors used classical Lie method to construct group invariant solutions. Moreover, variational iteration method (VIM) was applied to study the numerical solutions of BP equation [4].

Recently, fractional calculus has been extensively applied in many fields [5]. Many important phenomena are well described by fractional differential equations in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science. That is because of the fact that, a realistic modelling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. In particular, Wang [6, 7] employed the homotopy perturbation method (HPM) to solve the classical fractional KdV and KdV-Burgers equations, respectively. Momani [8] solved fractional KdV equation via the Adomian decomposition method (ADM) and Momani *et al.* [9] applied variational iteration method (VIM) to solve the space and time fractional KdV equation. In the above methods, HPM provides an effective procedure for explicit and numerical solutions of a wide and general class of differential systems representing real physical problems. Then HPM has been widely used by other authors [10, 11] as well as their referees to solve fractional order differential equations.

In this paper, we introduce fractional-order into the BP equation (3) by replacing the first-order time derivative by fractional derivative of order α , then

we obtain the fractional Burgers-Poisson (fBP) equation of the form

$$D_t^\alpha u - D_t^\alpha u_{xx} + u_x + uu_x - (3u_x u_{xx} + uu_{xxx}) = 0, \quad t > 0, 0 < \alpha \leq 1, \quad (4)$$

where α denotes the order of the fractional time-derivative in the Caputo sense. The function $u(x, t)$ is assumed to be a causal function of time and space, *i.e.* vanishing for $t < 0$ and $x < 0$. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$, the fractional Eq. (4) reduces to the classical BP equation (3). Furthermore, HPM will be employed to obtain both exact and approximate explicit solutions of fBP equation.

2. Preliminaries

We first give the definitions of fractional-order integration and fractional-order differentiation [12]. For the concept of fractional derivative, we will adopt Caputo's definition, which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems.

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ , $\mu \in R$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_n if and only if $f^n \in C_\mu$, $n \in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order α , J^α , of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0, t > 0). \quad (5)$$

Definition 3. The fractional derivative of $f(t)$ in Caputo's sense is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t - \tau)^{\alpha+1-m}} d\tau, \quad (6)$$

where $m - 1 < \alpha \leq m$, $m \in N$, $t > 0$, $f \in C_{-1}^m$.

Definition 4. For m to be the smallest integer that exceeds α , the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x, t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x, \tau)}{\partial \tau^m} d\tau, & \text{if } m-1 < \alpha < m, \\ \frac{\partial^m u(x, t)}{\partial t^m}, & \text{if } \alpha = m \in N. \end{cases} \quad (7)$$

Lemma 1. If $m-1 < \alpha \leq m$, $m \in N$, and $f \in C_\mu^m$, $\mu \geq -1$, then

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f(0^+) \frac{x^k}{k!}. \quad (8)$$

3. Application of HPM to fBP equation

Homotopy perturbation method (HPM) is first proposed by He [13]. It is a powerful mathematic tool to solve nonlinear problems, especially engineering problems. To further complement HPM, He has developed it [14, 15] recently.

In what follow, the HPM is used to study the fBP equation (4) with the initial condition

$$u(x, 0) = x. \quad (9)$$

The exact solution of BP equation (3), the special case of fBP equation (4) when $\alpha = 1$, is given by classical Lie method (see Ref.[3])

$$u(x, t) = \frac{1+x}{1+t} - 1. \quad (10)$$

In view of HPM, the homotopy is constructed as following

$$(1-p)D_t^\alpha u + p(D_t^\alpha u - D_t^\alpha u_{xx} + u_x + uu_x - (3u_x u_{xx} + uu_{xxx})) = 0, \quad (11)$$

or

$$D_t^\alpha u + p(-D_t^\alpha u_{xx} + u_x + uu_x - (3u_x u_{xx} + uu_{xxx})) = 0, \quad (12)$$

where $p \in [0, 1]$ is an embedding parameter. By utilizing the parameter p , the solution $u(x, t)$ can expanded in the following form

$$u(x, t) = u_0(x, t) + pu_1(x, t) + p^2 u_2(x, t) + p^3 u_3(x, t) + \dots \quad (13)$$

Setting $p = 1$ gives the approximate solution

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots \quad (14)$$

Now, substituting (14) into (12), and equating the terms with the identical powers of p , gives

$$\begin{aligned}
p^0 : D_t^\alpha u_0 &= 0, \quad u_0(x, 0) = x; \\
p^1 : D_t^\alpha u_1 - D_t^\alpha u_{0xx} + u_{0x} + u_0 u_{0x} - 3u_{0x} u_{0xx} - u_0 u_{0xxx} &= 0, \quad u_1(x, 0) = 0; \\
p^2 : D_t^\alpha u_2 - D_t^\alpha u_{1xx} + u_{1x} + (u_0 u_{1x} + u_1 u_{0x}) - 3(u_{0x} u_{1xx} + u_{1x} u_{0xx}) \\
&\quad - (u_0 u_{1xxx} + u_1 u_{0xxx}) = 0, \quad u_2(x, 0) = 0; \\
p^3 : D_t^\alpha u_3 - D_t^\alpha u_{2xx} + u_{2x} + (u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}) \\
&\quad - 3(u_{0x} u_{2xx} + u_{1x} u_{1xx} + u_{2x} u_{0xx}) - (u_0 u_{2xxx} + u_1 u_{1xxx} + u_2 u_{0xxx}) = 0, \\
&\quad u_3(x, 0) = 0; \\
&\vdots
\end{aligned} \tag{15}$$

Applying the operator J^α on both sides of equations in (15) and utilizing Lemma 1, yields

$$\begin{aligned}
u_0(x, t) &= x; \\
u_1(x, t) &= -\frac{1+x}{\Gamma(1+\alpha)} t^\alpha; \\
u_2(x, t) &= \frac{2(1+x)}{\Gamma(2+\alpha)} t^{2\alpha}; \\
u_3(x, t) &= -\frac{(1+x)(4\Gamma(1+\alpha)^2 + \Gamma(2+\alpha))}{\Gamma(1+\alpha)^2 \Gamma(3+\alpha)} t^{3\alpha} \\
&\vdots
\end{aligned} \tag{16}$$

Since

$$\begin{aligned}
D_t^\alpha u_j - D_t^\alpha u_{(j-1)xx} + u_{(j-1)x} + \sum_{i=0}^{j-1} u_i u_{(j-i-1)x} - 3 \sum_{i=0}^{j-1} u_{ix} u_{(j-i-1)xx} \\
- \sum_{i=0}^{j-1} u_i u_{(j-i-1)xxx} = 0
\end{aligned} \tag{17}$$

and $u_j(x, 0) = 0$, the other arbitrary $u_j(x, t)$ ($j \geq 4$) can be calculated in the same manner by symbolic software programme *Mathematica*. If only the first four approximations of equation (14) are sufficient. Then the approximate explicit solution of equation (4) will be expressed as

$$u(x, t) = x - \frac{1+x}{\Gamma(1+\alpha)} t^\alpha + \frac{2(1+x)}{\Gamma(2+\alpha)} t^{2\alpha} - \frac{(1+x)(4\Gamma(1+\alpha)^2 + \Gamma(2+\alpha))}{\Gamma(1+\alpha)^2 \Gamma(3+\alpha)} t^{3\alpha}. \tag{18}$$

In the other word, the exact solution for fBP equation when $\alpha = 1$ can be

obtained by HPM. In face, if $\alpha = 1$, equation (16) will read

$$\begin{aligned} u_0(x, t) &= x; \\ u_1(x, t) &= -(1+x)t; \\ u_2(x, t) &= (1+x)t^2; \\ u_3(x, t) &= -(1+x)t^3. \end{aligned} \tag{19}$$

According to (17), u_j satisfies

$$u_j(x, t) = (-1)^j(1+x)t^j. \tag{20}$$

Then the exact solution will be expressed as

$$u(x, t) = \lim_{n \rightarrow \infty} \sum_{j=0}^n u_j(x, t) = \frac{1+x}{1+t} - 1, \tag{21}$$

which is just the same as for the classical Lie method [3] and the VIM [4].

Table 1: Comparison with the exact solution and the four-term approximation solution.

(x, t)	u_{exact}	u_{HPM}	$ u_{exact} - u_{HPM} $
(2,0.3)	0.528462	0.526	0.002462
(2,0.35)	0.481481	0.45925	0.022231
(2,0.4)	0.428571	0.392	0.036571
(2,0.45)	0.37931	0.32275	0.05656
(2,0.5)	0.333333	0.25	0.083333
(0.9,0.2)	0.583333	0.5804	0.002933
(1.2,0.2)	0.833333	0.8304	0.002933
(1.5,0.2)	1.08333	1.08	0.00333
(1.8,0.2)	1.33333	1.3296	0.00373
(2,0.2)	1.5	1.496	0.004

4. Numerical simulation

In order to illustrate the approximate solution is efficiency and accuracy, we will give explicit value of the parameters x, t . Then calculate the two solutions

and make a comparison between them. Taking $x = 2$ for different value of t and $t = 0.2$ for different value of x , we calculate the numerical solutions of the two solutions which are given in (10) and (18). We list their numerical solutions in Table 1.

From the numerical solutions in the Table 1, it can be seen that at the same time x , the value of the approximate solutions and the exact solutions are quite close. Also, when the value of t decreases the approximate solutions are more and more closed to the exact solutions. This shows the approximate solution is efficiency. It is also suggested that HPM is a powerful method for solving fractional differential equation with fully nonlinear dispersion terms.

5. Conclusions

In this paper, both exact and approximate explicit solutions of fractional-order Burgers-Poisson equation are obtained by employing homotopy perturbation method. The comparison of numerical solution and exact solution demonstrates that the proposed method is very effective and simple for solving solutions of fractional differential equations. It should be pointed out that detailed studies of fractional-order Burgers-Poisson equation are only beginning. The periodical waves, peakons, fractional Hamilton structure and other properties are still open. We hope that this work is a step in this direction and some traditional analytic method for nonlinear differential equations of integer order can be extended to fractional-order equations.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 10871074).

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