

Symmetry Analysis of 2+1 dimensional Burgers equation with variable damping

D. Pandiaraja¹ and B. Mayil Vaganan²

¹Department of Mathematics, Thiagarajar College, Madurai-625009, India

²Department of Applied Mathematics and Statistics, Madurai Kamaraj University, Madurai-625021, India

Abstract The symmetry classification of the two dimensional Burgers equation with variable coefficient is considered. Symmetry algebra is found and a classification of its subalgebras, up to conjugacy, is obtained. Similarity reductions are performed for each class.

Keywords Lie symmetries, Symmetry analysis, Killing form.

AMS Classification Numbers 22E60, 27E70, 34A05, 35G20.

1. Introduction

Gandarias [5] has studied Type-II hidden symmetries of the two dimensional Burgers equation

$$u_t + uu_x - u_{xx} - u_{yy} = 0. \quad (1.1)$$

In [4], applications of (1.1) have been discussed.

In this paper we provide a detailed symmetry analysis of the two dimensional Burgers equation with variable damping, viz.,

$$u_t + uu_x + \alpha(t)u - u_{xx} - u_{yy} = 0. \quad (1.2)$$

One of the significant application of Lie symmetry groups to differential equations is to achieve a complete classification of its symmetry reductions. The symmetry properties and reductions of certain differential equations have been recently investigated (See references 10-12 in [1]). As described in [8], the classification of group invariant solutions requires a classification of subalgebras of the symmetry algebra into conjugacy classes under the adjoint action of the symmetry group.

This paper is organized as follows. In section 2, we perform a symmetry classification of (1.2) and a classification of one-dimensional, two-dimensional, three-dimensional subalgebras of the symmetry algebra. In section 3, we tabulate the reductions of (1.2) under one-dimensional, two-dimensional, three-dimensional subalgebras. We summarize the results in section 4.

2. The symmetry algebra and classification of subalgebras

Equation (1.2) is assumed to be invariant under Lie group of infinitesimal transformations (Olver [7], Blumen and Kumei [3])

$$x_i^* = x_i + \epsilon \xi_i(x, y, t, u) + O(\epsilon^2), \quad i = 1, 2, 3, 4, \quad (2.1)$$

where $\xi_1 = \xi, \xi_2 = \eta, \xi_3 = \tau, \xi_4 = \phi$. Then the fourth prolongation $pr^{(4)}V$ of

$$V = \tau(x, y, t; u) \partial_t + \xi(x, y, t; u) \partial_x + \eta(x, y, t; u) \partial_y + \phi(x, y, t; u) \partial_u, \quad (2.2)$$

must satisfy

$$pr^{(4)}V\Omega(x, y, t; u)|_{\Omega(x, y, t; u)=0} = 0, \quad (2.3)$$

where Ω is the RHS of (1.2). The following system of 12 determining equations are obtained from (2.3) (See [6]).

$$\begin{aligned} (\xi_1)_u &= 0, \\ (\xi_2)_u &= 0, \\ (\xi_3)_u &= 0, \\ (\phi_1)_{u,u} &= 0, \\ (\xi_3)_y &= 0, \\ (\xi_3)_x &= 0, \\ -(\xi_2)_t - u(\xi_2)_x + (\xi_2)_{x,x} + (\xi_2)_{y,y} - 2(\phi_1)_{y,u} &= 0, \\ \phi_1 - (\xi_1)_t - u(\xi_1)_x + 2u(\xi_2)_y + (\xi_1)_{x,x} + (\xi_1)_{y,y} - 2(\phi_1)_{x,u} &= 0, \\ u\alpha_t\xi_3 + \alpha\phi_1 + 2u\alpha(\xi_2)_y + (\phi_1)_t - u\alpha(\phi_1)_u + u(\phi_1)_x - (\phi_1)_{x,x} - (\phi_1)_{y,y} &= 0, \\ (\xi_1)_y + (\xi_2)_x &= 0, \\ -(\xi_1)_x + (\xi_2)_y &= 0, \\ 2(\xi_2)_y - (\xi_3)_t &= 0. \end{aligned}$$

Solving these determining equations we get

$$\xi = \frac{q}{2}x + c_2 + c_1 \int e^{-\int \alpha dt} dt, \quad (2.4)$$

$$\eta = \frac{q}{2}y + p, \quad (2.5)$$

$$\tau = qt + r, \quad (2.6)$$

$$\phi = -\frac{q}{2}u + c_1 e^{-\int \alpha dt}, \quad (2.7)$$

under the condition that

$$(qt + r)\alpha_t = 0 \quad (2.8)$$

Thus we have the following theorems:

Theorem 1. If $\alpha(t) = \alpha_0$, a real constant then symmetry algebra L_c of (1.2) can be written as $L_c = V_1 + V_2 + V_3 + V_4$ where $V_1 = -\frac{e^{-\alpha_0 t}}{\alpha_0} \partial_x + e^{-\alpha_0 t} \partial_u$, $V_2 = \partial_t$, $V_3 = \partial_y$ and $V_4 = \partial_x$ provided $q = 0, r \neq 0$

Table 1- Commutator table for the Lie algebra L_c :

	V_1	V_2	V_3	V_4
V_1	0	$\alpha_0 V_1$	0	0
V_2	$-\alpha_0 V_1$	0	0	0
V_3	0	0	0	0
V_4	0	0	0	0

Theorem 2. If $\alpha(t) \neq 0$ is an arbitrary function of time t , then the general element of the symmetry algebra L of (1.2) can be written as $L = V_1 + V_2 + V_3$ where $V_1 = \int e^{\int -\alpha dt} dt \partial_x + e^{\int -\alpha dt} \partial_u$, $V_2 = \partial_x$ and $V_3 = \partial_y$.

It should be noted here that the generators V_1, V_2, V_3 commute.

The problem of enumeration of all subalgebras of a given finite-dimensional Lie algebra L is essential for the group analysis of differential equations. For this group $\text{Int } L$ of the inner automorphisms can be found. Under the operation of an automorphism, every algebra transforms into a subalgebra of the same dimensionality.

During the construction of optimal systems for a given Lie algebra L a special role is played by the center Z . The point is that every vector $z \in Z$ is invariant relative to an automorphism $A \in \text{Int } L$. Thus the central elements generate subalgebras (ideals), which cannot be changed by any automorphisms. They can be included as a direct term in any subalgebra of the Lie algebra L . Thus if the optimal systems are known for the factor algebra L/Z , then they can be considered as known for the entire Lie algebra L . It is sufficient to describe methods for constructing optimal systems only for Lie algebras with null centers [8].

Case 1. α is a constant.

We assume that $\alpha = \alpha_0$. The commutator table gives the following information about the structure of the Lie algebra L_c . The Lie algebra L_c can be written as $L_c = Z \oplus L_2$ where $Z = \{V_3, V_4\}$ is the center of the Lie algebra and $L_2 = \{V_1, V_2\}$

We shall proceed to construct the optimal system now. First the mapping $\text{ad}(V)$ is calculated by the equation $\text{ad}(V) \langle x \rangle = [x, V]$, where V is the general vector, $V = V^1 V_1 + V^2 V_2$ and $x \in L_2$ and it turns out to be

$$\text{ad}(V) = \begin{pmatrix} \alpha_0 V^2 & -\alpha_0 V^1 \\ 0 & 0 \end{pmatrix}$$

The general automorphism of the group $\text{Int } L^2$ can be found by constructing one-parameter groups of Automorphisms $A_i(t)$ for every basis vector V_1, V_2

To find the group $A_1(t)$ corresponding to V_1 , it is necessary to solve the equation $\partial_t x' = [x', V]$, solving we get the matrix of automorphism

$$A_1(t) = \begin{pmatrix} 1 & -\alpha_0 t \\ 0 & 1 \end{pmatrix}$$

To find the group $A_2(t)$ corresponding to V_2 , it is necessary to solve the equation $\partial_t x' = [x', V]$, solving we get the matrix of automorphism

$$A_2(t) = \begin{pmatrix} e^{\alpha_0 t} & 0 \\ 0 & 1 \end{pmatrix}$$

Here it is convenient to assume $t = \frac{1}{\alpha_0} a$ in $A_1(t)$ and $t = \frac{1}{\alpha_0} \ln b$ in $A_2(t)$

$$A = A_1\left(\frac{1}{\alpha_0} a\right) \circ A_2\left(\frac{1}{\alpha_0} \ln b\right) \ln b \quad (2.9)$$

The general automorphism $A \in \text{Int}L^2$ is obtained and it depends on two parameters, a, b :

$$A = \begin{pmatrix} b & -a \\ 0 & 1 \end{pmatrix}$$

Now, it is possible to find the components of the vector $x' = A(x)$ in the basis $\{V_1, V_2\}$:

$$x'^1 = bx^1 - ax^2, \quad x'^2 = x^2 \quad (2.10)$$

The decomposition of the space L^2 , on the classes of similar vectors, can be found using the invariance property of the killing form.

The killing form is

$$\frac{1}{\alpha_0^2} K(x, x) = (x^2)^2, \quad (2.11)$$

and is invariant of the group $\text{Int}L^2$. Since $K(x, y) \neq 0$, it is not degenerate and the algebra L^2 is semi simple. The vectors are separated into two disjoint classes:

- 1) $K(x, x) > 0$, 2) $K(x, x) = 0$

A representative of class (1) is V_2 and a representative of class (2) is V_1 . Hence $\Theta_1 = \{V_1, V_2\}$. Thus the sub algebras of the symmetry algebra L_c is given in the following table:

Table 3 Classification of subalgebras of symmetry algebra L_c

Dimension	Subalgebra
-----------	------------

1-dimensional subalgebra	$L_1 = (aV_3 + bV_4 + V_1)$
	$L_2 = (aV_3 + bV_4 + V_2)$
	$L_3 = (aV_3 + bV_4)$
2-dimensional subalgebra	$L_4 = (aV_3 + bV_4, V_1)$
	$L_5 = (aV_3 + bV_4, V_2)$
	$L_6 = (V_3, V_4)$
3-dimensional subalgebra	$L_7 = (aV_3 + bV_4, V_1, V_2)$

Case 2. α is arbitrary

The commutator table gives the following information about the structure of the Lie algebra L . The Lie algebra L itself a center hence a maximal ideal. Thus the subalgebras of the symmetry algebra can be classified as in table 4.

Table 4 Classification of subalgebras of symmetry algebra L

Dimension	Subalgebra
1-dimensional subalgebra	$L_1 = (aV_1 + bV_2 + cV_3)$
2-dimensional subalgebra	$L_2 = (V_1, V_2)$
	$L_3 = (V_1, V_3)$
	$L_4 = (V_2, V_3)$
3-dimensional subalgebra	$L_5 = (V_1, V_2, V_3)$

3. Symmetry reduction for (1.2)

For brevity, we present only two representative reductions that too for the case when α is arbitrary below. The complete details of reduced equations for each subalgebra when α is either a constant or any function of t are presented in table forms. When α is arbitrary, the reductions to PDEs and then to ODEs are given respectively in Appendix-A and Appendix-B. Similarly, when α is a constant, the respective reductions are given in Appendix-C and Appendix-D.

3.1 Reduction under one dimensional subalgebra $V_1 + V_2 + V_3$

The characteristic equation for $aV_1 + bV_2 + cV_3$ is

$$\frac{dx}{1 + \int e^{\int -\alpha(t)dt}} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{e^{\int -\alpha(t)dt}} \quad (3.1)$$

Integrating the characteristic equation we get three similarity variables

$$\xi = x - y(1 + \int e^{-\int \alpha(t)dt}), \eta = t, u = ye^{-\int \alpha(t)dt} + F(\xi, \eta) \quad (3.2)$$

In terms of the similarity variables, (1.2) is reduced to

$$(1 + (1 + \int e^{-\int \alpha(\eta)d\eta})^2)F_{\xi\xi} - F_{\eta} - FF_{\xi} - \alpha(\eta)F = 0 \quad (3.3)$$

We give a complete table of reductions to PDEs for all one dimensional subalgebras in Appendix-A.

3.1 Reduction under two dimensional sub algebra $[V_2, V_3]$

Consider the algebra given by V_2 and V_3 . Since $[V_2, V_3] = 0$, we begin with $V_2 = \frac{\partial}{\partial x}$. The similarity variables for this generators are given by $\xi = y, \eta = t, u = F(\xi, \eta)$. Using these variables (1.2) reduces to a PDE

$$F_{\xi\xi} - F_{\eta} + \alpha(\eta)F = 0. \quad (3.4)$$

In order to perform second reduction of the above equation, we firstly write V_3 in terms of new variables ξ, η and $F(\xi, \eta)$ as $V_3 =$

Conclusions

The symmetry classification of the two dimensional Burgers equations with constant and variable coefficients, viz.,

$$u_t + uu_x + \alpha_0 u = u_{xx} - u_{yy} = 0, \quad (3.5)$$

$$u_t + uu_x + \alpha(t)u = u_{xx} - u_{yy} = 0, \quad (3.6)$$

have been carried out. For, the symmetry algebras and their subalgebras, up to conjugacy, are enumerated. Successive reductions to PDEs with two independent variables and then to ODEs of both second and first orders are performed for each subalgebra. As most of the reduced equations are of the standard forms or can be changed to canonical forms through simple transformations we dispense with the problem of writing solutions to these equations.

References

1. Azad H and Mustafa M T 2007 *J. Math. Anal. Appl.* **333** 1180-1188

2. Barbara Abraham Shrauner and Keshlan S Govinder 2006 *J. Nonlin. Math. Phys.*) **13** 612-622
3. Bluman G W and Kumei S *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989.
4. Edwards M P and Broadbridge P 1995 *Z. Angew. Math. Phys.* **46** 595 - 622
5. Gandarias M L 2008 *J. Math. Anal. Appl.* **348** 752-759
6. Gerd Baumann *Symmetry Analysis of Differential Equations with Mathematica*, Springer-Verlag, New York, 2000.
7. Olver P J *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.
8. Ovsiannikov L V *Group analysis of Differential Equations*, Academic Press, New York, 1982

Appendix-A

<i>Subalgebra</i>	<i>Reduced equation</i>
V_1	$F_{\xi\xi} - F_\eta - \alpha(\eta)F = 0$
V_2	$F_{\xi\xi} - F_\eta - \alpha(\eta)F = 0$
V_3	$F_{\xi\xi} - F_\eta - FF_\xi - \alpha(\eta)F = 0$
$V_1 + V_2$	$F_{\xi\xi} - F_\eta - \alpha(\eta)F = 0$
$V_2 + V_3$	$2F_{\xi\xi} - F_\eta - FF_\xi - \alpha(\eta)F = 0$
$V_1 + V_3$	$(1 + (\int e^{-\int \alpha(\eta)d\eta})^2)F_{\xi\xi} - F_\eta - FF_\xi - \alpha(\eta)F = 0$
$V_1 + V_2 + V_3$	$(1 + (1 + \int e^{-\int \alpha(\eta)d\eta})^2)F_{\xi\xi} - F_\eta - FF_\xi - \alpha(\eta)F = 0$

Appendix B

<i>Subalgebra</i>	<i>Reduced equation</i>
$[V_1, V_2]$	$W_r + \alpha(r)W = 0$
$[V_1, V_3]$	$W_r + \alpha(r)W = 0$
$[V_2, V_3]$	$W_r + \alpha(r)W = 0$
$[V_1, V_2, V_3]$	$W_r + \alpha(r)W = 0$

Appendix C

<i>Subalgebra</i>	<i>Condition</i>	<i>Reduced equation</i>
$aV_3 + bV_4 + V_1$	$a \neq 0, b \neq 0$	$(a^2 + (b + e^{-\alpha_0\eta})^2)F_{\xi\xi} - FF_\xi - F_\eta - \alpha_0F = 0$
	$a = 0, b \neq 0$	$F_{\xi\xi} - F_\eta - \alpha_0F = 0$
	$a \neq 0, b = 0$	$(a^2 + \frac{e^{-2\alpha_0\eta}}{\alpha_0^2})F_{\xi\xi} - FF_\xi - F_\eta - \alpha_0F = 0$
	$a = 0, b = 0$	$F_{\xi\xi} - F_\eta = 0$
$aV_3 + bV_4 + V_2$	$a \neq 0, b \neq 0$	$F_{\xi\xi} + F_{\eta\eta} - aFF_\xi + bF_\xi + aF_\eta - \alpha_0F = 0$
	$a = 0, b \neq 0$	$F_{\xi\xi} + F_{\eta\eta} - FF_\xi + bF_\xi - \alpha_0F = 0$
	$a \neq 0, b = 0$	$F_{\xi\xi} + F_{\eta\eta} - FF_\xi + aF_\eta - \alpha_0F = 0$
	$a = 0, b = 0$	$F_{\xi\xi} + F_{\eta\eta} - FF_\xi - \alpha_0F = 0$
$aV_3 + bV_4$	$a \neq 0, b \neq 0$	$(a^2 + b^2)F_{\xi\xi} - aFF_\xi - F_\eta - \alpha_0F = 0$
	$a = 0, b \neq 0$	$F_{\xi\xi} - FF_\xi - F_\eta - \alpha_0F = 0$
	$a \neq 0, b = 0$	$F_{\xi\xi} - F_\eta - \alpha_0F = 0$

Appendix D

<i>Subalgebra</i>	<i>Condition</i>	<i>Reduced equation</i>
$[aV_3 + bV_4, V_1]$	$a \neq 0, b \neq 0$	$W_r + \alpha_0W = 0$
	$a = 0, b \neq 0$	$W_r + \alpha_0W = 0$
	$a \neq 0, b = 0$	$W_r + \alpha_0W = 0$
$[aV_3 + bV_4, V_2]$	$a \neq 0, b \neq 0$	$(a^2 + b^2)W_{rr} - aWW_r - \alpha_0W = 0$
	$a = 0, b \neq 0$	$W_{rr} - WW_r - \alpha_0W = 0$
	$a \neq 0, b = 0$	$W_{rr} - \alpha_0W = 0$
$[V_3, V_4]$		$W_r + \alpha_0W = 0$
$[aV_3 + bV_4, V_1, V_2]$	$a \neq 0, b \neq 0$	$W_r + \alpha_0W = 0$