Symmetry Analysis for a Generalized Kadomtsev-Petviashvili Equation

B. Mayil Vaganan[1], D. Pandiaraja[2] and M. Senthilkumaran[2]
[1]Department of Applied Mathematics and Statistics, Madurai Kamaraj University, Madurai-625021, India
[2]Department of Mathematics, Thiagarajar College, Madurai-625009, India

Abstract

A generalized Kadomtsev-Petviashvili equation (GKPE) $(u_t+uu_x+\beta(t)u+\gamma(t)u_{xxx})_x+\sigma(t)u_{yy} = 0$ is shown to admit an infinite-dimensional Lie group of symmetries when $\beta(t)$, $\gamma(t)$ and $\sigma(t)$ are arbitrary. The Lie algebra of this symmetry group contains two arbitrary functions $f(t)$ and $g(t)$. Further, low-dimensional subalgebras and physically meaningful five dimensional Lie algebra containing translation and Galilei transformation are derived. A solution of GKPE involving two arbitrary functions of time t, in addition to $f(t)$ and $g(t)$, is obtained using an one-dimensional subalgebra.

Key Words: Generalised KP equation, Symmetry group, Symmetry algebra, Conjugacy classes.

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1. Introduction

Kadomtsev-Petviashvili (KP) equation

$$(u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx})_x + \frac{3}{4}u_{yy} = 0, \tag{1}$$

known also as the two-dimensional Korteweg-de Vries equation arises in the study of long gravity waves in a single layer, or multilayered shallow fluid, when the waves propagate predominantly in one direction with a small perturbation in the perpendicular direction. The mathematical interest of KP equation stems from the fact that it is associated with an infinite-dimensional Lie groups. It is integrable in the sense of allowing Lax pair, conservation laws, solitons, and periodic solutions (See [3] and references 1-11 in [3]).

A prototype example of the derivation of a generalized KP (GKP) equation from Euler equations in somewhat realistic conditions was given by David, Levi and Winternitz [5]. David, Levi and Winternitz [4] studied the symmetries and reductions for a generalized KP equation

$$(u_t + uu_x + u_{xxx})_x + \sigma(t)u_{yy} = 0. \tag{2}$$

Brugarino and Greco [2] studied VCKP equation

$$(u_t + a(x, y, t)u + b(x, y, t)u_x + f(x, y, t)uu_x + g(x, y, t)u_{xxx})_x + h(x, y, t)u_{yy} = k(x, y, t), \tag{3}$$

to determine the conditions on the coefficient functions under which (3) passes the Painlevé test.

Güngör and Winternitz [9] classified another VCKP equation

$$(u_t + f(x, y, t)uu_x + g(x, y, t)u_{xxx})_x + h(x, y, t)u_{yy} = 0, \quad (4)$$

into equivalence classes under fibre preserving point transformations with a nonzero Jacobian.

Güngör and Winternitz [10], using the allowed transformation, transformed yet another VCKP equation

$$(u_t+p(t)uu_x+q(t)u_{xxx})_x+\sigma(y,t)u_{yy}+a(y,t)u_y+b(y,t)u_{xy}+c(y,t)u_{xx}+e(y,t)u_x+f(y,t)u+h(y,t) = 0, \quad (5)$$

into the canonical form

$$(u_t + uu_x + u_{xxx})_x + \epsilon u_{yy} + a(y, t)u_y + b(y, t)u_{xy} + c(y, t)u_{xx} + f(y, t)u = 0, \quad \epsilon = \pm 1, \quad (6)$$

and investigated its group theoretical properties in order to establish the conditions on the coefficient functions a, b, c and f under which (5) admits an infinite-dimensional symmetry group having a Kac-Moody-Virasoro structure.

Here If we consider a GKPE

$$(u_t + \alpha'(t)uu_x + \beta'(t)u + \gamma'(t)u_{xxx})_x + \sigma'(t)u_{yy} \;=\; 0, \quad \gamma'(t), \sigma'(t) \neq 0. \quad (7)$$

The point transformation

$$\bar{t} = \int_{t_0}^{t} \alpha(s)ds, \quad (8)$$

replaces (7) by an equation of the form

$$(u_t + uu_x + \beta(t)u + \gamma(t)u_{xxx})_x + \sigma(t)u_{yy} \;=\; 0, \quad \gamma(t), \sigma(t) \neq 0. \quad (9)$$

Equation (2) is a special case of (9) when $\beta(t) = 0$ and $\gamma(t) = 1$. In this paper we study the symmetry properties of the GKPE (9) by closely following the works of David, Kamran, Levi and Winternitz [3] and Güngör [7-8]. To be precise, we shall show that the GKPE (9) admits an infinite-dimensional symmetry group and determine the corresponding Lie algebra, extend it by specifying the coefficient functions $\beta(t), \gamma(t), \sigma(t)$, and classify the one- and two-dimensional subalgebras of the symmetry algebra under the adjoint action of the symmetry group in order to reduce (9) to (1+1)-dimensional partial differential equations (PDEs) and then to ordinary differential equations (ODEs). The symmetry algebra is found to involve two arbitrary functions $f(t)$ and $g(t)$. It is shown that (9) reduces to a linear PDE $W_{yy}(y, t) = F(f(t), f'(t))$ and also to a VCKdVE (42). Several symbolic manipulation packages are available for calculating the symmetry group of PDEs (See Yao Ruo-Xia and Lou Sen-Yue [14] and references therein). In this work we use MathLie [6] to determine the symmetry group of GKPE (9).

This paper is organised as follows: In section 2 we derive the symmetry group and study the structure of the symmetry algebra of the GKPE (9). Section 3 is devoted to the determination of physically interesting finite-dimensional algebra by restricting $f(t)$ and $g(t)$ to first degree polynomials. In section 4 we give the classification of low-dimensional subalgebras of the GKPE algebra, namely those of dimension $n = 1, 2$ into conjugacy classes under the adjoint action of the symmetry group of the GKPE (9). This is done mainly to elucidate the structure of the considered infinite-dimensional Lie algebra and to establish the applicability of tools developed for classifying subalgebras of finite-dimensional Lie algebras. In section 5 we reduce the GKPE (9) into (1+1)-dimensional PDEs using the one-dimensional subalgebras of GKPE algebra. In section 6 we use two isomorphy classes of two-dimensional algebras, namely, Abelian and non-Abelian, to reduce the PDEs obtained in section 5 to ODEs. In section 7 we write down the general form of the reduced ODEs and are transformed to special cases of equations introduced by Mayil Vaganan and Senthilkumaran [11]. Finally in section 8 we summarise the results of the present work.

2. The symmetry group and Lie algebra of the GKPE (9)

If (9) is assumed to be invariant under Lie group of infinitesimal transformations (Olver [11], Bluman and Kumei [12])

$$x_i^* = x_i + \epsilon\xi_i(x, y, t, u) + O(\epsilon^2), \quad i = 1, 2, 3, 4, \tag{10}$$

where $\xi_1 = \xi, \xi_2 = \eta, \xi_3 = \tau, \xi_4 = \phi$, then the corresponding vector field V is

$$V = \tau(x, y, t; u)\ \partial_t + \xi(x, y, t; u)\ \partial_x + \eta(x, y, t; u)\ \partial_y + \phi(x, y, t; u)\ \partial_u. \tag{11}$$

Then the fourth prolongation of V must satisfy

$$pr^{(4)}V\Omega(x, y, t; u)|_{\Omega(x,y,t;u)=0} = 0. \tag{12}$$

where $\Omega(x, y, t; u) = 0$ is (9) and $pr^{(4)}$ stands for the fourth prolongation of the vector field V. The defining equations are obtained from (12) and solved for the infinitesimals ξ, η, τ, ϕ for the following five cases:

Case i. β, γ, σ are arbitrary.

The infinitesimals ξ, η, τ and ϕ are obtained as

$$\xi = f - \frac{y}{2}\left(\frac{g'}{\sigma}\right), \quad \eta = g, \quad \tau = 0, \quad \phi = f' - \frac{y}{2}\left(\frac{g'}{\sigma}\right)'. \tag{13}$$

The symmetry algebra of (9) is an infinite-dimensional Lie algebra $L_p = \{V\}$, where

$$\begin{aligned} V &= X(f) + Y(g), &(14)\\ X(f) &= f\partial_x + f'\partial_u, &(15)\\ Y(g) &= -\frac{y}{2}\left(\frac{g'}{\sigma}\right)\partial_x + g\partial_y - \frac{y}{2}\left(\frac{g'}{\sigma}\right)'\partial_u. &(16)\end{aligned}$$

Here $f(t)$ and $g(t)$ are arbitrary smooth function and satisfy commutation relations

$$[X(f_1), X(f_2)] = 0, \quad [X(f), Y(g)] = 0, \quad [Y(g_1), Y(g_2)] = X\left[\frac{1}{2\sigma}\left(g_2 g_1' - g_1 g_2'\right)\right]. \tag{17}$$

As ∂_t does not appear in V, the Lie algebra L_p is not of Virasoro type (cf. Güngör [7]). Each of the vector fields $X(f)$ and $Y(g)$ can be integrated separately to obtain the Lie group of transformations. Thus if $u(x, y, t)$ is any solution to (9), then so are

$$u'(x', y', t') = u(x - \epsilon f(t), y, t) + \epsilon f'(t), \tag{18}$$

$$u'(x', y', t') = u\left(x - \frac{g'}{2\sigma} y\epsilon - \frac{gg'}{4\sigma}\epsilon^2, y + g\epsilon, t\right) - \frac{1}{2}\left(\frac{g'}{\sigma}\right)'\left(y\epsilon + g\frac{\epsilon^2}{2}\right). \tag{19}$$

Now we shall show that the algebra L_p becomes larger when we specify the functions β, γ, σ. We list below 3 such extensions of L_p. In the foregoing analysis $c_1,\ lambda \in R$.

Case ii. $\beta(t) = \beta, \gamma(t) = \gamma, \sigma(t) = \sigma$, where β, γ, σ are constants.

It is found that τ is no longer zero, but is given by $\tau = c_1$. Therefore, in this case, the symmetry algebra L_1 is represented by (14) and $T_0 = \partial_t$.

Now the Lie algebra L_1 with the basis $X(f), Y(g)$ and T_0 can be written as a semidirect sum

$$L_1 = \{X(f), Y(g)\} \oplus_s \{T_0\}.$$

Case iii. β, γ are constants and $\sigma(t) = e^{\lambda t}$

The infinitesimals which undergo changes are η and τ. Indeed, we find that

$$\eta = c_1\ y\ + g(t) \quad \text{and} \quad \tau = \frac{2}{\lambda} c_1. \tag{20}$$

The Lie algebra L_2 has an additional generator

$$D_\lambda = \frac{\lambda}{2}\ y\ \partial_y + \partial_t, \tag{21}$$

which is a scaling in the y-direction and translation in time t. Thus the basis of L_2 is $X(f), Y(g)$ and D_λ. In this case we may write L_2 as

$$L_2 = \{X(f), Y(g)\}\ \oplus_s\ \{D_\lambda\}.$$

Case iv. β, σ are constants and $\gamma(t) = e^{\lambda t}$.

Here the infinitesimals are

$$\xi = f + \frac{c_1}{2\beta}x - \frac{1}{2\sigma}yg', \quad \eta = \frac{c_1 y}{4\beta} + g, \quad \tau = \frac{3c_1}{2\beta\lambda}, \quad \phi = \frac{c_1 u}{2\beta} + f' - \frac{yg''}{2\sigma}. \tag{22}$$

Hence the basis of the Lie algebra L_3 is now given by the three generators

$$X(f), Y(g) \quad \text{and} \quad E_\lambda = \frac{\lambda}{3}x\partial_x + \frac{\lambda}{6}y\partial_y + \partial_t + \frac{\beta\lambda}{3}u\partial_u. \tag{23}$$

The gererator E_λ contains scalings in x, y and u directions and translation in t. We write the Lie algebra L_3 as

$$L_3 = \{X(f), Y(g)\} \ \oplus_s \ \{E_\lambda\}.$$

It is now easy to infer the following facts:

(i) When β, γ and σ are arbitrary functions of time t, the Lie algebra $L_p = \{X(f), Y(g)\}$, is of infinite-dimensional with the basis given by two generators $X(f), Y(g)$.

(ii) If we restrict β, γ and σ to constants then the Lie algebra L_p gets enlarged to L_1 as L_1 is found to be the semi-direct sum of L_p and T_0.

(iii) If we only take β, γ to be constants and $\sigma(t) = e^{\lambda t}$, then Lie algebra L_2, in addition to $X(f)$ and $Y(g)$, contain another basis element D_λ.

(iv) If $\gamma(t) = e^{\lambda t}$ and β, σ are taken as constants, then Lie algebra L_3 is shown to be generated by the three infinitesimal generators $X(f), Y(g)$, and E_λ.

The commutator table amongst $X(f), Y(g), T_0, D_\lambda, E_\lambda$ is given below:

	$X(f)$	$Y(g)$	T_0	D_λ	E_λ
$X(f)$	0	0	$-X(f')$	$-X(f')$	$X(\frac{\lambda}{3}f - f')$
$Y(g)$	0	0	$-Y(g')$	$X(\frac{\lambda}{2}\frac{yg'}{\sigma}) + Y(\frac{\lambda}{2}g - g')$	$Y(\frac{\lambda}{6}g - g')$
T_0	$X(f')$	$Y(g')$	0	0	0
D_λ	$X(f')$	$-X(\frac{\lambda}{2}\frac{yg'}{\sigma}) - Y(\frac{\lambda}{2}g - g')$	0	0	0
E_λ	$-X(\frac{\lambda}{3}f - f')$	$-Y(\frac{\lambda}{6}g - g')$	0	0	0

Table- 1.

3. A finite-dimensional subalgebra of physical transformations

We shall now systematically classify L_p into finite-dimensional subalgebras of physical interest. If we choose $f(t) = g(t) = 1$ and $f(t) = g(t) = t$ respectively, then we have

$$X(1) = \partial_x = X, \quad Y(1) = \partial_y = Y, \tag{24}$$

and

$$X(t) = t\partial_x + \partial_u = B, \quad Y(t) = -\frac{y}{2\sigma}\partial_x + t\partial_y = R. \tag{25}$$

Here X and Y are translations in x and y respectively and B is a Galilei transformation in the x direction. Finally R is a combination of a Galilei transformation in the y direction and a pseudo-rotation.

Now the Lie algebra L_0 corresponding to the GKPE

$$(u_t + uu_x + \beta u + \gamma u_{xxx})_x + \sigma u_{yy} \ = \ 0, \tag{26}$$

where β, γ and σ are constants, is

$$L_0 = \{X, B, R, Y, T_0\} \tag{27}$$

which is of dimension five. The commutator table for L_0 is

	X	B	R	Y	T_0
X	0	0	0	0	0
B	0	0	0	0	$-X$
R	0	0	0	$-\frac{X}{2\sigma}$	$-Y$
Y	0	0	$\frac{X}{2\sigma}$	0	0
T_0	0	X	Y	0	0

Table-2

4. Low-dimensional subalgebras of the symmetry algebra of GKPE (9)

In order to obtain the solutions of the GKPE (9) by symmetry reduction, it is essential to identify the low-dimensional subalgebras of the GKPE symmetry algebra. In particular, we need to find subalgebras that correspond to Lie groups having orbits of codimension 2 or 1 in the four-dimensional space coordinated by (x, y, t, u). We therefore classify the one-dimensional subalgebras into conjugacy classes under the adjoint action of the symmetry group of the GKPE (9). In the foregoing analysis the results given in (17) and Table-1 are used.

Case 1. β, γ, σ - arbitrary functions of time t

If we take conjugation of $V = X(f) + Y(g)$ by $Y(G)$, where $G(t)$ is to be determined, then, in view of the commutation relation (17), we have

$$\begin{aligned}
Ad\{\exp(\epsilon Y(G))\}V &= V-\epsilon\,[y(G),V] \\
&= V-\epsilon\,[Y(G),X(f)+Y(g)] \\
&= V-\epsilon\,[Y(G),X(f)]-\epsilon[Y(G),Y(g)] \\
&= V-\epsilon\,X\,\left(\frac{1}{2\sigma}(gG'-Gg')\right) \\
&= X(f)+Y(g)-X\left(\frac{\epsilon}{2\sigma}(gG'-Gg')\right) \\
&= X\left(f-\frac{\epsilon}{2\sigma}(gG'-Gg')\right)+Y(g).
\end{aligned} \tag{28}$$

Now we fix $G(t)$ as

$$G(t)=2bg(t)\int_1^t\frac{\sigma(t)f(t)}{[g(t)]^2}dt+cg(t), \tag{29}$$

where b and c are arbitrary constants. We choose $G(t)$ given by (29) as the function labelling the generator $Y(G)$ of the symmetry algebra of the GKPE (9), and $\epsilon=b^{-1}$ as the value of the parameter ϵ of the one-parameter subgroup associated with $Y(G)$. Then it is evident that V is conjugate to $Y(g)$ if $g\neq 0$ and V is conjugate to $X(f)$ if $g=0$. Therefore it is enough to consider the two one-dimensional subalgebras namely $L_{p,1}=\{X(f)\}$ and $L_{p,2}=\{Y(g)\}$ instead of the full symmetry algebra L_p itself.

Case 2. β,γ,σ - arbitrary constants.
If we take conjugation of $V_1=X(f)+Y(g)+aT_0$, $a\neq 0$ by $X(F)+Y(G)$ we obtain

$$\begin{aligned}
Ad\{\exp(\epsilon X(F)+\delta Y(G))\}V_1 &= V_1-\epsilon\,[X(F),V_1]-\delta[Y(G),V_1] \\
&= V_1-\epsilon\,[X(F),aT_0]-\delta[Y(G),Y(g)]-\delta[Y(G),aT_0] \\
&= aT_0+X(f+a\epsilon F'-\frac{\delta}{2\sigma}[gG'-Gg'])+Y(g+a\delta G').
\end{aligned} \tag{30}$$

If we choose $a=0$, $\delta=1/b$ and $G(t)$ as in (29), then V_1 is conjugate to $Y(g)$. On the other hand if we set $a\neq 0$, $\delta=1/b$, $\epsilon=1/c$ and define $F(t)$ and $G(t)$ as

$$F(t)=\frac{c}{2a^2\sigma}\int\left[-g^2+g'\int g(t)dt-f(t)\right]dt+c_1,\quad G(t)=-\frac{b}{a}\int g(t)dt+c_2, \tag{31}$$

where c_1 and c_2 are arbitrary constants, then V_1 is conjugate to T_0. If $a=g=0$ then V_1 is conjugate to $X(f)$.

Case 3. β,γ are arbitrary constants and $\sigma=e^{\lambda t}$.

Conjugating the general element $V_2 = X(f) + Y(g) + aD_\lambda, a \neq 0$ by $X(F) + Y(G)$ we obtain

$$\begin{aligned}
& Ad\,\{\exp(\epsilon X(F) + \delta Y(G))\}\, V_2 \\
= \;& V_2 - \epsilon\, [X(F), aD_\lambda] - \delta[Y(G), Y(g)] - \delta[Y(G), aD_\lambda], \\
= \;& aD_\lambda + X(f + a\epsilon F' - \frac{\delta}{2\sigma}[gG' - Gg']) - a\delta Y(\frac{\lambda}{2}g - g') - a\delta X(\frac{\lambda}{2}\frac{yg'}{\sigma}), \\
= \;& aD_\lambda + X(f + a\epsilon F' - \frac{\delta}{2\sigma}[gG' - Gg'] - \frac{a\delta\lambda}{2}\frac{yg'}{\sigma}) + Y(g - \frac{a\delta\lambda}{2}g + a\delta g'). \qquad (32)
\end{aligned}$$

If we choose $a \neq 0, \epsilon = 1/d$, $g = 0$ and fix $F(t) = -\frac{d}{a}\int f(t)dt + c_1$, then V_2 is conjugate to D_λ. If $a = 0$, $G(t)$ as in (29), then V_2 is conjugate $Y(g)$. If $a = g = 0$, then V_2 is conjugate to $X(f)$.

Case 4. β, σ are arbitrary constants and $\gamma(t) = e^{\lambda t}$.

Conjugating the general element $V_4 = X(f) + Y(g) + aE_\lambda,\ a \neq 0$ by $X(F) + Y(G)$ we obtain

$$\begin{aligned}
& Ad\,\{\exp(\epsilon X(F) + \delta Y(G))\}\, V_4 \\
= \;& V_4 - \epsilon\, [X(F), aE_\lambda] - \delta[Y(G), Y(g)] - \delta[Y(G), aE_\lambda], \\
= \;& aE_\lambda + X(f - \frac{a\epsilon\lambda}{3}F + a\epsilon F' - \frac{\delta}{2\sigma}(gG' - Gg')) + Y(g - \frac{a\delta\lambda G}{6} + \delta a G'). \qquad (33)
\end{aligned}$$

Again we can shown that V_4 is conjugate to either one of the generators $X(f), Y(g), E_\lambda$.

5. Reductions to (1+1) dimensional PDEs.

The general method for performing the symmetry reduction using some specific subgroup G_0 of the symmetry group G is to first find the invariants of G_0 and rewrite (9) in terms of these invariants. The invariants are obtained by solving the system of PDEs $X_i I(x, y, t, u) = 0,\ i = 1, ..., r$, where $X_1, X_2, ..., X_r$ is a basis for the Lie algebra of the symmetry group G_0.

5.1 Subalgebra $L_{s,1} = \{X(f)\}$. Integration of the one-dimensional vector field $X(f)$, where $f(t)$ is arbitrary leads to

$$u(x, y, t) = \frac{f'(t)}{f(t)}x + W(y, t). \qquad (34)$$

Insertion of (34) into (9) yields the PDE

$$\left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2 + \beta\left(\frac{f'}{f}\right) + \sigma W_{yy} = 0 \qquad (35)$$

If we denote f'/f by $F(t)$, then equation (35) can be integrated to yield

$$W(y, t) = -\frac{1}{\sigma}(F' + F^2 + \beta F)\frac{y^2}{2} + h(t)y + k(t). \qquad (36)$$

Thus we obtain the following family of solutions of (9) which involve three arbitrary functions $f(t), h(t)$ and $k(t)$ of time t, by inserting (36) into (34):

$$u = \frac{f'}{f}x - \frac{1}{\sigma}(F' + F^2 + \beta F)\frac{y^2}{2} + h(t)y + k(t). \qquad (37)$$

5.2 Subalgebra $L_{s,2} = \{Y(g)\}$
We use the ansatz

$$u = W(\xi, \eta) - \frac{y^2}{4g}\left(\frac{g'}{\sigma}\right)', \quad \xi = \frac{y^2}{2} + \frac{2g\sigma}{g'}x, \quad \eta = t, \tag{38}$$

into (9) and obtain the PDE

$$G^2 W W_\xi + \beta G W + \gamma G^4 W_{\xi\xi\xi} + \sigma W + G W_\eta + G'\xi W_\xi = 0, \quad G(\eta) = \frac{2g\sigma}{g'}. \tag{39}$$

If we choose

$$\sigma + \beta G = G', \tag{40}$$

then (39) admits a first integral

$$\left(\frac{1}{2}G^2W^2 + G'\xi W + \gamma(\eta)G^4 W_{\xi\xi}\right)_\xi + G W_\eta = 0. \tag{41}$$

Further if we assume that $G = c$ where c is a constant, then (41) reduces to

$$W_\eta + cWW_\xi + c^3\gamma(\eta)W_{\xi\xi\xi} = 0. \tag{42}$$

which is a variable coefficient K-dV equation. We note that a generalized version of (42) in the form

$$u_t + u^n u_x + \alpha(t)u + \beta(t)u_{xxx} = 0, \tag{43}$$

has recently been studied for its symmetry group and similarity solution by Senthilkumaran, Pandiaraja and Mayil Vaganan [13]. Equation (42) is a special case of (43) if α is a constant.

The two conditions $G(\eta) = 2g\sigma/g'$ and $G = c$ lead to the determination of $g(t)$ and $\beta(t)$ in terms of $\sigma(t)$

$$g(t) = g_0 e^{\frac{2}{c}\int \sigma(t)} dt, \quad \beta(t) = -\frac{1}{c}\sigma(t). \tag{44}$$

5.3 Subalgebra $L_{s,3} = \{T_0\}$
The change of variables $u = W(\xi, \eta),\ \xi = x,\ \eta = y$ replaces (9) by

$$(WW_\xi + \beta W + \gamma W_{\xi\xi\xi})_\xi + \sigma W_{\eta\eta} = 0. \tag{45}$$

5.4 Subalgebra $L_{s,4} = \{D_\lambda\}$

Insertion of $u = W(\xi, \eta), \xi = x,\ \eta = ye^{-\frac{\lambda}{2}t}$ into (9) changes the latter to

$$\left(-\frac{\lambda}{2}\eta W_\eta + WW_\xi + \beta W + \gamma W_{\xi\xi\xi}\right)_\xi + W_{\eta\eta} = 0. \tag{46}$$

5.5 Subalgebra $L_{s,5} = \{E_\lambda\}$

Under the transformation $u = e^{\lambda t/3} W(\xi, \eta),\ \xi = x e^{-\lambda t/3},\ \eta = y e^{-\lambda t/6}$, (9) becomes

$$-\frac{\lambda}{3}\xi W_{\xi\xi} - \frac{\lambda}{6}\eta W_{\eta\xi} + W_{\xi}^2 + W W_{\xi\xi} + \beta W_{\xi} + W_{\xi\xi\xi\xi} + \sigma W_{\eta\eta} = 0. \qquad (47)$$

6. Reduction to ODEs

We shall now reduce the PDEs (45), (46), (47) to ODEs by imbedding T_0, D_λ and E_λ into two dimensional subalgebras of the the symmetry algebra of the GKPE. For, we commute T_0, D_λ and E_λ with $V = X(f) + Y(g)$ and require that they form a two-dimensional subalgebra. As a consequence, the function $f(t)$ and $g(t)$ get defined in terms of t. As there are two isomorphy classes of two-dimensional Lie algebras,namely, Abelian and non-Abelian,we shall take this fact into account in the foregoing analysis.

6.1 Abelian Subalgebras

6.1.1 Abelian Subalgebra. $L_{a,1} = \{T_0, X(1) + Y(1)\}$

Now we reduce the PDE (45) to an ODE by imbedding T_0 into two-dimensional Abelian subalgebra $L_{a,1}$ of the the symmetry algebra of the GKPE (9). Indeed, the transformation $W = H(\rho), \rho = \xi - \eta$ replaces (45) by the third order ODE

$$HH' + \beta H + \gamma H''' - \sigma H' = 0. \qquad (48)$$

6.1.2 Abelian Subalgebra $L_{a,2} = \{D_\lambda, X(1)\}$

Now we reduce the PDE (46) through the transformation $W = H(\rho),\ \rho = \eta$ to

$$H'' = 0. \qquad (49)$$

6.1.3 Abelian Subalgebra $L_{a,3} = \left\{E_\lambda, X(e^{\frac{\lambda}{3}t}) + Y(e^{\frac{\lambda}{6}t} f)\right\}$

Now we reduce the PDE (47) to a ODE by imbedding E_λ into two-dimensional Abelian subalgebra $L_{a,3}$ of the the symmetry algebra of the GKPE. The transformation

$$W = \frac{\lambda}{3}\eta - \frac{\lambda^2}{144\sigma}\eta^2 + H(\rho), \quad \rho = \xi - \eta + \frac{\lambda}{24\sigma}\eta^2 \qquad (50)$$

reduces (47) to the fourth order ODE

$$H^{iv} + HH'' + (-\frac{\lambda}{3}\rho + \sigma)H'' + H'^2 + (\beta + \frac{\lambda}{12})H' - \frac{\lambda^2}{72} = 0 \qquad (51)$$

Integrating (51) with respect to ρ, we get

$$H''' + HH' + \sigma H' - \frac{\lambda}{3}\rho H' + (\beta + \frac{5\lambda}{12})H - \frac{\lambda^2\rho}{72} = c_1 \qquad (52)$$

Equation (52) can again be integrated to

$$H'' + \frac{H^2}{2} + \sigma H - \frac{\lambda}{3}\rho H - \frac{\lambda^2}{144}\rho^2 + c_1\rho + c_2 = 0, \quad \text{if} \quad \beta = -\frac{3\lambda}{4}. \tag{53}$$

6.2 Non-Abelian Subalgebras

6.2.1 Non Abelian Subalgebra $L_{n,1} = \{T_0, X(e^t) + Y(e^t)\}$

Now we reduce the PDE (45) to an ODE by imbedding T_0 into two dimensional non-Abelian subalgebra $L_{n,1}$ of the the symmetry algebra of the GKPE (9).

Invariance under the two dimensional subalgebra $L_{n,1}$ gives

$$W = H(\rho) + \xi, \quad \rho = \xi - \eta + \frac{\eta^2}{4\sigma}, \tag{54}$$

where $H(\rho)$ satisfies the fourth order ODE

$$H^{(iv)} + H''\rho + \sigma H'' + H'^2 + HH'' + \left(\beta + \frac{3}{2}\right)H' + (1+\beta) = 0. \tag{55}$$

Integration of (55) results in

$$H''' + H'\rho + \sigma H' + HH' + \left(\beta + \frac{3}{2}\right)H + (1+\beta)\rho = c_1, \tag{56}$$

which under the condition $\beta = -\frac{1}{2}$, changes to

$$H'' + \rho H + \sigma H + \frac{H^2}{2} + \frac{1}{4}\rho^2 + c_1\rho + c_2 = 0. \tag{57}$$

6.2.2 Non-Abelian Subalgebra $L_{n,2} = \{D_\lambda, X(e^t)\}$

Now under $W = \xi + H(\rho),\ \rho = \eta$ changes to $H'' + (1+\beta) = 0$.

6.2.3 Non-Abelian Subalgebra $L_{n,3} = \left\{E_\lambda,\ X(e^{(1+\frac{\lambda}{3})t}) + Y(e^{(1+\frac{\lambda}{6})t})\right\}$

Now we reduce the PDE (47) to a ODE by imbedding E_λ into two-dimensional Abelian subalgebra $L_{n,3}$ of the the symmetry algebra of the GKPE (9). Equation (47), under the similarity transformation

$$W = \frac{3+\lambda}{3}\eta - \frac{6+\lambda^2}{144\sigma}\eta^2 + H(\rho), \quad \rho = \xi - \eta + \frac{6+\lambda}{24\sigma}\eta^2, \tag{58}$$

reduces to

$$H''' + HH' - \frac{\lambda}{3}\rho H' + H' + (\beta + \frac{\lambda}{3})H - \frac{(6+\lambda)^2}{72}\rho = c_1, \tag{59}$$

If $\beta = -\frac{2\lambda}{3}$, then (59) can be integrated to yield

$$H'' + \frac{\lambda}{2}H^2 - \frac{\lambda}{3}\rho H + H - \frac{(6+\lambda)^2}{144}\rho^2 + c_1\rho + c_2 = 0. \tag{60}$$

7. The general form of reductions of GKPE (9)

The transformation $H(\rho) = f^{-1}(\rho)$ replaces the ODEs (53), (57) and (60), respectively, by

$$ff'' - 2f'^2 - \frac{1}{2}f - \sigma f^2 + \frac{\lambda}{3}\rho f^2 + \left(\frac{\lambda^2}{144}\rho^2 - c_1\rho - c_2\right) f^3 = 0, \tag{61}$$

$$ff'' - 2f'^2 - \frac{1}{2}f - (\rho + \sigma) f^2 + \left(\frac{1}{4}\rho^2 + c_1\rho + c_2\right) f^3 = 0 \tag{62}$$

$$ff'' - 2f'^2 - \frac{\lambda}{2}f - \left(\frac{\lambda}{3}\rho - 1\right) f^2 + \left(\frac{(\lambda+6)^2}{144} - c_1\rho - c_2\right) f^3 = 0. \tag{63}$$

We may write the general form of the equations (61), (62), (63) as

$$ff'' + af'^2 + bf + g(\rho)f^2 + h(\rho)f^3 = 0. \tag{64}$$

which is a special case of the equation introduced by Mayil Vaganan and Senthilkumaran [11], viz.,

$$ff'' + a(\rho)f'^2 + b(\rho)ff' + c(\rho)f^2 + d(\rho)f' + g(\rho)f^3 + kf = 0. \tag{65}$$

8.Conclusions

We now summarize the results of the present work,below:

As emphasized by David, Karman, Levi and Winternitz [1] and Güngör [2] that it is of great interest to identify all nonolinear PDEs that admit infinite-dimensional symmetry groups and Lie algebras containing arbitrary functions.

In this paper we have shown that the GKPE (9)is one such equation. When all the four functions $\beta(t), \gamma(t)$ and $\sigma(t)$ are kept arbitrary. The GKPE (9) is shown to admit an infinite-dimensional symmetry group with a Lie algebra L_p involving two arbitraray functions $f(t)$ and $g(t)$. Further we extend the Lie algebra L_p into four Lie algebras $L_i, i = 1, 2, 3, 4$ by taking σ, γ to be equal to $e^{\lambda t}$.

The classification of one-dimensional subalgebras of the symmetry algebra under the adjoint action of the symmetry group is carried out. Then by commuting $T_0, D_\lambda, E_\lambda$ with $V = X(f)+Y(f)$ two-dimensional subalgebras are constructed.

The GKPE (9) is also shown to reduce to a linear PDE of the form $W_{yy} = F(f(t), f'(t))$ (cf.(49)), a variable coefficient-KdV equation (42).

The reduction of the GKPE (9) into ODEs (61), (62), (63) under Abelian subalgebras and non-Abelian subalgebras are of the form (64).

We also have found a new solution (37) of (9) involving two arbitrary functions.

A rigorous analysis of the equation (64) or its generalized version (65) is yet to be studied.

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