Hodograph solutions of the dispersionless coupled KdV hierarchies, critical points and the Euler-Poisson-Darboux equation

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Abstract

It is shown that the hodograph solutions of the dispersionless coupled KdV (dcKdV) hierarchies describe critical and degenerate critical points of a scalar function which obeys the Euler-Poisson-Darboux equation. Singular sectors of each dcKdV hierarchy are found to be described by solutions of higher genus dcKdV hierarchies. Concrete solutions exhibiting shock type singularities are presented.

Key words: Integrable systems. Hodograph equations. Euler-Poisson-Darboux equation.
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1 Introduction

In the present paper we study hierarchies of hydrodynamical systems describing quasiclassical deformations of hyperelliptic curves [1, 2]

$$p^2 = u(\lambda), \quad u(\lambda) := \lambda^m - \sum_{i=0}^{m-1} \lambda^i\, u_i, \quad m \geq 1. \tag{1}$$

These hierarchies are of interest for several reasons. First, there are hierarchies of important hydrodynamical type systems among them. For $m = 1$ one has the Burgers-Hopf hierarchy [3, 4] associated with the dispersionless KdV equation $u_t = \dfrac{3}{2}\, u\, u_x$. For $m = 2$ it is the hierarchy of higher equations for the 1-layer Benney system (classsical long wave equation)

$$\begin{cases} u_t + u\, u_x + v_x = 0 \\ \\ v_t + (u\, v)_x = 0. \end{cases} \tag{2}$$

The system (2) and the corresponding hierarchy are quasiclassical limits of the nonlinear Schrödinger (NLS) equation and the NLS hierarchy [5]. For $m \geq 3$ these hierarchies turn to describe the singular sectors of the above $m = 1, 2$ hierarchies [1].

Second, all these hierarchies are the dispersionless limits of integrable coupled KdV (cKdV) hierarchies [6]-[8] associated to Schrödinger spectral problems

$$\partial_{xx}\, \psi = v(\lambda, x)\, \psi, \tag{3}$$

with potentials which are polynomials in the spectral parameter λ

$$v(\lambda, x) := \lambda^m - \sum_{i=0}^{m-1} \lambda^i\, v_i(x) \quad m \geq 1,$$

The cKdV hierarchies have been studied in [6]-[8], they have bi-Hamiltonian structures and, as a consequence of this property, the dispersionless expansions of their solutions possess interesting features such as the quasi-triviality property [9]-[10]. Moreover, the cKdV hierarchies arise also in the study of the singular sectors of the KdV and AKNS hierarchies [11, 12]. Henceforth we will refer to the hierarchies of hydrodynamical systems associated with the curves (1) for a fixed m as the m-th dispersionless coupled KdV (dcKdV_m) hierarchies. The Hamiltonian structures of the dcKdV_m hierarchies have been studied in [13]. At last, it should be noticed that the dcKdV_m hierarchies are closely connected with the higher genus Whitham hierarchies introduced in [14].

In our analysis of the hodograph equations for the dcKdV_m hierarchies we use Riemann invariants β_i (roots of the polynomial $u(\lambda)$ in (1)) which provide a specially convenient system of coordinates. We show that the dcKdV_m hodograph equations have the form

$$\frac{\partial W_m(\boldsymbol{t}, \boldsymbol{\beta})}{\partial \beta_i} = 0, \quad i = 1, \ldots, m, \tag{4}$$

where $\boldsymbol{t} = (t_1, t_2, \ldots)$ are times of the hierarchy and

$$W_m(\boldsymbol{t}, \boldsymbol{\beta}) := \oint_\gamma \frac{\mathrm{d}\lambda}{2\, i\, \pi}\, \frac{\sum_{n\geq 0} t_n\, \lambda^n}{\sqrt{\prod_{i=1}^m (1 - \beta_i/\lambda)}}. \tag{5}$$

Here γ denotes a large positively oriented circle $|\lambda| = r$. Thus, the hodograph solutions of the dcKdV_m hierarchies describe critical points of the functions $W_m(\boldsymbol{t}, \boldsymbol{\beta})$. These functions turn to be very special as they satisfy a well-known system of equations in differential geometry: the Euler-Poisson-Darboux (EPD) equations [15]

$$2\,(\beta_i - \beta_j)\,\frac{\partial^2 W_m}{\partial\beta_i\,\partial\beta_j} = \frac{\partial W_m}{\partial\beta_i} - \frac{\partial W_m}{\partial\beta_j}. \tag{6}$$

The system (6) has also appeared in the theory of the Whitham equations arising in the small dispersion limit of the KdV equations [17]-[19], and in the theory of hydrodynamic chains [20].

We also study the singular sectors $\mathcal{M}_m^{\mathrm{sing}}$ of the spaces of hodograph solutions for the dcKdV_m hierarchies. They are given by the points $(\boldsymbol{t}, \boldsymbol{\beta})$ such that

$$\mathrm{rank}\Big(\frac{\partial^2 W_m(\boldsymbol{t}, \boldsymbol{\beta})}{\partial\beta_i\,\partial\beta_j}\Big) < m. \tag{7}$$

The varieties $\mathcal{M}_m^{\mathrm{sing}}$ provide us with special classes of degenerate critical points of the function W_m within the general theory of critical points developed by V. I. Arnold and others about fourty years ago [23, 24]. The use of equations (4)-(6) simplify drastically the analysis of the structure of these singular sectors. In particular, we prove that there is a nested sequence of subvarieties

$$\mathcal{M}_m^{\mathrm{sing}} \supset \mathcal{M}_{m,1}^{\mathrm{sing}} \supset \mathcal{M}_{m,2}^{\mathrm{sing}} \supset \cdots \mathcal{M}_{m,q}^{\mathrm{sing}} \supset \cdots, \tag{8}$$

which represents subsets of the singular sector $\mathcal{M}_m^{\mathrm{sing}}$ of the dcKdV_m hierarchy with increasing singular degree q, such that each $\mathcal{M}_{m,q}^{\mathrm{sing}}$ is determined by a class of hodograph solutions of the $\mathrm{dcKdV}_{m+2\,q}$ hierarchy.

The paper is organized as follows. The dcKdV_m hierarchies are described in Section 2. Equations (4)-(6) are derived in Section 3. Section 4 deals with the analysis of the singular sectors of the dcKdV_m hierarchies in terms of their associated hodograph equations. The relation between singular points of the dcKdV_m hodograph equations and solutions of higher $\mathrm{dcKdV}_{m+2\,q}$ hodograph equations is stated in Section 4. Some concrete examples involving shock singularities of the Burgers-Hopf equation and the 1-layer Benney system are presented in Section 5.

2 The dcKdV_m hierarchies

Given a positive integer $m \geq 1$ we consider the set M_m of algebraic curves (1). For $m = 2\,g+1$ (odd case) and $m = 2\,g+2$ (even case) these curves are, generically, hyperelliptic Riemann surfaces of genus g. We will denote by $\boldsymbol{q} = (q_1, \ldots, q_m)$ any of the two sets of parameters $\boldsymbol{u} := (u_0, \ldots, u_{m-1})$ or $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_m)$ which determine the curves (1)

$$u(\lambda) = \lambda^m - \sum_{i=0}^{m-1} \lambda^i\, u_i = \prod_{i=1}^{m} (\lambda - \beta_i). \tag{9}$$

Obviously, for any fixed $\boldsymbol{\beta}$ all the permutations $\sigma(\boldsymbol{\beta}) := (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(m)})$ represent the same element of M_m. Note also that

$$u_i = (-1)^{m-i-1} \mathrm{s}_{m-i}(\boldsymbol{\beta}), \tag{10}$$

where s_k are the elementary symmetric polynomials

$$\mathrm{s}_k = \sum_{1\leq i_1<\ldots<i_k\leq m} \beta_{i_1}\cdots\beta_{i_k}.$$

We next introduce the dcKdV$_m$ hierarchy as a particular systems of commuting flows

$$\boldsymbol{q}(\boldsymbol{t}),\quad \boldsymbol{t}:=(x:=t_0,t_1,t_2,\ldots),$$

on M_m. In order to define these flows we use the set $\mathcal{L}$ of formal power series

$$f(z)=\sum_{n=-\infty}^{+\infty} c_n\, z^n,$$

where

$$z:=\lambda^{1/2}\text{ for } m=2\,g+1;\quad z:=\lambda \text{ for } m=2\,g+2.$$

For any given $m\geq 1$ a distinguished element of $\mathcal{L}$ is provided by the branch of $p=\sqrt{u(\lambda)}$ such that as $z\to\infty$ has an expansion of the form

$$\begin{cases} p(z,\boldsymbol{q})=z^{2\,g+1}\Big(1+\sum_{n\geq1}\dfrac{b_n(\boldsymbol{q})}{z^{2\,n}}\Big), & m=2\,g+1,\\ \\ p(z,\boldsymbol{q})=z^{g+1}\Big(1+\sum_{n\geq1}\dfrac{b_n(\boldsymbol{q})}{z^{n}}\Big), & m=2\,g+2.\end{cases}\tag{11}$$

We define the following splittings $\mathcal{L}=\mathcal{L}_{(+,\boldsymbol{q})}\bigoplus\mathcal{L}_{(-,\boldsymbol{q})}$

$$f_{(+,\boldsymbol{q})}(z):=\Big(\frac{f(z)}{p(z,\boldsymbol{q})}\Big)_{\oplus}p(z,\boldsymbol{q}),\quad f_{(-,\boldsymbol{q})}(z):=\Big(\frac{f(z)}{p(z,\boldsymbol{q})}\Big)_{\ominus}p(z,\boldsymbol{q}),\tag{12}$$

where $f_\oplus$ and $f_\ominus$ stand for the standard projections on positive and strictly negative powers of z, respectively

$$f_\oplus(z):=\sum_{n=0}^{N}c_n\,z^n,\quad f_\ominus(z):=\sum_{n=-\infty}^{-1}c_n\,z^n.$$

The dcKdV$_m$ flows $\boldsymbol{q}(\boldsymbol{t})$ are characterized by the following condition: There exists a family of functions $S(z,\boldsymbol{t},\boldsymbol{q}(\boldsymbol{t}))$ in $\mathcal{L}$ satisfying

$$\partial_{t_n}S(z,\boldsymbol{t},\boldsymbol{q}(\boldsymbol{t}))=\Omega_n(z,\boldsymbol{q}(\boldsymbol{t})),\quad n\geq 0.\tag{13}$$

where

$$\Omega_n(z,\boldsymbol{q}):=(\lambda(z)^{n+m/2})_{(+,\boldsymbol{q})}=\begin{cases}(z^{2\,n+2\,g+1})_{(+,\boldsymbol{q})}, & m=2\,g+1\\ \\ (z^{n+g+1})_{(+,\boldsymbol{q})}, & m=2\,g+2,\end{cases}\qquad n\geq 0.\tag{14}$$

We notice that

$$\Omega_n(z,\boldsymbol{q})=\Big(\lambda^n\,R(\lambda(z),\boldsymbol{q})\Big)_{\oplus}p.\tag{15}$$

where R is the generating function

$$R(\lambda, \boldsymbol{q}) := \sqrt{\frac{\lambda^m}{u(\lambda)}} = \sum_{n\geq 0} \frac{R_n(\boldsymbol{q})}{\lambda^n}, \quad \lambda \to \infty. \tag{16}$$

The coefficients $R_n(\boldsymbol{q})$ are polynomials in the coordinates $\boldsymbol{q}$, for example

$$R_0 = 1, \quad R_1 = \frac{1}{2}\, u_{m-1}, \quad R_2 = \frac{1}{2}\, u_{m-2} + \frac{3}{8}\, u_{m-1}^2, \quad \ldots$$

Functions S which satisfy (13) will be referred to as *action functions* of the dcKdV$_m$ hierarchy. This kind of generating functions S has been already used in the theory of dispersionless integrable systems (see e.g. [14]). It can be proved [1] that (13) is a compatible system of equations for S. In fact its general solution will be determined in the next section. We notice that for $n = 0$ the equation (13) reads

$$\partial_x\, S(z, \boldsymbol{t}, \boldsymbol{q}(\boldsymbol{t})) = p(z, \boldsymbol{q}(\boldsymbol{t})), \tag{17}$$

so that (13) is equivalent to the system

$$\partial_{t_n} p(z, \boldsymbol{q}(t)) = \partial_x\, \Omega_n(z, \boldsymbol{q}(\boldsymbol{t})), \quad n \geq 0. \tag{18}$$

We will henceforth refer to the dcKdV$_m$ hierarchy for $m = 2\,g{+}1$ and $m = 2\,g{+}2$ as the Burgers-Hopf (BH$_g$) and the dispersionless Jaulent-Miodek (dJM$_g$) hierarchies, respectively. Observe that both hierarchies, BH$_g$ and dJM$_g$ determine deformations of hyperelliptic Riemann surfaces of genus g. In our work we will always consider an arbitrary but finite number of these flows.

Since $u = u(\lambda(z), \boldsymbol{q}) = p(z, \boldsymbol{q})^2$, the operator $J = J(\lambda, u)$ defined by

$$J := 2\, p \cdot \partial_x \cdot p = 2\, u\, \partial_x + u_x,$$

$$J = \sum_{i=0}^{m} \lambda^i\, J_i, \quad J_m = 2\, \partial_x, \quad J_i = -(2\, u_i\, \partial_x + u_{i,x}), \quad u_m := -1,$$

satisfies $J\, R = 0$. Then from (18) it follows that

$$\partial_n\, u = J\left(\lambda^n\, R(\lambda, \boldsymbol{u})\right)_{\oplus} = -J\left(\lambda^n\, R(\lambda, \boldsymbol{u})\right)_{\ominus}, \tag{19}$$

which constitutes the dcKdV$_m$ hierarchy in terms of the coordinates u_i

$$\partial_n\, u_i = \sum_{l-k=i,\, k\geq 1} J_l\, R_{n+k}(\boldsymbol{u}), \quad i = 0, \ldots, m-1. \tag{20}$$

From (18) it also follows that

$$\partial_{t_n}\, \log p(z, \boldsymbol{q}) = \frac{\partial_x \left[\left(\lambda(z)^n\, R(\lambda(z), \boldsymbol{q})\right)_{\oplus} p\right]}{p(z, \boldsymbol{q})},$$

and then, identifying the residues of both sides at $\lambda = \beta_i$, we get

$$\partial_n\, \beta_i = \omega_{n,i}(\boldsymbol{\beta})\, \partial_x\, \beta_i, \quad i = 1, \ldots, m, \tag{21}$$

where

$$\omega_{n,i}(\boldsymbol{\beta}) := (\lambda^n\, R(\lambda, \boldsymbol{\beta}))_{\oplus}|_{\lambda=\beta_i}. \tag{22}$$

The systems (21) are the equations of the dcKdV_m hierarchy in terms of the coordinates β_i. Observe that we have two dcKdV_m hierarchies, BH_g and dJM_g, which determine deformations of hyperelliptic Riemann surfaces of genus g. It can be shown [2, 13] that the dcKdV_m flows are bi-Hamiltonian systems.

We next present some examples of interesting flows in the dcKdV_m hierarchies. The dcKdV_1 hierarchy is associated to the curve

$$p^2 - u(\lambda) = 0, \quad u(\lambda) = \lambda - v, \quad v := u_0 = \beta_1.$$

The corresponding flows are given by

$$\partial_{t_n} v = c_n\, v^n\, v_x, \quad c_n := \frac{(2\,n+1)!!}{2^n\, n!}, \quad n \geq 1,$$

and constitute the Burgers-Hopf hierarchy BH_0. In particular the t_1-flow is the Burgers-Hopf equation

$$\partial_t\, v = \frac{3}{2}\, v\, v_x,$$

which is in turn the dispersionless limit of the KdV equation.

The dcKdV_2 (dJM_0) hierarchy is associated to the curve

$$p^2 - u(\lambda) = 0, \quad u(\lambda) = \lambda^2 - \lambda\, u_1 - u_0 = (\lambda - \beta_1)\,(\lambda - \beta_2),$$

$$u_1 = \beta_1 + \beta_2, \quad u_0 = -\beta_1\, \beta_2.$$

The t_1-flow of this hierarchy is given by the disperssionless Jaulent-Miodek system

$$\begin{cases} \partial_{t_1}\, u_0 = u_0\, u_{1\,x} + \dfrac{1}{2}\, u_1\, u_{0\,x}, \\ \\ \partial_{t_1}\, u_1 = u_{0\,x} + \dfrac{3}{2}\, u_1\, u_{1\,x}, \end{cases} \tag{23}$$

which under the changes of dependent variables

$$u = -u_1, \quad v = u_0 + \frac{u_1^2}{4},$$

becomes the 1-layer Benney system (2). In terms of the Riemann invariants β_1 and β_2

$$u = -(\beta_1 + \beta_2), \quad v = (\beta_1 - \beta_2)^2/4,$$

the system (2) takes the well-known form

$$\begin{cases} \partial_{t_1}\, \beta_1 = \dfrac{1}{2}\,(3\,\beta_1 + \beta_2)\, \beta_{1\,x}, \\ \\ \partial_{t_1}\, \beta_2 = \dfrac{1}{2}\,(3\,\beta_2 + \beta_1)\, \beta_{2\,x}. \end{cases} \tag{24}$$

For $v > 0$ the 1-layer Benney system is hyperbolic while for $v < 0$ it is elliptic.

Finally, we consider the BH_1 hierarchy. Its associated curve is given by

$$p^2 - u(\lambda) = 0, \quad u(\lambda) = \lambda^3 - \lambda^2 u_2 - \lambda u_1 - u_0 = (\lambda - \beta_1)(\lambda - \beta_2)(\lambda - \beta_3),$$

$$u_1 = \beta_1 + \beta_2 + \beta_3, \quad u_2 = -(\beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3), \quad u_3 = \beta_1 \beta_2 \beta_3.$$

The first flow takes the forms

$$\begin{cases} \partial_{t_1} u_0 = \dfrac{1}{2} u_2 u_{0x} + u_0 u_{2x}, \\ \partial_{t_1} u_1 = u_{0x} + \dfrac{1}{2} u_2 u_{1x} + u_1 u_{2x}, \\ \partial_{t_1} u_2 = u_{1x} + \dfrac{3}{2} u_2 u_{2x}. \end{cases} \Longleftrightarrow \begin{cases} \partial_{t_1} \beta_1 = \dfrac{1}{2} (3\beta_1 + \beta_2 + \beta_3) \beta_{1x}, \\ \partial_{t_1} \beta_2 = \dfrac{1}{2} (\beta_1 + 3\beta_2 + \beta_3) \beta_{2x}, \\ \partial_{t_1} \beta_3 = \dfrac{1}{2} (\beta_1 + \beta_2 + 3\beta_3) \beta_{3x}. \end{cases} \tag{25}$$

3 Hodograph equations for dcKdV_m hierarchies and the Euler-Poisson-Darboux equation

Let us introduce the function

$$W_m(\boldsymbol{t}, \boldsymbol{q}) := \oint_\gamma \frac{\mathrm{d}\lambda}{2 i \pi} U(\lambda, \boldsymbol{t}) R(\lambda, \boldsymbol{q}) = \sum_{n \geq 0} t_n R_{n+1}(\boldsymbol{q}), \tag{26}$$

where γ denotes a large positively oriented circle $|\lambda| = r$, $U(\lambda, \boldsymbol{t}) := \sum_{n \geq 0} t_n \lambda^n$ and $R(\lambda, \boldsymbol{q})$ is the function defined in (16).

Theorem 1. *If the functions $\boldsymbol{q}(\boldsymbol{t}) = (q_1(\boldsymbol{t}, \ldots, q_m(\boldsymbol{t}))$ satisfy the system of hodograph equations*

$$\frac{\partial W_m(\boldsymbol{t}, \boldsymbol{q})}{\partial q_i} = 0, \quad i = 1, \ldots, m, \tag{27}$$

then $\boldsymbol{q}(\boldsymbol{t})$ is a solution of the $dcKdV_m$ hierarchy.

Proof. We are going to prove that the function

$$S(z, \boldsymbol{t}, \boldsymbol{q}(\boldsymbol{t})) = \sum_{n \geq 0} t_n \Omega_n(z, \boldsymbol{q}(\boldsymbol{t})) = \Big(U(\lambda(z), \boldsymbol{t}) R(\lambda(z), \boldsymbol{q}(\boldsymbol{t})) \Big)_\oplus p(z, \boldsymbol{q}(\boldsymbol{t})), \tag{28}$$

is an action function for the dcKdV_m hierarchy. By differentiating (28) with respect to t_n we have that

$$\partial_n S = \Omega_n + (U \partial_n R)_\oplus p + (U R)_\oplus \partial_n p, \tag{29}$$

We now use the coordinates $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$ so that we may take advantage of the identities

$$\partial_{\beta_i} p = -\frac{1}{2} \frac{p}{\lambda - \beta_i}, \quad \partial_{\beta_i} R = \frac{1}{2} \frac{R}{\lambda - \beta_i}. \tag{30}$$

Thus we deduce that

$$(U\,\partial_n\,R)_\oplus\,p+(U\,R)_\oplus\,\partial_n\,p=\frac{1}{2}\sum_{i=1}^{m}\Big[\Big(\frac{U\,R}{\lambda-\beta_i}\Big)_\oplus-\frac{(U\,R)_\oplus}{\lambda-\beta_i}\Big]\,p\,\partial_n\,\beta_i. \tag{31}$$

On the other hand

$$\frac{\partial W_m(\boldsymbol{t},\boldsymbol{\beta})}{\partial\beta_i}=\frac{1}{2}\oint_\gamma\frac{\mathrm{d}\lambda}{2\,i\,\pi}\,\frac{U(\lambda,\boldsymbol{t})\,R(\lambda,\boldsymbol{\beta})}{\lambda-\beta_i}=\frac{1}{2}\oint_\gamma\frac{\mathrm{d}\lambda}{2\,i\,\pi}\,\frac{(U(\lambda,\boldsymbol{t})\,R(\lambda,\boldsymbol{\beta}))_\oplus}{\lambda-\beta_i}. \tag{32}$$

Hence the hodograph equations (27) can be written as

$$(U(\lambda,\boldsymbol{t})\,R(\lambda,\boldsymbol{\beta}(\boldsymbol{t})))_\oplus|_{\lambda=\beta_i}=0,\quad i=1,\dots,m. \tag{33}$$

Thus we have that $(U(\lambda,\boldsymbol{t})\,R(\lambda,\boldsymbol{\beta}(\boldsymbol{t}))_\oplus$ is a polynomial in λ which vanish at $\lambda=\beta_i(\boldsymbol{t})$ for all i. As a consequence

$$\frac{(U\,R)_\oplus}{\lambda-\beta_i}=\Big(\frac{(U\,R)_\oplus}{\lambda-\beta_i}\Big)_\oplus=\Big(\frac{U\,R}{\lambda-\beta_i}\Big)_\oplus.$$

Then from (29) and (31) we deduce that $\partial_n\,S=\Omega_n$ and therefore the statement follows.

□

Using (26) we obtain that the hodograph equations (27) can be expressed as

$$\sum_{n\geq 0}t_n\,\frac{\partial R_{n+1}(\boldsymbol{q})}{\partial q_i}=0,\quad i=1,\dots,m. \tag{34}$$

Furthermore, from (21), (22) and (33) the hodograph equations (27) can be also written as [1]

$$\sum_{n\geq 0}t_n\,\omega_{n,i}(\boldsymbol{\beta})=0,\quad i=1,\dots,m, \tag{35}$$

which represent the hodograph transform for the dcKdV$_m$ hierarchy of flows in hydrodynamic form.

Notice also that we may shift the time parameters $t_n\to t_n-c_n$ in (34) to get solutions depending on an arbitrary number of constants.

It is easy to see that the generating function

$$R(\lambda,\boldsymbol{\beta}):=\sqrt{\frac{\lambda^m}{u(\lambda)}}=\sqrt{\frac{\lambda^m}{\prod_{i=1}^m(\lambda-\beta_i)}},$$

is a symmetric solution of the EPD equation

$$2\,(\beta_i-\beta_j)\,\frac{\partial^2 R}{\partial\beta_i\,\partial\beta_j}=\frac{\partial R}{\partial\beta_i}-\frac{\partial R}{\partial\beta_j}. \tag{36}$$

Consequently, the same property is satisfied by $W(\boldsymbol{t},\boldsymbol{\beta})$ for all $\boldsymbol{t}$. Thus, we have proved

Theorem 2. *The solutions* $(\boldsymbol{t},\boldsymbol{\beta})$ *of the hodograph equations*

$$\frac{\partial W_m(\boldsymbol{t},\boldsymbol{\beta})}{\partial \beta_i}=0,\quad i=1,\dots,m, \tag{37}$$

are the critical points of the solution

$$W_m(\boldsymbol{t},\boldsymbol{\beta}):=\oint_\gamma \frac{\mathrm{d}\lambda}{2\,i\,\pi}\,\frac{U(\lambda,\boldsymbol{t})}{\sqrt{\prod_{i=1}^m(1-\beta_i/\lambda)}}$$

of the EPD equation

$$2\,(\beta_i-\beta_j)\,\frac{\partial^2 W_m}{\partial\beta_i\,\partial\beta_j}=\frac{\partial W_m}{\partial\beta_i}-\frac{\partial W_m}{\partial\beta_j}. \tag{38}$$

Let us denote by $\mathcal{M}_m$ the *variety* of points $(\boldsymbol{t},\boldsymbol{\beta})\in\mathbb{C}^\infty\times\mathbb{C}^m$ which satisfy the hodograph equations (37). From (32) it is clear that for any permutation σ of $\{1,\dots,m\}$ the functions

$$F_i(\boldsymbol{t},\boldsymbol{\beta}):=\frac{\partial W_m(\boldsymbol{t},\boldsymbol{\beta})}{\partial\beta_i}, \tag{39}$$

satisfy

$$F_i(\boldsymbol{t},\sigma(\boldsymbol{\beta}))=F_{\sigma(i)}(\boldsymbol{t},\boldsymbol{\beta}). \tag{40}$$

Then, it is clear that $\mathcal{M}_m$ is invariant under the action of the group of permutations

$$(\boldsymbol{t},\boldsymbol{\beta})\in\mathcal{M}_m\Longrightarrow(\boldsymbol{t},\sigma(\boldsymbol{\beta}))\in\mathcal{M}_m.$$

If $(\boldsymbol{t},\boldsymbol{\beta})$ is a solution of (37) such that $\beta_i\neq\beta_j$ for all $i\neq j$ then it will be called an *unreduced* solution of (37). In this case the EPD equation (38) implies that

$$\frac{\partial^2 W_m(\boldsymbol{t},\boldsymbol{\beta})}{\partial\beta_i\,\partial\beta_j}=0,\quad \forall i\neq j. \tag{41}$$

Given $2\le r\le m$, a solution $(\boldsymbol{t},\boldsymbol{\beta})$ of (37) such that exactly r of its components are equal will be called a *r-reduced solution* of (37).

The formulation (27) of the hodograph equations for the dcKdV$_m$ hierarchies allows us to apply the theory of critical points of functions to analyze the solutions of these hierarchies, while (38) indicates that the functions W_m are of a very special class.

The EPD equation (38) arose in the study of cyclids [15], where solutions W of the above form have been found too. Much later it appeared in the theory of Whitham equations describing the small dispersion limit of the KdV equation [17, 19].

We note that hodograph equations of a form close to (27) have been presented in [20] and [22]. Furthermore, linear equations of the EPD type and their connection with hydrodynamic chains have been studied in [21] too.

Finally, we emphasize that the functions W_m depend on the parameters $t_1,t_2,\dots$ (times of the hierarchy). Since "degenerate critical points appear naturally in cases when the functions depend on parameters " [23, 24], one should expect the existence of families of degenerate critical points for the functions W_m. Their connection with the singular sectors in the spaces of solutions for dcKdV$_m$ will be considered in the next section.

To illustrate the statements given above we next present some simple examples. For the dcKdV$_2$ hierarchy we have

$$\begin{aligned} W_2(\boldsymbol{t},\boldsymbol{\beta}) = \frac{x}{2}(\beta_1+\beta_2) + \frac{t_1}{8}(3\beta_1^2+2\beta_1\beta_2+3\beta_2^2) + \frac{t_2}{16}\left(5\beta_1^3+3\beta_1^2\beta_2+3\beta_1\beta_2^2+5\beta_2^3\right) \\ + \frac{t_3}{128}(35\beta_1^4+20\beta_1^3\beta_2+18\beta_1^2\beta_2^2+20\beta_1\beta_2^3+35\beta_2^4)+\cdots \end{aligned}$$

The hodograph equations with $t_n = 0$ for $n \geq 4$, take the form

$$\begin{cases} 8x+4t_1(3\beta_1+\beta_2)+3t_2\left(5\beta_1^2+2\beta_1\beta_2+\beta_2^2\right)+\dfrac{t_3}{8}(140\beta_1^3+60\beta_1^2\beta_2+36\beta_1\beta_2^2+20\beta_2^3)=0, \\ \\ 8x+4t_1(\beta_1+3\beta_2)+3t_2\left(\beta_1^2+2\beta_1\beta_2+5\beta_2^2\right)+\dfrac{t_3}{8}(140\beta_2^3+60\beta_2^2\beta_1+36\beta_2\beta_1^2+20\beta_1^3)=0. \end{cases} \tag{42}$$

For the dcKdV$_3$ hierarchy we have

$$\begin{aligned} W_3(\boldsymbol{t},\boldsymbol{\beta}) = \frac{x}{2}(\beta_1+\beta_2+\beta_3) + \frac{t_1}{8}\left(3\beta_1^2+3\beta_2^2+3\beta_3^2+2\beta_1\beta_2+2\beta_1\beta_3+2\beta_2\beta_3\right) \\ + \frac{t_2}{16}\Big(5\beta_1^3++5\beta_2^3+5\beta_3^3+3\beta_1^2\beta_2+3\beta_1^2\beta_3+3\beta_1\beta_2^2+3\beta_2^2\beta_3+3\beta_1\beta_3^2 \\ +3\beta_2\beta_3^2+2\beta_1\beta_2\beta_2\Big)+\cdots \end{aligned}$$

The hodograph equations with $t_n = 0$ for $n \geq 3$ are

$$\begin{cases} 8\,x+4\,t_1\,(3\,\beta_1+\beta_2+\beta_3)+t_2\,(15\,\beta_1^2+3\,\beta_2^2+3\,\beta_3^2+6\,\beta_1\,\beta_2+6\,\beta_1\,\beta_3+2\,\beta_2\,\beta_3)=0, \\ \\ 8\,x+4\,t_1\,(\beta_1+3\,\beta_2+\beta_3)+t_2\,(3\,\beta_1^2+15\,\beta_2^2+3\,\beta_3^2+6\,\beta_1\,\beta_2+2\,\beta_1\,\beta_3+6\,\beta_2\,\beta_3)=0, \\ \\ 8\,x+4\,t_1\,(\beta_1+\beta_2+3\,\beta_3)+t_2\,(3\,\beta_1^2+3\,\beta_2^2+15\,\beta_3^2+2\,\beta_1\,\beta_2+6\,\beta_1\,\beta_3+6\,\beta_2\,\beta_3)\;=\;0. \end{cases} \tag{43}$$

4 Singular sectors of dcKdV$_m$ hierarchies

We say that $(\boldsymbol{t},\boldsymbol{\beta}) \in \mathcal{M}_m$ is a regular point if it is a nondegenerate critical point of the function W_m. That it is to say, if it satisfies [23, 24]

$$\det\Big(\frac{\partial^2 W_m(\boldsymbol{t},\boldsymbol{\beta})}{\partial\beta_i\,\partial\beta_j}\Big) \neq 0. \tag{44}$$

The set of regular points of $\mathcal{M}_m$ will be denoted by $\mathcal{M}_m^{\mathrm{reg}}$ and the points of its complementary set $\mathcal{M}_m^{\mathrm{sing}} := \mathcal{M}_m - \mathcal{M}_m^{\mathrm{reg}}$, where the second differential of W_m is a degenerate quadratic form, will be called singular points. We will also refer to $\mathcal{M}_m^{\mathrm{reg}}$ and $\mathcal{M}_m^{\mathrm{sing}}$ as the regular and singular sectors of the dcKdV$_m$ hierarchy. So $\mathcal{M}_m^{\mathrm{sing}}$ describes families of degenerate critical points of the function W_m. Near a regular point the variety $\mathcal{M}_m^{\mathrm{reg}}$ can be uniquely described as $(\boldsymbol{t}, \boldsymbol{\beta}(\boldsymbol{t}))$ where $\boldsymbol{\beta}(\boldsymbol{t})$ is a solution of the dcKdV$_m$ hierarchy.

The aim of this section is to analyze the structure of $\mathcal{M}_m^{\mathrm{sing}}$ by taking advantage of the special properties of the set of coordinates $\boldsymbol{\beta}$.

In general, the singular sectors of dcKdV$_m$ hierarchies with $m \geq 2$ contain both reduced and unreduced points. For example, the hodograph equations (42) of the dcKdV$_2$ hierarchy have reduced singular points given by $(x, t_1, t_2, t_3, \beta_1 = \beta_2)$ where

$$72 x t_3^2 = -9t_2^2 + 36 t_1 t_2 t_3 + (8t_1 t_3 - 3t_2^2)\sqrt{9t_2^2 - 24 t_1 t_3},$$

and

$$\beta_1 = \beta_2 = -\frac{3t_2 + \sqrt{9t_2^2 - 24t_1 t_3}}{12 t_3}.$$

Furthermore, there are also unreduced singular points $(x, t_1, t_2, t_3, \beta_1, \beta_2)$ determined by the constraint

$$360 x t_3^3 = -45 t_3 t_2^3 + 180 t_1 t_3^2 t_2 + \sqrt{15}\,(8t_1 t_3 - 3t_2^2)\,\sqrt{t_3^2\left(3t_2^2 - 8t_1 t_3\right)},$$

and

$$\beta_1 = \frac{-3t_2 t_3 + \sqrt{15}\sqrt{t_3^2\left(3t_2^2 - 8t_1 t_3\right)}}{12 t_3^2}, \quad \beta_2 = -\frac{5t_2 t_3 + \sqrt{15}\sqrt{t_3^2\left(3t_2^2 - 8t_1 t_3\right)}}{20 t_3^2}$$

From (41) it follows at once that

Theorem 3. *Let $(\boldsymbol{t}, \boldsymbol{\beta})$ be an unreduced solution of the hodograph equations* (37)*, then $(\boldsymbol{t}, \boldsymbol{\beta})$ is a singular point if and only if at least one of the derivatives*

$$\frac{\partial^2 W_m(\boldsymbol{t}, \boldsymbol{\beta})}{\partial \beta_i^2}, \quad i = 1, \ldots, m,$$

vanishes.

Notice that since the function W_m satisfies the EPD equation (38) , its partial derivatives at unreduced points $(\boldsymbol{t}, \boldsymbol{\beta})$

$$\frac{\partial^q W_m(\boldsymbol{t}, \boldsymbol{\beta})}{\partial \beta_1^{q_1} \cdots \partial \beta_m^{q_m}}, \quad q := q_1 + \cdots + q_m,$$

can always be expressed as a linear combination of *diagonal* derivatives $\partial_{\beta_i}^{k_i} W_m$ with $k_i \leq q_i$. Thus, for each vector $\boldsymbol{q} = (q_1, \ldots, q_m) \in \mathbb{N}^m$ with at least one $q_i \geq 1$ it is natural to introduce an associated subvariety $\mathcal{M}_{m,\boldsymbol{q}}^{\mathrm{sing}}$ of $\mathcal{M}_m^{\mathrm{sing}}$ defined as the set of unreduced solutions $(\boldsymbol{t}, \boldsymbol{\beta})$ of the hodograph equations (37) such that

$$\frac{\partial^{k_i} W_m(\boldsymbol{t}, \boldsymbol{\beta})}{\partial \beta_i^{k_i}} = 0, \quad \forall k_i \leq q_i + 1. \tag{45}$$

In particular, for $\boldsymbol{q} = (0, \ldots, 0, q)$ with $q \geq 1$ we denote by $\mathcal{M}_{m,q}^{\mathrm{sing}}$ the subvariety associated to $\boldsymbol{q} = (0, \ldots, 0, q)$. That is to say, $\mathcal{M}_{m,q}^{\mathrm{sing}}$ is the set of solutions $(\boldsymbol{t}, \boldsymbol{\beta})$ of the hodograph equations (37) such that

$$\frac{\partial^2 W_m(\boldsymbol{t}, \boldsymbol{\beta})}{\partial \beta_m^2} = \frac{\partial^3 W_m(\boldsymbol{t}, \boldsymbol{\beta})}{\partial \beta_m^3} = \ldots = \frac{\partial^{q+1} W_m(\boldsymbol{t}, \boldsymbol{\beta})}{\partial \beta_m^{q+1}} = 0. \tag{46}$$

These subvarieties define a nested sequence

$$\mathcal{M}_m^{\text{sing}} \supset \mathcal{M}_{m,1}^{\text{sing}} \supset \mathcal{M}_{m,2}^{\text{sing}} \supset \cdots \mathcal{M}_{m,q}^{\text{sing}} \supset \cdots, \tag{47}$$

and represent sets of points whose singular degree increases with q. Moreover, due to the covariance of the functions $F_i = \partial_{\beta_i} W_m$ under permutations there is no need of introducing alternative sequences of the form (46) based on systems of equations corresponding to the remaining coordinates β_j for $j \neq m$.

The next result states that the varieties $\mathcal{M}_{m,q}^{\text{sing}}$ of the dcKdV$_m$ hierarchy are closely related to the $(2\,q+1)$-reduced solutions of the dcKdV$_{m+2\,q}$ hierarchy.

Notice that given $2 \leq r \leq m$, the hodograph equations for r-reduced solutions

$$\beta_{m-r+1} = \beta_{m-r+2} = \ldots = \beta_m,$$

of the dcKdV$_m$ hierarchy reduce to the system

$$F_i(\boldsymbol{t}, \boldsymbol{\beta}) = 0, \quad i = 1, \ldots, m-r+1,$$

of $m-r+1$ equations for the $m-r+1$ unknowns $(\beta_1, \ldots, \beta_{m-r+1})$. Now we prove

Theorem 4. *If $(\boldsymbol{t}, \boldsymbol{\beta}) \in \mathcal{M}_{m,q}^{sing}$ where $\boldsymbol{t} = (t_0, t_1, \ldots)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$, then if we define*

$$\boldsymbol{t}^{(m+2\,q)} := (t_q, t_{q+1}, \ldots), \quad \boldsymbol{\beta}^{(m+2\,q)} := (\beta_1, \ldots, \beta_m, \overbrace{\beta_m, \ldots, \beta_m}^{2\,q}),$$

it follows that $(\boldsymbol{t}^{(m+2\,q)}, \boldsymbol{\beta}^{(m+2\,q)})$ is a $(2\,q+1)$-reduced solution of the hodograph equations for the dcKdV$_{m+2\,q}$ hierarchy.

Proof. To proof this statement we will use superscripts (m) and $(m+2\,q)$ to distinguish objects corresponding to different hierarchies. By assumption we have that

$$(\boldsymbol{t}^{(m)}, \boldsymbol{\beta}^{(m)}) \in \mathcal{M}_{m,q}^{\text{sing}}.$$

Thus, taking (30) into account, we have that (46) can be rewritten as

$$\begin{cases} F_i^{(m)}(\boldsymbol{t}^{(m)}, \boldsymbol{\beta}^{(m)}) := \displaystyle\oint_\gamma \frac{\mathrm{d}\lambda}{2\,i\,\pi} \frac{U^{(m)}(\lambda, \boldsymbol{t}^{(m)})\, R^{(m)}(\lambda, \boldsymbol{\beta}^{(m)})}{\lambda - \beta_i^{(m)}} = 0, \quad i = 1, \ldots, m \\ \\ F_{m,j}^{(m)}(\boldsymbol{t}^{(m)}, \boldsymbol{\beta}^{(m)}) := \displaystyle\oint_\gamma \frac{\mathrm{d}\lambda}{2\,i\,\pi} \frac{U^{(m)}(\lambda, \boldsymbol{t}^{(m)})\, R^{(m)}(\lambda, \boldsymbol{\beta}^{(m)})}{(\lambda - \beta_m^{(m)})^j} = 0, \quad j = 2, \ldots, q+1. \end{cases} \tag{48}$$

Now a $(2\,q+1)$-reduced solution of the hodograph equations for the dcKdV$_{m+2\,q}$ is characterized by

$$F_i^{(m+2\,q)}(\boldsymbol{t}^{(m+2\,q)}, \boldsymbol{\beta}^{(m+2\,q)}) := \oint_\gamma \frac{\mathrm{d}\lambda}{2\,i\,\pi} \frac{U^{(m+2\,q)}(\lambda, \boldsymbol{t}^{(m+2\,q)})\, R^{(m+2\,q)}(\lambda, \boldsymbol{\beta}^{(m+2\,q)})}{\lambda - \beta_i^{(m+2\,q)}} = 0, \tag{49}$$

where $i = 1, \dots, m$. But it is clear that

$$R^{(m+2\,q)}(\lambda, \boldsymbol{\beta}^{(m+2\,q)}) = \frac{\lambda^q}{(\lambda - \beta_m^{(m)})^q}\, R^{(m)}(\lambda, \boldsymbol{\beta}^{(m)}) \tag{50}$$

Hence if we set

$$t_i^{(m+2\,q)} := t_{i+q}^{(m)}, \quad i \geq 0,$$

we have

$$U^{(m)}(\lambda, \boldsymbol{t}^{(m)}) = x^{(m)} + \lambda\, t_1^{(m)} + \cdots + \lambda^{q-1}\, t_{q-1}^{(m)} + \lambda^q\, U^{(m+2\,q)}(\lambda, \boldsymbol{t}^{(m+2\,q)}). \tag{51}$$

Then it follows that

$$F_i^{(m+2\,q)}(\boldsymbol{t}^{(m+2\,q)}, \boldsymbol{\beta}^{(m+2\,q)}) = \oint_\gamma \frac{\mathrm{d}\lambda}{2\,i\,\pi}\, \frac{U^{(m)}(\lambda, \boldsymbol{t}^{(m)})\, R^{(m)}(\lambda, \boldsymbol{\beta}^{(m)})}{(\lambda - \beta_i^{(m)})(\lambda - \beta_m^{(m)})^q}, \quad i = 1, \dots, m. \tag{52}$$

Furthermore, for any given $i = 1, \dots, m$ we have

$$F_i^{(m)}(\boldsymbol{t}^{(m)}, \boldsymbol{\beta}^{(m)}) = \oint_\gamma \frac{\mathrm{d}\lambda}{2\,i\,\pi}\, \frac{U^{(m)}(\lambda, \boldsymbol{t}^{(m)})\, R^{(m)}(\lambda, \boldsymbol{\beta}^{(m)})}{\lambda - \beta_i^{(m)}}$$

$$= \oint_\gamma \frac{\mathrm{d}\lambda}{2\,i\,\pi}\, \frac{(\lambda - \beta_m^{(m)})^q\, U^{(m)}(\lambda, \boldsymbol{t}^{(m)})\, R^{(m)}(\lambda, \boldsymbol{\beta}^{(m)})}{(\lambda - \beta_i^{(m)})\,(\lambda - \beta_m^{(m)})^q}$$

$$= \sum_{k=0}^{q} c_{1,k}(\boldsymbol{\beta}^{(m)})\, I_{i,k}(\boldsymbol{t}^{(m)}, \boldsymbol{\beta}^{(m)}),$$

and

$$F_{m,j}^{(m)}(\boldsymbol{t}^{(m)}, \boldsymbol{\beta}^{(m)}) = \oint_\gamma \frac{\mathrm{d}\lambda}{2\,i\,\pi}\, \frac{U^{(m)}(\lambda, \boldsymbol{t}^{(m)})\, R^{(m)}(\lambda, \boldsymbol{\beta}^{(m)})}{(\lambda - \beta_m^{(m)})^j}$$

$$= \oint_\gamma \frac{\mathrm{d}\lambda}{2\,i\,\pi}\, \frac{(\lambda - \beta_i^{(m)})\,(\lambda - \beta_m^{(m)})^{q-j}\, U^{(m)}(\lambda, \boldsymbol{t}^{(m)})\, R^{(m)}(\lambda, \boldsymbol{\beta}^{(m)})}{(\lambda - \beta_i^{(m)})\,(\lambda - \beta_m^{(m)})^q}$$

$$= \sum_{k=0}^{q-j+1} c_{j,k}(\boldsymbol{\beta}^{(m)})\, I_{i,k}(\boldsymbol{t}^{(m)}, \boldsymbol{\beta}^{(m)}), \quad j = 2, \dots, q+1.$$

where the functions $c_{j,k}(\boldsymbol{\beta}^{(m)})$ are the coefficients of the polynomials

$$\begin{cases} (\lambda - \beta_m^{(m)})^q = \sum_{k=0}^{q} c_{1k}(\boldsymbol{\beta}^{(m)})\, \lambda^k; \\ \\ (\lambda - \beta_i^{(m)})\,(\lambda - \beta_m^{(m)})^{q-j} = \sum_{k=0}^{q-j+1} c_{jk}(\boldsymbol{\beta}^{(m)})\, \lambda^k, \quad j = 2, \dots, q+1. \end{cases} \tag{53}$$

and

$$I_{i,k}(\boldsymbol{t}^{(m)},\boldsymbol{\beta}^{(m)}) := \oint_\gamma \frac{d\lambda}{2\,i\,\pi}\,\frac{\lambda^k\,U^{(m)}(\lambda,\boldsymbol{t}^{(m)})\,R^{(m)}(\lambda,\boldsymbol{\beta}^{(m)})}{(\lambda-\beta_i^{(m)})\,(\lambda-\beta_m^{(m)})^q} \tag{54}$$

Now, for any given $i = 1,\dots,m$ the system (46) implies

$$\begin{cases} F_i^{(m)}(\boldsymbol{t}^{(m)},\boldsymbol{\beta}^{(m)}) = 0, \\ \\ F_{m,j}^{(m)}(\boldsymbol{t}^{(m)},\boldsymbol{\beta}^{(m)}) = 0, \quad j = 2,\dots,q+1, \end{cases}$$

and, as a consequence, we deduce the following system of q homogeneous linear equations for the q functions $I_{i,k}(\boldsymbol{t}^{(m)},\boldsymbol{\beta}^{(m)})$

$$\sum_{k=0}^{q-j+1} c_{j,k}(\boldsymbol{\beta}^{(m)})\,I_{i,k}(\boldsymbol{t}^{(m)},\boldsymbol{\beta}^{(m)}) = 0, \quad j = 1,\dots,q+1.$$

Because of the linear independence of the polynomials (53) these equations are linearly independent and, therefore, all the functions $I_{i,k}(\boldsymbol{t}^{(m)},\boldsymbol{\beta}^{(m)})$ vanish. Finally, from (52) we conclude that $I_{i,0}(\boldsymbol{t}^{(m)},\boldsymbol{\beta}^{(m)}) = 0$ is equivalent to $F_i^{(m+2\,q)}(\boldsymbol{t}^{(m+2\,q)},\boldsymbol{\beta}^{(m+2\,q)}) = 0$ and the statement follows. $\square$

5 Examples

dcKdV$_1$ hierarchy

The hodograph equation for the dcKdV$_1$ hierarchy with $t_n = 0$ for all $n \geq 3$ reduce to

$$8\,x + 12\,t_1\,\beta_1 + 15\,t_2\,\beta_1^2 = 0. \tag{55}$$

The singular variety $\mathcal{M}_{1,1}^{\text{sing}}$ for (55) is determined by adding to (55) the equation

$$2\,t_1 + 5\,t_2\,\beta_1 = 0, \tag{56}$$

so that for $t_2 \neq 0$ we have $\beta_1 = -\dfrac{2\,t_1}{5\,t_2}$. Substituting this result in (55) we find a constraint for the flow parameters

$$x = \frac{3}{10}\,\frac{t_1^2}{t_2},$$

which is the shock region for the solution of (55) given by

$$\beta_1 = \frac{2}{15\,t_2}\Big(-3\,t_1 + \sqrt{3\,(3\,t_1^2 - 10\,t_2\,x)}\Big). \tag{57}$$

There are two sectors $\mathcal{M}_{1,1,k}^{\text{sing}}\,(k = 1,2)$ in $\mathcal{M}_{1,1}^{\text{sing}}$

$$\begin{aligned} &\mathcal{M}_{1,1,1}^{\text{sing}}: \quad x = t_1 = t_2 = 0, \quad \beta_1 \text{ arbitrary;} \\ &\mathcal{M}_{1,1,2}^{\text{sing}}: \quad (x,t_1,t_2,\beta_1) \text{ such that } t_2 \neq 0,\ x = \frac{3}{10}\,\frac{t_1^2}{t_2} \text{ and } \beta_1 = -\frac{2}{5}\,\frac{t_1}{t_2} \end{aligned} \tag{58}$$

To see the relationship with the dcKdV$_3$ hierarchy we notice that

$$x^{(3)} = t_1, \quad t_1^{(3)} = t_2,$$

and

$$\boldsymbol{\beta}^{(3} = (\beta_1, \beta_1, \beta_1) = -\frac{2}{5}\,\frac{x^{(3)}}{t_1^{(3)}}\,(1,1,1),$$

which is a 3-reduced solution of the first flow (25) of the dcKdV$_3$ hierarchy.

The dcKdV$_1$ hodograph equation with $t_n = 0$ for all $n \geq 6$ is

$$693\,t_5\,\beta_1^5 + 630\,t_4\,\beta_1^4 + 560\,t_3\,\beta_1^3 + 480\,t_2\,\beta_1^2 + 384\,t_1\,\beta_1 + 256\,x = 0.$$

Let us first consider the singular variety $\mathcal{M}_{1,1}^{\rm sing}$ with $t_n = 0$ for all $n \geq 4$. It is is determined by the equations

$$560\,t_3\,\beta_1^3 + 480\,t_2\,\beta_1^2 + 384\,t_1\,\beta_1 + 256\,x = 0,$$

$$1680\,t_3\,\beta_1^2 + 960\,t_2\,\beta_1 + 384\,t_1 = 0.$$

Thus an open subset of $\mathcal{M}_{1,1}^{\rm sing}$ can be parametrized by the equations

$$x = \frac{-25\,t_2^3 + 105\,t_1\,t_2\,t_3 + \sqrt{5}\,\sqrt{125\,t_2^6 - 1050\,t_1\,t_3\,t_2^4 + 2940\,t_1^2\,t_2^2\,t_3^2 - 2744\,t_1^3\,t_3^3}}{245\,t_3^2},$$

$$\beta_1 = -\frac{2\left(-25\,t_2^3 + 70\,t_1\,t_2\,t_3 + \sqrt{5}\,\sqrt{\left(5\,t_2^2 - 14\,t_1\,t_3\right)^3}\right)}{35\,t_3\,\left(14\,t_1\,t_3 - 5\,t_2^2\right)}.$$

It determines the following 3-reduced solution of the two first flows of the dcKdV$_3$ hierachy ($x^{(3)} = t_1$, $t_1^{(3)} = t_2$, $t_2^{(3)} = t_3$)

$$\beta_1^{(3)} = \beta_2^{(3)} = \beta_3^{(3)} = -\frac{2\left(-25\,(t_1^{(3)})^3 + 70\,x^{(3)}\,t_1^{(3)}\,t_2^{(3)} + \sqrt{5}\,\sqrt{\left(5\,(t_1^{(3)})^2 - 14\,x^{(3)}\,t_2^{(3)}\right)^3}\right)}{35\,t_2^{(3)}\,\left(14\,x^{(3)}\,t_2^{(3)} - 5\,(t_1^{(3)})^2\right)}.$$

Next, for the sector $\mathcal{M}_{1,2}^{\rm sing}$ if we set $t_n = 0$ for all $n \geq 5$, we obtain the equations

$$630\,t_4\,\beta_1^4 + 560\,t_3\,\beta_1^3 + 480\,t_2\,\beta_1^2 + 384\,t_1\,\beta_1 + 256\,x = 0,$$

$$2520\,t_4\,\beta_1^3 + 1680\,t_3\,\beta_1^2 + 960\,t_2\,\beta_1 + 384\,t_1 = 0,$$

$$7560\,t_4\,\beta_1^2 + 3360\,t_3\,\beta_1 + 960\,t_2 = 0.$$

From these equations we find

$$t_1 = \frac{5\left(-49\,t_3^3 + 189\,t_2\,t_3\,t_4 + \sqrt{7}\sqrt{343\,t_3^6 - 2646\,t_2\,t_4\,t_3^4 + 6804\,t_2^2\,t_3^2\,t_4^2 - 5832\,t_2^3\,t_4^3}\right)}{1701\,t_4^2},$$

$$x = \frac{5\left(-98\,t_3^4 + 378\,t_2\,t_4\,t_3^2 + 2\sqrt{7}\sqrt{\left(7\,t_3^2 - 18\,t_2\,t_4\right)^3}\,t_3 - 243\,t_2^2\,t_4^2\right)}{10206\,t_4^3},$$

$$\beta_1 = -\frac{2\left(-49\,t_3^3 + 126\,t_2\,t_3\,t_4 + \sqrt{7}\sqrt{\left(7\,t_3^2 - 18\,t_2\,t_4\right)^3}\right)}{63\,t_4\left(18\,t_2\,t_4 - 7\,t_3^2\right)}.$$

Then the associated 5-reduced solution of the two first flows of the dcKdV$_5$ hierarchy ($x^{(5)} = t_2$, $t_1^{(5)} = t_3$, $t_2^{(5)} = t_4$) is given by

$$\beta_i = -\frac{2\left(-49\,(t_1^{(5)})^3 + 126\,x^{(5)}\,t_1^{(5)}\,t_2^{(5)} + \sqrt{7}\sqrt{\left(7\,(t_1^{(5)})^2 - 18\,x^{(5)}\,t_2^{(5)}\right)^3}\right)}{63\,t_2^{(5)}\left(18\,x^{(5)}\,t_2^{(5)} - 7\,(t_1^{(5)})^2\right)}, \quad i = 1,\ldots,5.$$

dcKdV$_2$ hierarchy

Let us consider the hodograph equations for the dcKdV$_2$ hierarchy with $t_n = 0$ for all $n \geq 3$. From (42) we have that they take the form

$$\begin{cases} 8x + 4t_1(3\beta_1 + \beta_2) + 3t_2\left(5\beta_1^2 + 2\beta_1\beta_2 + \beta_2^2\right) = 0, \\ \\ 8x + 4t_1(\beta_1 + 3\beta_2) + 3t_2\left(\beta_1^2 + 2\beta_1\beta_2 + 5\beta_2^2\right) = 0. \end{cases} \tag{59}$$

The singular variety $\mathcal{M}_2^{\text{sing}}$ is determined by (59) together with the additional condition $(\det(\partial_{\beta_i\beta_j} W_m(\boldsymbol{t},\boldsymbol{\beta})) = 0)$

$$-(2t_1 + 3t_2(\beta_1 + \beta_2))^2 + 9(2t_1 + t_2(5\beta_1 + \beta_2))(2t_1 + t_2(\beta_1 + 5\beta_2)) = 0. \tag{60}$$

There elements of $\mathcal{M}_2^{\text{sing}}$ are

$$\begin{aligned} & x = t_1 = t_2 = 0, \quad (\beta_0, \beta_1) \text{ arbitrary;} \\ & (x, t_1, t_2, \beta_1, \beta_2) \text{ such that } t_2 \neq 0,\ x = \frac{t_1^2}{3\,t_2} \text{ and } \beta_1 = \beta_2 = -\frac{t_1}{3\,t_2} \end{aligned} \tag{61}$$

The subvarieties $\mathcal{M}_{2,q}^{\text{sing}}$ are all equal and given by

$$x = t_1 = t_2 = 0, \quad (\beta_0, \beta_1) \text{ arbitrary with } \beta_0 \neq \beta_1.$$

Notice that the constraint $x = \dfrac{t_1^2}{3\,t_2}$ determines the shock region for the following solution of (59)

$$\beta_1 \,=\, \frac{-\,t_1 \,+\, \sqrt{2}\,\sqrt{t_1^2 \,-\, 3\,t_2\,x}}{3\,t_2}, \quad \beta_2 \,=\, \frac{-\,t_1 \,-\, \sqrt{2}\,\sqrt{t_1^2 \,-\, 3\,t_2\,x}}{3\,t_2}. \tag{62}$$

Let us now consider the system of hodograph equations (42) for the dcKdV$_2$ hierarchy with $t_n = 0$ for all $n \geq 4$. The singular variety $\mathcal{M}_2^{\rm sing}$ is now determined by (42) and the condition $(\det(\partial_{\beta_i\beta_j} W_m(\boldsymbol{t}, \boldsymbol{\beta})) = 0)$

$$32t_1^2 + 96t_2(\beta_1+\beta_2)t_1 + 702t_3^2\beta_1^2\beta_2^2 + 72\left(3t_2^2+t_1t_3\right)\beta_1\beta_2 + 12\left(3t_2^2+13t_1t_3\right)\left(\beta_1^2+\beta_2^2\right) +$$

$$486t_2t_3\left(\beta_2\beta_1^2+\beta_2^2\beta_1\right) + 90t_2t_3\left(\beta_1^3+\beta_2^3\right) + 180t_3^2\left(\beta_2\beta_1^3+\beta_2^3\beta_1\right) + 45t_3^2\left(\beta_1^4+\beta_2^4\right) = 0.$$

One finds the following six sectors in $\mathcal{M}_2^{\rm sing}$

1. $$x \,=\, \frac{-9t_2^3+36t_1t_3t_2+(8t_1t_3-3t_2^2)\sqrt{9t_2^2-24t_1t_3}}{72t_3^2}, \qquad \beta_1 \,=\, \beta_2 \,=\, -\frac{3t_2+\sqrt{9t_2^2-24t_1t_3}}{12t_3},$$

2. $$x \,=\, \frac{-9t_2^3+36t_1t_3t_2-(8t_1t_3-3t_2^2)\sqrt{9t_2^2-24t_1t_3}}{72t_3^2}, \qquad \beta_1 \,=\, \beta_2 \,=\, \frac{-3t_2+\sqrt{9t_2^2-24t_1t_3}}{12t_3},$$

3. $$x \,=\, \frac{-45t_3t_2^3+180t_1t_3^2t_2+\sqrt{15}(8t_1t_3-3t_2^2)\sqrt{t_3^2\left(3t_2^2-8t_1t_3\right)}}{360t_3^3},$$

$$\beta_1 \,=\, -\frac{5t_2t_3+\sqrt{15}\sqrt{t_3^2\left(3t_2^2-8t_1t_3\right)}}{20t_3^2}, \qquad \beta_2 \,=\, \frac{-3t_2t_3+\sqrt{15}\sqrt{t_3^2\left(3t_2^2-8t_1t_3\right)}}{12t_3^2},$$

4. $$x \,=\, \frac{-45t_3t_2^3+180t_1t_3^2t_2-\sqrt{15}(8t_1t_3-3t_2^2)\sqrt{t_3^2\left(3t_2^2-8t_1t_3\right)}}{360t_3^3},$$

$$\beta_1 \,=\, -\frac{3t_2t_3+\sqrt{15}\sqrt{t_3^2\left(3t_2^2-8t_1t_3\right)}}{12t_3^2}, \qquad \beta_2 \,=\, \frac{-5t_2t_3+\sqrt{15}\sqrt{t_3^2\left(3t_2^2-8t_1t_3\right)}}{20t_3^2},$$

5. $$x = \frac{-45t_3t_2^3 + 180t_1t_3^2t_2 - \sqrt{15}(8t_1t_3 - 3t_2^2)\sqrt{t_3^2\left(3t_2^2 - 8t_1t_3\right)}}{360t_3^3},$$

$$\beta_1 = \frac{-5t_2t_3 + \sqrt{15}\sqrt{t_3^2\left(3t_2^2 - 8t_1t_3\right)}}{20t_3^2}, \qquad \beta_2 = -\frac{3t_2t_3 + \sqrt{15}\sqrt{t_3^2\left(3t_2^2 - 8t_1t_3\right)}}{12t_3^2}$$

6. $$x = \frac{-45t_3t_2^3 + 180t_1t_3^2t_2 + \sqrt{15}(8t_1t_3 - 3t_2^2)\sqrt{t_3^2\left(3t_2^2 - 8t_1t_3\right)}}{360t_3^3},$$

$$\beta_1 = \frac{-3t_2t_3 + \sqrt{15}\sqrt{t_3^2\left(3t_2^2 - 8t_1t_3\right)}}{12t_3^2}, \qquad \beta_2 = -\frac{5t_2t_3 + \sqrt{15}\sqrt{t_3^2\left(3t_2^2 - 8t_1t_3\right)}}{20t_3^2}.$$

It is easy to see that $\mathcal{M}_{2,1}^{\text{sing}}$ is given by the sectors 5 and 6. To check the connection between these sectors and the dcKdV$_4$ hierarchy it is enough to set

$$x^{(4)} = t_1, \quad t_1^{(4)} = t_2, \quad t_2^{(4)} = t_3, \quad \boldsymbol{\beta}^{(4)} = (\beta_1, \beta_2, \beta_2, \beta_2),$$

and it is immediate to prove that $\boldsymbol{\beta}^{(4)}(\boldsymbol{t}^{(4)})$ verifies the equations of the first flow of the dcKdV$_4$ hierarchy

$$\frac{\partial \beta_i}{\partial t_1^{(4)}} = \Big(\beta_i + \frac{1}{2}\sum_{k=1}^{4}\beta_k\Big)\frac{\partial \beta_i}{\partial x^{(4)}}, \quad i = 1, \ldots, 4.$$

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