ON STABILITY OF ROLLS NEAR THE ONSET OF CONVECTION IN A LAYER WITH STRESS-FREE BOUNDARIES

Olga Podvigina*

International Institute of Earthquake Prediction Theory
and Mathematical Geophysics,
84/32 Profsoyuznaya St, 117997 Moscow, Russian Federation;

UNS, CNRS, Laboratoire Cassiopée, Observatoire de la Côte d'Azur
BP 4229, 06304 Nice Cedex 4, France

Abstract

We consider a classical problem of linear stability of convective rolls in a plane layer with stress-free horizontal boundaries near the onset of convection. The problem has been studied by a number of authors, who have shown that rolls of wave number k are unstable with respect to perturbations of different types, if some inequalities relating k and the Rayleigh number R are satisfied. The perturbations involve a large-scale mode. Certain asymptotic dependencies between wave numbers of the mode and overcriticality are always assumed in the available proofs of instability. We analyse the stability analytically following the approach of Podvigina (2008) without making a priori assumptions concerning asymptotic relations between small parameters characterising the problem. Instability of rolls to short-scale modes is also considered. Therefore, our analytical results on stability to space-periodic perturbations are exhaustive; they allow to identify the areas in the (k, R) plane, where convective rolls are stable near the onset. The analytical results are compared with numerical solutions to the eigenvalue problem determining stability of rolls.

Keywords: Boussinesq convection; onset; stability; rolls; stress-free boundaries

*Email: olgap@mitp.ru

1 Introduction

We consider Boussinesq convection in a plane horizontal layer heated from below with stress-free horizontal boundaries. For small Rayleigh numbers R, i.e. for small temperature differences between the lower and upper boundaries, the fluid is not moving and the heat is transported by thermal diffusion only. As R exceeds the critical value R^s, the fluid motion sets in. The motion takes the form of rolls. We denote by k_s the horizontal wave number of the mode becoming unstable the first. (By a horizontal wave number we understand the length of the horizontal component of the wave vector.) In this paper we study analytically and numerically stability of rolls of a horizontal wave number close to k_s for the Rayleigh number slightly above R^s.

Instability of rolls in a convective layer was studied analytically by a number of authors. An instability of rolls specific for stress-free boundaries is known, which does not occur in a layer whose one or both horizontal boundaries are rigid. Its presence relies on existence of a slowly decaying large-scale mode. Zippelius and Siggia (1982, 1983) were the first to study the instability of this kind. In the leading order the unstable mode is a sum of a large-scale mode and of two short-scale modes with wave vectors close to the one of the perturbed rolls. In the study of stability of rolls in a rotating convective layer it was called *small angle* instability (Cox and Matthews 2000).

By deriving amplitude equations, Zippelius and Siggia obtained sufficient conditions for instability of rolls, in particular, they found that for $P < 0.782$ (P denoting the Prandtl number) no stable rolls existed near the onset. Their results were questioned by Busse and Bolton (1984), who found boundaries for instabilities of rolls by direct calculations of the unstable mode, and claimed that no stable rolls were present near the onset only for $P < 0.543$. Their results were confirmed by Bernoff (1994), who studied instability of rolls applying Ginzburg-Landau equations.

The conflict between the results of Zippelius and Siggia (1982, 1983) and Busse and Bolton (1984) was resolved by Mielke (1997), who studied stability of rolls by means of Lyapunov-Schmidt reduction and showed that instability boundaries had been found in these papers for different unstable modes. The problem of stability of rolls involves five small parameters: two wave numbers of the large scale mode, overcriticality, the difference between the wave number of perturbed rolls and the critical wave number, and the growth rate (depending on the first four). Zippelius and Siggia postulated asymptotic relations between the parameters, different from those postulated by Busse and Bolton, and hence different instability modes were examined.

However, in all these studies some asymptotic relations between the small parameters of the problem were assumed, and thus stability only to selected types of perturbations was studied. Hence, only *instability* of rolls was proven (as it is discussed in introduction and conclusion in Mielke (1997) and also section 3 of Bernoff (1994). The question, whether other unstable modes corresponding to other asymptotic scalings exist, remains open. Another question asked by Busse and Bolton (1984) and Bernoff (1994) and not answered by previous studies is whether enough terms of asymptotic expansions were taken into account. For four independent small parameters this is a hard question!

In the present paper both difficulties are resolved. We do not assume any asymptotic relations between the small parameters, hence instability with respect to all perturbations of the small-angle type is examined. An unstable mode is represented as a series in small parameters, with estimates for the remainder. (Estimates for omitted terms were not given before.) A condition for instability has the form of inequalities. We demonstrate that in these inequalities the omitted terms are asymptotically smaller than the terms retained in the analysis.

We also study instability to perturbations of a different kind, which are in the leading order convective rolls with the horizontal wave number close to the critical one, and a finite angle between wave vectors of the perturbed rolls and of the perturbation, which we call a *finite angle* instability.

Furthermore, we show that stability to all considered perturbations implies stability to a much wider class of doubly periodic in horizontal directions perturbations. We show, that the domain of the linearisation operator, acting on vector fields satisfying the assumed boundary conditions on the horizontal boundaries and doubly periodic in horizontal directions, splits into a direct sum of invariant subspaces. Thus investigation of instability is reduced to detection of instability modes in these invariant subspaces. Any instability mode in such a subspace is either of a small angle or finite angle type. Therefore, our study of instabilities of rolls with respect to perturbations which are doubly periodic in some directions in the (x, y) plane is complete: we demonstrate *stability* of rolls for which instability is not detected in our analysis. This is also a novel feature of our investigation: all previous papers focused exclusively on instability.

We estimate analytically the asymptotics of the most unstable mode and its growth rate on different parts of the (k, R) plane and calculate the instability boundaries estimating the orders of the neglected terms (which was not yet done in literature).

We are using notation and some results of Podvigina (2008), referred to as OP2008, where instability of flows in a rotating convective layer with stress-free boundaries was studied.

Stability of rolls in a layer with stress-free boundaries was studied by direct computations of dominant eigenvalues by Bolton and Busse (1985). Their numerical results agree well with the theoretical predictions of Busse and Bolton (1984), in particular, they found that for $P = 0.71$ rolls are stable near the onset. However, Mielke (1997) claimed that stable rolls near the onset for $P < 0.782$ are absent; he did not comment on the disagreement with the numerical results of Bolton and Busse (1985). In contrast to (Bolton and Busse 1985), our computations of stability modes indicate that rolls are unstable at the onset for $P < 0.782$, albeit in a small neighbourhood of the point (k_s, R^s) on the (k, R) plane. For P decreasing below 0.782, the right boundary of the area of stable rolls slowly moves to the left, away from the point (k_s, R^s).

2 The onset of convection

Boussinesq thermal convection satisfies the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{v}) + P\nabla^2 \mathbf{v} + PR\theta \mathbf{e}_z - \nabla p, \tag{1}$$

the incompressibility condition

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

and the heat transfer equation

$$\frac{\partial \theta}{\partial t} = -(\mathbf{v} \cdot \nabla)\theta + v_z + \nabla^2 \theta \tag{3}$$

where $\mathbf{v}$ is the flow velocity and θ is the deviation of the flow temperature from the linear profile. R and P are dimensionless parameters, the Rayleigh and Prandtl numbers, respectively. Stress-free horizontal boundaries held at fixed temperature are assumed:

$$\frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial z} = v_z = 0, \qquad \theta = 0 \qquad \text{at } z = 0, 1. \tag{4}$$

The trivial solution $(\mathbf{v}, \theta) = (\mathbf{0}, 0)$ describing pure thermal conduction loses stability to perturbations of wave number k at $R = R_c(k)$, where

$$R_c(k) = a^3 k^{-2}, \quad a = k^2 + \pi^2. \tag{5}$$

The critical horizontal wave number k_s for the onset of convection is $\pi/\sqrt{2}$, the respective critical value $R^s = 27\pi^4/4$.

We employ four-dimensional vectors

$$\mathbf{W} \equiv (\mathbf{W}^{\text{flow}}, \mathbf{W}^{\text{temp}}) = (\mathbf{v}, \theta). \tag{6}$$

For a Rayleigh number slightly above the critical value,

$$R = R_c + \varepsilon^2, \tag{7}$$

a solution to (1)-(4) can be represented as a power series

$$\mathbf{U} = \sum_{j=1}^{\infty} \varepsilon^j \mathbf{U}_j. \tag{8}$$

The first two terms of the solution representing rolls are

$$\mathbf{U}_1 = b \begin{pmatrix} -\pi k^{-1} \cos \pi z \sin kx \\ 0 \\ \sin \pi z \cos kx \\ a^{-1} \sin \pi z \cos kx \end{pmatrix} \tag{9}$$

($\mathbf{U}_1$ is an eigenvector of the linearisation of (1)-(3)),

$$\mathbf{U}_2 = b^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -(8\pi a)^{-1} \sin 2\pi z \end{pmatrix}, \tag{10}$$

where

$$8a = R_c b^2. \tag{11}$$

3 Invariant subspace

We study stability of rolls of wave number k, which is close to the critical one;

$$\alpha = k - k_s \tag{12}$$

is thus a small parameter.

Stability of (8) is controlled by eigenvalues of the linear operator L, a linearisation of (1)-(3) near the steady state. The operator can be expanded in a series

$$L = \sum_{j=0}^{\infty} \varepsilon^j L_j. \tag{13}$$

Here the first term is

$$L_0(\mathbf{v}, \theta) = (P\nabla^2\mathbf{v} + PR_c\theta\mathbf{e}_z - \nabla p, v_z + \nabla^2\theta). \tag{14}$$

We consider the eigenvalue problem

$$L\mathbf{W} = \lambda\mathbf{W}. \tag{15}$$

In OP2008 a three-dimensional invariant subspace of L was considered, and the problem of stability of rolls was reduced to the analysis of eigenvalues of L in this subspace. Denote by $\mathbf{W}_j$, $j = 1, 2, 3$, a basis in this subspace and by $\mathcal{A}$ the matrix A_{ij} of the restriction of L on the subspace:

$$L\mathbf{W}_j = \sum_{i=1}^{3} A_{ij}\mathbf{W}_i, \quad j = 1, 2, 3. \tag{16}$$

We expand the basis and the matrix in power series in ε:

$$\mathbf{W}_j = \sum_{l=0}^{\infty} \varepsilon^l \mathbf{W}_{j,l}, \tag{17}$$

$$A_{ij} = \sum_{l=0}^{\infty} \varepsilon^l A_{ij,l}. \tag{18}$$

The coefficients in the series (17) and (18) depend on δ_x and δ_y, which are small perturbations of the horizontal component of the wave vector $(k, 0, \pi)$: $\delta_x \ll k$ and $\delta_y \ll k$. (The two small parameters are involved in the definition of the three-dimensional invariant subspace of L, see OP2008.)

Vector fields $\mathbf{W}_{j,0}$ are eigenfunctions of L_0:

$$L_0\mathbf{W}_{j,0} = \lambda_{j,0}\mathbf{W}_{j,0}, \tag{19}$$

the leading order approximations of $\mathbf{W}_{j,0}$ in δ_x and δ_y were derived in OP2008:

$$\mathbf{W}_{1,0} = \begin{pmatrix} -\delta_y \sin(\delta_x x + \delta_y y) \\ \delta_x \sin(\delta_x x + \delta_y y) \\ 0 \\ 0 \end{pmatrix}, \tag{20}$$

$$\mathbf{W}_{2,0}=\left(\begin{array}{c}-\pi s_+k_+^{-1}\cos\pi z\sin((k+\delta_x)x+\delta_y y)\\ -\pi\delta_y k_+^{-2}\cos\pi z\sin((k+\delta_x)x+\delta_y y)\\ \sin\pi z\cos((k+\delta_x)x+\delta_y y)\\ a_+^{-1}\sin\pi z\cos((k+\delta_x)x+\delta_y y)\end{array}\right)+\mathrm{O}((k_+^2-k^2)^2,\alpha(k_+^2-k^2)),\quad(21)$$

$$\mathbf{W}_{3,0}=\left(\begin{array}{c}-\pi s_-k_-^{-1}\cos\pi z\sin((k-\delta_x)x-\delta_y y)\\ \pi\delta_y k_-^{-2}\cos\pi z\sin((k-\delta_x)x-\delta_y y)\\ \sin\pi z\cos((k-\delta_x)x-\delta_y y)\\ a_-^{-1}\sin\pi z\cos((k-\delta_x)x-\delta_y y)\end{array}\right)+\mathrm{O}((k_-^2-k^2)^2,\alpha(k_-^2-k^2))\quad(22)$$

(the expression for $\mathbf{W}_{1,0}$ is exact). Here it is denoted $k_\pm=((k\pm\delta_x)^2+\delta_y^2)^{1/2}$, $s_\pm=(k\pm\delta_x)k_\pm^{-1}$ and $a_\pm=k_\pm^2+\pi^2$. $\mathbf{W}_{1,0}$ is a large-scale horizontal mode.

The paper is mainly concerned with the eigenvalue problem (15) in the three-dimensional invariant subspace, discussed above, where the eigenfunction $\mathbf{W}$ and the operator L take the forms (17) and (18), respectively. The case of an unstable mode from this subspace is called *small angle* instability, because the angles between the wave vector $(k,0,\pi)$ of perturbed rolls and the wave vectors $(k+\delta_x,\delta_y,\pi)$ of short-scale components of perturbation ($\mathbf{W}_{2,0}$ and $\mathbf{W}_{3,0}$ in the leading order) are small. In section 5 we also study stability with respect to perturbations of the form of rolls at an angle ξ to the perturbed ones, where ξ is finite (and not asymptotically small). In the remaining part of the section it is shown that any growing mode of L belongs to one of these two classes.

Consider the space $\mathcal{F}(\delta_x,\delta_y)$ of 4-component vector fields of the form (6), such that $\mathbf{W}^{\rm flow}$ and $\mathbf{W}^{\rm temp}$

(i) are linear combinations of harmonics with wave vectors $(m_1k+\delta_x,\delta_y,m_2\pi)$ or $(m_1k-\delta_x,-\delta_y,m_2\pi)$, where m_1 and m_2 are integer,

(ii) are symmetric about the vertical axis, i.e.

$$(\mathbf{v}(x,y,z),\theta(x,y,z))=(-v_x(-x,-y,z),-v_y(-x,-y,z),v_z(-x,-y,z),\theta(-x,-y,z)),$$

(iii) satisfy the boundary conditions (4),

(iv) $\mathbf{W}^{\rm flow}$ is divergence-free.

It was shown in OP2008 that $\mathcal{F}$ is L-invariant for any (not necessarily small) δ_x and δ_y.

Let the subspace $\tilde{\mathcal{F}}(\delta_x,\delta_y)$ be defined like we have defined $\mathcal{F}(\delta_x,\delta_y)$, but omitting the condition (ii). Such subspaces also are L-invariant. We consider perturbations which are doubly periodic on the (x,y) plane. Let the domain of L be comprised of 4-component vector fields, for which (iii) and (iv) are satisfied, and which have the same double periodicity on the (x,y) plane. The domain can be split into a sum of invariant spaces $\tilde{\mathcal{F}}(\delta_x,\delta_y)$; hence we can assume that a mode belongs to such an invariant subspace. Any mode can be represented as a sum of a symmetric and an antisymmetric vector field; each of these vector fields is itself a mode, since the subspaces of symmetric and an antisymmetric vector fields are L-invariant. Moreover, in a coordinate system with the origin shifted by half a period, π/δ_y, in the $y-$direction (this shift does not affect rolls, since they are independent of the y coordinate) the antisymmetric modes become symmetric. In the case $\delta_y=0$, if integer m and n such that $\delta_x/k=(2m+1)/(2n)$ exist, the shift of the origin by

$l = 2\pi n/k = \pi(3m+1)/\delta_x$ along the $x-$direction turns a symmetric mode into an antisymmetric one. For a given k and δ_x the ratio $(2m+1)/(2n)$ can be arbitrary close to δ_x/k. Consequently, without any loss of generality we consider henceforth only modes belonging to $\mathcal{F}$.

Eigenvalues of (13) are perturbations of the ones of L_0. Positive or slightly negative eigenvalues of L_0 are associated with eigenvectors, whose wave vectors are either $(\delta_x, \delta_y, 0)$ (large-scale mode) or $(k\cos\xi, k\sin\xi, \pi)$ with k close to k_s (see, e.g., discussion in Bernoff 1994). The latter eigenvectors are (9) rotated by the angle ξ about a vertical axis; we denote them by $\mathbf{U}(k,\xi)$. Consider an eigenvector $\mathbf{W}$ of L which belongs to some $\mathcal{F}$. The vector field $\mathbf{W}_0 = \lim_{\varepsilon\to 0}\mathbf{W}$ can be one of the following:

(a) $\mathbf{W}_0 = a_1\mathbf{W}_{1,0} + a_2\mathbf{W}_{2,0} + a_3\mathbf{W}_{3,0}$ (with at least one $a_j \neq 0$ and $\mathbf{W}_{j,0}$ defined by (20)-(22));

(b) $\mathbf{W}_0 = \mathbf{U}(k,\xi)$ with k close to k_s, $\xi = \mathrm{O}(1)$ and $\xi \pm 2\pi/3 = \mathrm{O}(1)$;

(c) $\mathbf{W}_0 = a_1\mathbf{U}(k,\xi_1)+a_2\mathbf{U}(k,\xi_2)$ with k close to k_s, $\xi_1-2\pi/3 = \mathrm{o}(1)$ and $\xi_2+2\pi/3 = \mathrm{o}(1)$.

In case (a) small-angle instability takes place studied in section 4, in cases (b) and (c) finite angle instability considered in section 5. Therefore, we examine all types of possibly growing perturbations.

4 Stability of rolls: analytical results

It is shown in Appendix B that in order to study stability of rolls with respect to perturbations from the subspace constructed above, it suffices to check, whether (for a given P, ε and α) there exist such δ_x and δ_y that the following inequalities are satisfied:

$$\det\mathcal{A} > 0 \tag{23}$$

or

$$S(\mathcal{A})\mathrm{tr}\,\mathcal{A} - \det\mathcal{A} > 0 \tag{24}$$

where

$$S(\mathcal{A}) = A_{11}A_{22} - A_{12}A_{21} + A_{11}A_{33} - A_{13}A_{31} + A_{22}A_{33} - A_{23}A_{32}$$

is the sum of the three second order minors. The matrix $\mathcal{A}$ has an eigenvalue with a positive real part, if and only if at least one of the inequalities is satisfied for some δ's.

It is calculated in Appendix A, that in the leading order

$$\det\mathcal{A} = D_0 + \varepsilon^2 D_1 + \varepsilon^4 D_2 + \alpha D_3 + \alpha^2 D_4 + \alpha\varepsilon^2 D_5, \tag{25}$$

where

$$D_0 = d_{01}\delta_x^6 + d_{02}\delta_x^4\delta_y^2 + d_{03}\delta_x^2\delta_y^6 + d_{04}\delta_y^{10},$$

$$D_1 = (\delta_x^2 + \delta_y^2)^{-1}(d_{11}\delta_x^6 + d_{12}\delta_x^4\delta_y^2 + d_{13}\delta_x^2\delta_y^4 + d_{14}\delta_y^8),$$

$$D_2 = (\delta_x^2 + \delta_y^2)^{-1}(d_{21}\delta_x^2\delta_y^2 + d_{22}\delta_y^4),$$

$$D_3 = d_{31}\delta_x^2\delta_y^4 + d_{32}\delta_y^8,$$

$$D_4 = (\delta_x^2 + \delta_y^2)(d_{41}\delta_x^2 + d_{42}\delta_y^4),$$

$$D_5 = d_{51}\delta_x^2\delta_y^2(\delta_x^2 + \delta_y^2)^{-1} + d_{52}\delta_y^4;$$

$$S(\mathcal{A})\mathrm{tr}\,\mathcal{A} - \det\mathcal{A} = E_0 + \varepsilon^2 E_1 + \varepsilon^4 E_2 + \alpha^2 E_3 + \alpha\varepsilon^2 E_4, \qquad (26)$$

where

$$E_0 = e_{01}\delta_x^6 + e_{02}\delta_x^4\delta_y^2 + e_{03}\delta_x^2\delta_y^4 + e_{04}\delta_y^8,$$

$$E_1 = (\delta_x^2 + \delta_y^2)^{-1}(e_{11}\delta_x^6 + e_{12}2\delta_x^4\delta_y^2 + e_{13}\delta_x^2\delta_y^4 + e_{14}\delta_y^6),$$

$$E_2 = (\delta_x^2 + \delta_y^2)^{-1}(e_{21}\delta_x^4 + e_{22}\delta_x^2\delta_y^2 + e_{23}\delta_y^4),$$

$$E_3 = e_3\delta_x^4$$

$$E_4 = e_4\delta_x^2\delta_y^2(\delta_x^2 + \delta_y^2)^{-1}.$$

The values of d_{ij} and e_{ij} are given by expressions (67) and (69) in Appendix A.

To investigate stability of rolls we consider exhaustively different asymptotic relations between α and ε and different signs of α. We also consider two limit values of the Prandtl number.

Examples of areas on the (k, R) plane where rolls are stable are shown on Fig. 1 for several values of P. The area of stable rolls found numerically (see section 8) is shaded; the instability boundaries determined analytically are shown by lines. We use the standard notation (Busse and Bolton 1984; Bernoff 1994; Mielke 1997; Getling 1998) for the instability modes and respective instability boundaries. For a skew-varicose (SV) mode the associated eigenvalue is real, and for an oscillatory skew-varicose (OSV) mode the associated eigenvalues are complex; both modes exist for $\delta_x \sim \delta_y$. A zigzag (ZZ) mode emerges for $\delta_x = 0$ and the associated eigenvalue is real. The mode responsible for the instability for $P < 0.782$ exists for $\delta_y \gg \delta_x \neq 0$ and the associated eigenvalue is real; it is also called a skew-varicose mode (see, e.g., Mielke 1997). To distinguish this skew-varicose mode from the SV mode, we label the former SV2.

4.1 The case $\alpha^2 \ll \varepsilon^4$

Let $\delta_x^2 \gg \delta_y^2$ and $\delta_x^2 \ll \varepsilon^2$. Then in the leading order

$$\det\mathcal{A} = \varepsilon^4 d_{21}\delta_y^2. \qquad (27)$$

d_{21} is positive for all P (see (67)), hence $\det\mathcal{A} > 0$ as well, implying that for the assumed δ's the matrix has a positive real eigenvalue and the rolls suffer from monotonous instability (it can be shown that oscillatory instability does not emerge in this case).

4.2 The case $\alpha > 0$

Assume again $\delta_x^2 \gg \delta_y^2$ and $\delta_x^2 \ll \varepsilon^2$. In the leading order the determinant now includes $\alpha-$dependent terms:

$$\det\mathcal{A} = \varepsilon^4 d_{21}\delta_y^2 + \alpha D_3 + \alpha^2 D_4 + \alpha\varepsilon^2 D_5.$$

However, for $\alpha > 0$ the terms involving α are positive. The first term is positive, as discussed in the previous subsection, hence rolls are also monotonously unstable in this case.

4.3 The case $\alpha < 0$, $P < P_1 \approx 0.782$, $\alpha^2 \sim \varepsilon^4$

Denote by P_1 the Prandtl number which is the solution to the equation

$$d_{13} = 2(d_{02}d_{22})^{1/2},$$

i.e.

$$-P^2 + 2P + 2 = 2(2P^2(P+1))^{1/2}. \tag{28}$$

The solution is

$$P_1 = (3-2\sqrt{2})(1+\sqrt{7+4\sqrt{2}}) \approx 0.782$$

(cf. Zippelius and Siggia 1982). If $\delta_x^2 \ll \delta_y^2$, $\delta_x^2 \gg \delta_y^4$ and $\alpha^2 \ll \varepsilon^2$, the sum of asymptotically largest terms in (25) is

$$d_{02}\delta_x^4\delta_y^2 + \varepsilon^2 d_{13}\delta_x^2\delta_y^2 + \varepsilon^4 d_{22}\delta_y^2, \tag{29}$$

which is positive for $P < P_1$. Hence, for $P < P_1$ and the assumed α's rolls are unstable.

4.4 The case $\alpha < 0$, $P > P_1$, $\alpha^2 \sim \varepsilon^4$

Re-write (25) as $\det\mathcal{A} = \mathcal{D}_1 + \mathcal{D}_2$, where

$$\mathcal{D}_1 = D_0 + \varepsilon^2 D_1 + \varepsilon^4 d_{22}\delta_y^4(\delta_x^2+\delta_y^2)^{-1} + \alpha D_3 + \alpha^2 D_4 + \alpha\varepsilon^2 d_{52}\delta_y^4 \tag{30}$$

$$\mathcal{D}_2 = \varepsilon^2(\varepsilon^2 d_{21} + \alpha d_{51})\delta_x^2\delta_y^2(\delta_x^2+\delta_y^2)^{-1}. \tag{31}$$

As proved in Appendix C, $\mathcal{D}_1 < 0$ for $P > P_1$ and the assumed α's. $\mathcal{D}_2 > 0$ if

$$\mathrm{SV}: \quad \varepsilon^2 > -\alpha f_1, \quad f_1 = \frac{d_{51}}{d_{21}} = \frac{108}{7}\pi^2 k. \tag{32}$$

For $\varepsilon^2 \gg \delta_x^2 \gg \delta_y^2$ $\mathcal{D}_2$ is asymptotically larger than $\mathcal{D}_1$, hence (32) yields a boundary for monotonous instability of rolls.

Represent (26) as $S(\mathcal{A})\mathrm{tr}\,\mathcal{A} - \det\mathcal{A} = \mathcal{E}_1 + \mathcal{E}_2$, where

$$\mathcal{E}_1 = E_0 + \varepsilon^2 E_1 + \alpha^2 E_3 \tag{33}$$

$$\mathcal{E}_2 = \varepsilon^2(\varepsilon^2 E_2 + \alpha E_4). \tag{34}$$

$\mathcal{E}_1 < 0$ for $P > P_1$ and assumed α's. $\mathcal{E}_2 > 0$ if

$$e_{22} + e_4 \frac{\alpha}{\varepsilon^2} > 2(e_{21}e_{23})^{1/2},$$

i.e.

$$\mathrm{OSV}: \quad \varepsilon^2 < -\alpha f_2, \quad f_2 = -\frac{e_4}{2(e_{21}e_{23})^{1/2} - e_{22}} = \tag{35}$$

$$\frac{108(P+1)^2\pi^2 k}{(P+3)(3P^2+2P+2) + 3P^2(P+1)^{1/2}(P+5)^{1/2}}.$$

For $\varepsilon^2 \gg \delta_x^2$ and $\varepsilon^2 \gg \delta_y^2$, $\mathcal{E}_2$ is asymptotically larger than $\mathcal{E}_1$, hence (35) is a condition for instability.

4.5 The case $\alpha < 0$, $\alpha^2 \gg \varepsilon^4$

Assuming, as above, $\varepsilon^2 \gg \delta_x^2$ and $\varepsilon^2 \gg \delta_y^2$, we find that $\varepsilon^2\alpha E_4 > 0$ is asymptotically the largest term in (34). Hence, rolls are unstable.

4.6 The case $\alpha < 0$, $\alpha^2 \sim \varepsilon^2$ and large P

For $P \to \infty$, f_1 in (32) has a finite limit, and f_2 in (35) vanishes. Hence, for P sufficiently large the instability under the condition (35) can compete with the instability occurring for $\alpha \sim \varepsilon$. As shown in Appendix D, for large P, $\det\mathcal{A} > 0$, if

$$\mathrm{ZZ}: \quad \varepsilon^2 < f_3\alpha^2, \quad f_3 = \frac{9\pi^2P^2}{2(P+1)}. \tag{36}$$

4.7 The case $\alpha < 0$, $\alpha^2 \sim \varepsilon^4$ and P slightly smaller than P_1

Mielke (1997) showed that for $P < P_1$ rolls near the onset are always unstable. More precisely, the following has been proved: for such P there exists a neighbourhood of the point (k_s, R^s) in the (k, R) plane, such that for a given R rolls of horizontal wave number k (where they exist) are unstable. The question, how the area where rolls are stable is modified, as P becomes smaller than P_1, has not been addressed in literature. We show below that the area of stable rolls does not disappear abruptly and it still exists near the onset, but its boundary does not include the point (k_s, R^s) (cf. figs. 1a,b and figs. 1c-f). As P decreases, the area of stable rolls moves away from this point, because the SV2 boundary moves to the left.

In the search of the horizontal scale ratios for which (25) is positive for $P < P_1$, it was assumed in section 4.3 that $\delta_y^2 \gg \delta_x^2 \gg \delta_y^4$ and then in the leading order $\det\mathcal{A}$

is given by (29). For P slightly smaller than P_1 the expression (29) can be of the same order as other terms in (25) not far from the onset. Consider a new small parameter $\beta = P_1 - P$. The maximum of (29) is admitted for

$$\delta_x^2 = -\frac{d_{13}\varepsilon^2}{2d_{02}} \equiv q\varepsilon^2, \tag{37}$$

and the maximum is equal to

$$-\varepsilon^4\delta_y^2\frac{4(-(-P^2+2P+2)^2+8P^2(P+1))}{81\pi^4P(P+1)^2} \approx d_6\varepsilon^4\delta_y^2\beta,$$

where

$$d_6 = \frac{4(-2(-2P_1+2)(-P_1^2+2P_1+2)+24P_1^2+16P_1)}{81\pi^4P_1(P_1+1)^2} \approx 5.0\cdot10^{-3}.$$

For δ_x defined by (37) in the leading order the determinant is

$$\varepsilon^2(\varepsilon^2(\varepsilon^2(d_{01}+d_{02})q^3+\varepsilon^2d_{12}q^2+\varepsilon^2d_{21}q+\alpha d_{51}q)+\varepsilon^2d_6\delta_y^2\beta+(d_{03}q+d_{14})\delta_y^6) \tag{38}$$

which is a cubic polynomial of δ_y^2. Its maximum is admitted, when

$$\delta_y^4 = -\frac{\varepsilon^2d_6\beta}{3(d_{03}q+d_{14})}, \tag{39}$$

and the maximum of (38) is

$$\varepsilon^4s_3(\varepsilon^2s_1+\varepsilon\beta^{3/2}s_2+\alpha),$$

where

$$s_3 = d_{51}q \approx 0.0299,\ \ s_1 = s_3^{-1}((d_{01}+d_{02})q^3+d_{12}q^2+d_{21}q) \approx 0.00279,$$

$$s_2 = \frac{2}{3}d_6s_3^{-1}\Big(\frac{-d_6}{3(d_{03}q+d_{14})}\Big)^{1/2} \approx 0.0876.$$

The maximum is positive for

$$\alpha > -\varepsilon^2s_1-\varepsilon\beta^{3/2}s_2. \tag{40}$$

However, the boundary of the SV2 instability defined by (40) turns out to be in a poor agreement with the numerical results discussed in section 8. For $P = 0.6$ and $P = 0.7$ (figs. 1a,b) the right boundary (40) of the area of stable rolls is shifted far to the left compared to the computed one. The SV2 boundary defined by the condition $\max_{\delta_x,\delta_y}\det\mathcal{A} = 0$ with all leading terms in (66) retained is still shifted too far to the left. The asymptotics fails because the values (37) and (39) of δ_x and δ_y, respectively, are of the order of 0.1 for the considered overcriticalities $\varepsilon^2 \sim 1$, while the asymptotic analysis is applicable for infinitesimally small δ_x and δ_y. (For example, as we have found numerically, for $P = 0.7$ the intersection of the SV and

SV2 boundaries is at $\alpha = -0.00172$ and $\varepsilon^2 = 0.583$. The respective values (37) and (39) are $\delta_x = 0.126$ and $\delta_y = 0.413$. For the SV2 dominant mode on the stability boundary the computed values are also large, $\delta_x = 0.124$ and $\delta_y = 0.332$, which surprisingly do not differ much from the values obtained analytically.)

Consequently, we follow an alternative approach and assume that the SV2 instability boundary can be described as an equation, where it suffices to retain two first non-vanishing terms in the Taylor expansion in ε and β. The condition for the instability thus takes the form

$$\text{SV2}: \quad \alpha > \varepsilon^2 h_1 + \varepsilon\beta h_2, \tag{41}$$

for some coefficients h_1 and h_2, which can be determined numerically. The intersections of the SV2 boundary with the SV and OSV boundaries have been computed for $P = 0.6$ and $P = 0.7$ (see figs. 1a,b). The minimum (over the coefficients h_i) of the maximum (over the four points of intersection) relative error is equal to 0.14, it is admitted for

$$h_1 = -0.0012 \text{ and } h_2 = -0.018\,. \tag{42}$$

By the relative error we understand the ratio $|(\alpha_c - \alpha_t)/\alpha_c|$, where α_c is the computed value (see section 8) and α_t is found from (41) at the points of intersection. We have also computed several points on the SV2 boundary in the regions of other instabilities of rolls, employing the fact that the respective (local) maximum of λ over δ_x and δ_y is admitted for δ_y much larger than for the other instabilities. The computed values agree well with (41), (42) (see figs. 1a,b). Fitting of s_1 and s_2 in (40) yields a much higher (about 0.5) minimum of the maximum over the four points relative error.

5 The finite angle case

To analyse stability of rolls of horizontal wave number k to rolls of wave number k_p, which are rotated by angle ξ with finite ξ, we use center manifold reduction with the center eigenspace spanned by eigenvectors (9) with wave vectors $(k, 0, \pi)$ and $(k_p \cos\xi, k_p \sin\xi, \pi)$. We perform the reduction like in (Podvigina and Ashwin 2007). Here only results of calculations are presented. Periodicity in horizontal directions of the considered rolls implies that periodicity cells are parallelograms (and not squares as *ibid*).

Restricted to the two-dimensional ($\mathbf{C}^2$) center manifold, the system has the form

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 + z_1(A_1|z_1|^2 + A_2|z_2|^2), \\ \dot{z}_2 &= \lambda_2 z_2 + z_2(A_3|z_1|^2 + A_4|z_2|^2), \end{aligned} \tag{43}$$

where z_1 and z_2 are coordinates in the center manifold, along the directions $(k, 0, \pi)$ and $(k_p \cos\xi, k_p \sin\xi, \pi)$, respectively. The reduction is performed for $R = R_c(k)$. We are interested in k_p close to k (otherwise λ_2 is of the order of one and negative and thus the rolls $(z_1, 0)$ are stable near the onset).

For ε defined by (7) and k_p close to k, the coefficients of linear terms in (43) are

$$\lambda_1 = P(P+1)^{-1}k^2a^{-2}\varepsilon^2,$$
$$\lambda_2 = P(P+1)^{-1}\Big(k^2a^{-2}\varepsilon^2 - 4(k_p-k)^2 - 8\alpha(k_p-k)\Big).$$

For small $k_p - k$ the differences $A_1 - A_4$ and $A_2 - A_3$ are small, coefficients of cubic terms in (43) are

$$A_1 \sim A_4 = -\frac{0.125P}{(P+1)},$$

$$A_3 \sim A_2 = -\frac{0.125P}{(P+1)} - \frac{a(1-\cos^2\xi)}{3(P+1)}\Big((1-\cos\xi)\frac{2aq_+ + 2Pa^2}{P\Delta_+} + (1+\cos\xi)\frac{2aq_- + 2Pa^2}{P\Delta_-}\Big)$$
$$-\frac{\pi^2}{2(P+1)}\Big((1-\cos\xi)^2\frac{Pq_+ + 3h_+a}{\Delta_+} + (1+\cos\xi)^2\frac{Pq_- + 3h_-a}{\Delta_-}\Big),$$

where

$$h_\pm = 2k^2(1\pm\cos\xi),\ q_\pm = 4\pi^2 + h_\pm,\ \Delta_\pm = q_\pm^3 - Rh_\pm.$$

The amplitude of emerging rolls is

$$|z_1|^2 = -\lambda_1/A_1,$$

and four eigenvalues of (43) linearised around the steady state are

$$-2\lambda_1, \quad \lambda_2 + A_3|z_1|^2, \quad 0, \quad 0, \tag{44}$$

hence the instability condition is

$$\varepsilon^2k^2a^{-2}(1 - A_3^{max}/A_1) - 4(k-k_p)^2 - 8\alpha(k-k_p) > 0,$$

i.e. instability occurs if

$$\varepsilon^2 < f_5\alpha^2, \quad f_5 = -\frac{4A_1a^2}{k^2(A_1 - A_3^{max})},$$

where by A_3^{max} we have denoted the maximum of A_3 in ξ.

For a finite P the instability boundary is below the boundary defined by (35). For large P, the limits of A_1 and A_3 are finite, hence $f_5 < f_3$, and finite-angle instabilities do not affect the area of stability of rolls. The instability with respect to rolls rotated by $\xi = \pi/2$ is called the cross-roll instability. Note that the maximum of A_3 can be admitted for a $\xi \neq \pi/2$, but we do not consider here the problem of maximisation of A_3 in ξ.

For ξ close to $2\pi/3$ the center eigenspace also involves rolls with the direction of the axes rotated by $-2\pi/3$. The system restricted to the three-dimensional ($\mathbf{C}^3$) center manifold is

$$\begin{aligned}\dot z_1 &= \lambda_1z_1 + z_1(A_1|z_1|^2 + A_2|z_2|^2 + B_1|z_3|^2),\\ \dot z_2 &= \lambda_2z_2 + z_2(A_3|z_1|^2 + A_4|z_2|^2 + B_2|z_3|^2),\\ \dot z_3 &= \lambda_2z_3 + z_3(A_5|z_1|^2 + A_6|z_2|^2 + B_3|z_3|^2).\end{aligned}\tag{45}$$

However, the eigenvalues determining stability of rolls are (44), examined above.

6 Growth rates

In this section we find orders of growth rates of the dominant unstable modes. If entries of the matrix $\mathcal{A}$ are of different orders, it is possible to calculate dominant eigenvalues, like it was in the case of rotating layer in OP2008. In the present problem often almost all entries of $\mathcal{A}$ turn out to have the same asymptotics, and only orders of growth rates can be determined. Also we find orders of coefficients a_j, j =1,2,3, of the most unstable mode

$$\mathbf{W} = a_1\widetilde{\mathbf{W}}_1 + a_2\widetilde{\mathbf{W}}_2 + a_3\widetilde{\mathbf{W}}_3. \tag{46}$$

Unstable modes (or instabilities) can be roughly categorised into five different types[2]:

$$\begin{aligned} &\mathrm{SV}: && \delta_x \sim \delta_y \\ &\mathrm{SV2}: && \delta_x \ll \delta_y \\ &\mathrm{OSV}: && \delta_x \sim \delta_y \\ &\mathrm{ZZ}: && \delta_x = 0 \\ &\mathrm{E-l}: && 2k\delta_x + \delta_y^2 = -2k\alpha \end{aligned}$$

Similarly to section 4, we consider different asymptotic relations between α and ε. Our findings are summarised in Table 1, where

$$\xi_1 = \frac{(P^2-2P-2)^2 - 8P^2(P+1)}{9\pi^2(P+1)^2P},\ \xi_2 = \frac{4(2k)^{1/2}}{3\pi(P+1)^{1/2}},\ \xi_3 = \frac{4P}{(P+1)},\ \xi_4 = \frac{8}{9\pi^2P}.$$

[2]**E-l** stands for Eckhaus-like instability. Maximisation of the growth rate in δ_x and δ_y yields only the horizontal wave number of the most unstable mode, see OP2008 and section 6.5. The conventional Eckhaus instability is a particular case of the **E-l** instability for $\delta_y = 0$.

Table 1. Possibly dominant instability modes for various asymptotic relations between α and ε and values of P. The last column presents eigenvalues, when they can be calculated, or their orders of magnitude otherwise. (Hence, often it remains unclear which mode is dominant.)

Relations between α and ε	Conditions for existence	Type of the mode	δ_x and δ_y	Eigenvalues
$\alpha^2 \ll \varepsilon^4$	none	SV	$\delta_x \sim \delta_y \sim \varepsilon$	$\lambda \sim \varepsilon^2$
	$P < P_1$	SV2	$\delta_x \ll \delta_y,\ \delta_x \sim \varepsilon$	$\lambda = \xi_1 \varepsilon^2$
$\alpha^2 \sim \varepsilon^4$	$\varepsilon^2 > -f_1\alpha$	SV	$\delta_x \sim \delta_y \sim \varepsilon$	$\lambda \sim \varepsilon^2$
	$P < P_1$	SV2	$\delta_x \ll \delta_y,\ \delta_x \sim \varepsilon$	$\lambda = \xi_1 \varepsilon^2$
	$\varepsilon^2 < -f_2\alpha$	OSV	$\delta_x \sim \delta_y \sim \varepsilon$	$\mathrm{Re}(\lambda) \sim \varepsilon^2$
$\varepsilon^4 \ll \alpha^2 \ll \varepsilon^{4/3}$	$\alpha > 0$	SV	$\delta_x^2 \sim \delta_y^2 \ll \varepsilon\alpha^{1/2}$	$\lambda = \xi_2 \varepsilon \alpha^{1/2}$
or	$\alpha < 0$,	OSV	$\delta_x^2 \sim \delta_y^2 \sim \varepsilon\alpha^{1/2}$	$\mathrm{Re}(\lambda) \sim \varepsilon\alpha^{1/2}$
$\alpha^2 \sim \varepsilon^{4/3}$	$\varepsilon^2 < -f_2\alpha$			
$\alpha^2 \sim \varepsilon^2$	$\alpha < 0$,	ZZ	$\delta_x = 0,\ \delta_y \sim \varepsilon$	$\lambda = -\xi_4\varepsilon^2 + \xi_3\alpha^2$
	$\varepsilon^2 > -f_2\alpha$,			
	$\varepsilon^2 < f_3\alpha^2$			
$\alpha^2 \gg \varepsilon^{4/3}$	none	E-l	$2k\delta_x + \delta_y^2 = -2k\alpha$	$\lambda = \xi_3\alpha^2$

6.1 The case $\alpha^2 \ll \varepsilon^4$

In this case there exists a growing mode

$$\mathrm{SV}: \quad \lambda \sim \varepsilon^2, \text{ for } \delta_x^2 \sim \delta_y^2 \sim \varepsilon^2,$$

since it can be easily shown that $\det \mathcal{A} > 0$ for some

$$\delta_x^2 \sim \delta_y^2 \sim \varepsilon^2. \tag{47}$$

If (47) holds, $\mathrm{tr}\, \mathcal{A} \sim \varepsilon^2$, $S\mathcal{A} \sim \varepsilon^4$ and $\det \mathcal{A} \sim \varepsilon^6$, implying that eigenvalues are $\sim \varepsilon^2$. For the assumed dependence of δ's on ε, after the change of variables $\widetilde{\mathbf{W}}_2 \to \varepsilon\widetilde{\mathbf{W}}_2$ all coefficients (except for $\widetilde{A}_{32}$) become of the same order in ε, implying that for the associated eigenmode $a_1/a_2 \sim a_3/a_2 \sim \varepsilon$.

For $P < P_1$ the expression (66) can be positive also, if $\delta_y^2 \gg \delta_x^2 \sim \varepsilon^2$. For $\delta_x^2 + \delta_y^2 \gg \varepsilon$, the matrix (65) has an eigenvalue close to $-P(\delta_x^2 + \delta_y^2)$. In the leading order the associated eigenvector is $\widetilde{\mathbf{W}}_1+\xi_2\widetilde{\mathbf{W}}_2+\xi_3\widetilde{\mathbf{W}}_3$, where $\xi_2=\widetilde{A}_{21}/(\widetilde{A}_{11}-\widetilde{A}_{22})$ and $\xi_3=\widetilde{A}_{31}/(\widetilde{A}_{11}-\widetilde{A}_{33})$. Two remaining eigenvalues are eigenvalues of the matrix

$$\left[\begin{array}{cc}\widetilde{A}_{22}-\xi_2\widetilde{A}_{12} & \widetilde{A}_{23}-\xi_2\widetilde{A}_{13}\\ \widetilde{A}_{32}-\xi_3\widetilde{A}_{12} & \widetilde{A}_{33}-\xi_3\widetilde{A}_{13}\end{array}\right]. \tag{48}$$

In the leading order they are

$$\lambda=-C_4(4k^2\delta_x^2+\delta_y^4+4\alpha k\delta_y^2)-\varepsilon^2C_3+\varepsilon^2C_5\delta_y^2\frac{(3\delta_x^2-\delta_y^2)}{(\delta_x^2+\delta_y^2)^2} \tag{49}$$

$$\pm\left[\varepsilon^4\left(C_3+C_5\delta_y^2\frac{(3\delta_x^2-\delta_y^2)}{(\delta_x^2+\delta_y^2)^2}\right)^2+4C_4k^2\delta_x^2(\delta_y^2+2\alpha k)\left(\varepsilon^2C_5\frac{\delta_y^2}{(\delta_x^2+\delta_y^2)^2}-4C_4(\delta_y^2+2\alpha k)\right)\right]^{1/2},$$

where

$$C_5=\frac{b^2\pi^2}{2k^2P}.$$

Calculating their maxima in δ_x and δ_y we find the most unstable mode:

$$\mathrm{SV2}:\quad \lambda=\varepsilon^2\frac{(P^2-2P-2)^2-8P^2(P+1)}{9\pi^2(P+1)^2P},\ \text{for}$$

$$\delta_x^2=\varepsilon^2\frac{8(P+1)^2-(P^2-2P-2)^2}{18(P+1)P^2},\ \delta_y\gg\delta_x.$$

These relations between δ's, ε and λ imply that the associated eigenvector (a_1,a_2,a_3) of the matrix $\widetilde{\mathcal{A}}$ (65) has components with the asymptotics $a_1/a_2\sim\varepsilon^2\delta_y^{-1}$ and $a_3/a_2\sim\varepsilon$.

6.2 The case $\alpha^2\sim\varepsilon^4$

Dominant eigenvalues and asymptotic relations between the coefficients a_1,a_2,a_3 for the eigenmodes SV and SV2 are the same as above. The SV mode is growing if (32) holds, and SV2 if $P<P_1$.

A growing oscillatory mode can exist, if (35) holds true. As discussed in Appendix B, condition (24) does not guarantee its existence. However, if such mode exists for all $\alpha^2\gg\varepsilon^4$, $\alpha<0$, (see section 6.4), by continuity it exists for some $\alpha^2\sim\varepsilon^4$. The maximal growth rate of the mode

$$\mathrm{OSV}:\quad \mathrm{Re}(\lambda)\sim\varepsilon^2$$

is admitted for

$$\delta_x^2\sim\delta_y^2\sim\varepsilon^2.$$

By the same arguments as for the SV mode, for the OSV eigenmode with the maximal growth rate and $a_1/a_2\sim a_3/a_2\sim\varepsilon$.

6.3 The case $\varepsilon^{4/3} \gg \alpha^2 \gg \varepsilon^4$, $\alpha > 0$

We employ the same change of variables as in section 6.1, and maximisation in δ_x and δ_y yields the maximal growth rate for SV modes

$$\mathrm{SV}: \quad \lambda = \frac{1}{2P^{1/2}} d_{51}^{1/2} \varepsilon \alpha^{1/2} = \frac{4(2k)^{1/2}}{3\pi(P+1)^{1/2}} \varepsilon \alpha^{1/2}, \text{ for } \delta_x^2 = \delta_y^2 \ll \varepsilon \alpha^{1/2}. \tag{50}$$

The eigenmode coefficients have the asymptotics $a_1/a_2 \sim \varepsilon\alpha^{-1/2}$ and $a_3/a_2 \sim \varepsilon^2(\alpha^{1/2}\delta_x)^{-1}$. The SV2 mode has the growth rate $\mathrm{O}(\varepsilon^2, \alpha^2)$, which is asymptotically smaller than (50).

6.4 The case $\varepsilon^{4/3} \gg \alpha^2 \gg \varepsilon^4$, $\alpha < 0$

For the dominant oscillatory mode the maximal growth rate and (δ_x, δ_y) for which it is admitted are:

$$\mathrm{OSV}: \quad \mathrm{Re}(\lambda) \sim \varepsilon\alpha^{1/2}, \text{ at } \delta_x^2 \sim \delta_y^2 \sim \varepsilon\alpha^{1/2}. \tag{51}$$

This can be obtained by the following arguments. Assume

$$\delta_x^2 \sim \delta_y^2 \ll \varepsilon\alpha^{1/2}. \tag{52}$$

Consider the cubic equation $\det(\lambda\mathbf{I} - \mathcal{A}) = 0$. The assumption (52) implies, by virtue of the standard formulae for roots of cubic equations, existence of complex roots with a positive $\mathrm{Re}\lambda \sim (\det \mathcal{A})^{1/3} \sim \varepsilon^2\alpha\delta_x^2\delta_y^2(\delta_x^2+\delta_y^2)^{-1}$. Since this holds true for any (δ_x, δ_y) satisfying (52), this relation remains true for a $\delta_x^2 \sim \delta_y^2 \sim \varepsilon\alpha^{1/2}$. The last two asymptotic relations imply that the associated eigenvector (a_1, a_2, a_3) of the matrix $\widetilde{\mathcal{A}}$ (65) has components with the asymptotics $a_1/a_2 \sim \alpha^{1/2}$ and $a_3/a_2 \sim \alpha^{3/4}\varepsilon^{-1/2}$. For the SV2 mode the growth rate is $\sim \varepsilon^2$ or $\sim \alpha^2$, i.e. it is asymptotically smaller than (51).

6.5 The case $\alpha^2 \gg \varepsilon^{4/3}$

Maximisation of (49) in δ_x and δ_y yields that the maximal growth rate

$$\mathrm{E-I}: \quad \lambda = 4C_4k^2\alpha^2 \tag{53}$$

is admitted for

$$(k_\pm^2 - k^2) = -2\alpha k;$$

the associated eigenvectors are either $\mathbf{W}_2$ or $\mathbf{W}_3$. (Note that growth rates are asymptotically smaller than (53), unless $\delta_x^2 + \delta_y^2 \gg \varepsilon$; if this asymptotic relation is satisfied, (49) employed in maximisation is valid, see section 6.1.) Alternatively, (53) can be obtained directly from (64). For the OSV mode $\mathrm{Re}(\lambda) \sim \varepsilon^{4/3}$, which is asymptotically smaller, than (53).

6.6 The case $\alpha^2 \sim \varepsilon^{4/3}$

The maximal growth rate is $\mathrm{O}(\alpha^2) \sim \mathrm{O}(\varepsilon^{4/3})$. For $\alpha > 0$ it is given by (50), if

$$\varepsilon^2 > \alpha^3 \frac{9\pi^2 P}{2k(P+1)},$$

or by (53) otherwise. For $\alpha < 0$ it is either (51), or (53).

6.7 The case $\alpha^2 \sim \varepsilon^2$, $\alpha < 0$ and large P

As noted in section 4.7, in the limit $P \to \infty$ the ZZ instability with $\delta_x = 0$ competes with the OSV instability and becomes of importance near the onset. If $\delta_x = 0$ and $\delta_y \gg \varepsilon$, the eigenvalues of the matrix $\mathcal{A}$ are

$$\lambda_1 = -P\delta_y^2, \quad \lambda_2 = -2\varepsilon^2 C_3 - C_4\xi, \quad \lambda_3 = -\varepsilon^2 b^2 \frac{\pi^2}{4Pk^2} - C_4\xi,$$

where

$$\xi = \alpha k \delta_y^2 + \delta_y^4/4.$$

For large P, $\lambda_3 > \lambda_2$, maximisation of λ_3 in δ_y yields the maximal growth rate

$$\lambda_{\max} = -\varepsilon^2 b^2 \frac{\pi^2}{4Pk^2} + 4C_4 k^2 \alpha^2.$$

The associated eigenvector has asymptotics $a_1/a_3 \sim 1$ and $a_2 = 0$.

7 Asymptotics of neglected terms in equations for stability boundaries

Expressions (32), (35) and (36) determining stability of rolls are only asymptotically correct. In this section we estimate the asymptotic order of errors in calculation of boundaries, relying on the known orders of the remainder terms in (66) and (68).

In the course of derivation of an equation defining the SV instability boundary, $\det \mathcal{A}$ has been expressed in section 4.4 as a sum of $\mathcal{D}_1$ (30) and $\mathcal{D}_2$ (31), where $\mathcal{D}_1$ is negative and involves terms $\mathrm{O}(\delta^6 + \varepsilon^2\delta^4 + \varepsilon^4\delta_y^4\delta^{-2})$ (here $\delta^2 = \delta_x^2 + \delta_y^2$) and $\mathcal{D}_2$ can be positive and involves terms $\mathrm{O}(\varepsilon^4\delta_x^2\delta_y^2\delta^{-2})$. The inequality (32) is a restatement of the condition $\mathcal{D}_2 > 0$. Near the boundary $\varepsilon^2 \gg \delta_x^2 \gg \delta_y^2$ must be satisfied so that the sum (31) were positive. Under this condition $\mathcal{D}_1$ is asymptotically smaller than $\mathcal{D}_2$ and hence asymptotic corrections to $\mathcal{D}_1$ do not affect the boundary. Upon reintroduction of the terms omitted in (66), that are not asymptotically smaller than $\mathcal{D}_1$, (31) becomes

$$\mathcal{D}_2 = \varepsilon^2\Big(\varepsilon^2(d_{21} + \mathrm{O}(\varepsilon^2, \alpha)) + \alpha(d_{51} + \mathrm{O}(\varepsilon^2, \alpha))\Big)\delta_x^2\delta_y^2(\delta_x^2 + \delta_y^2)^{-1}$$

and thus the equation for the boundary takes the form

$$\mathrm{SV}: \quad \varepsilon^2 > -\alpha f_1 + \mathrm{O}(\alpha^2). \tag{54}$$

Since at the boundary the factor in front of $\delta_x^2\delta_y^2\delta^{-2}$ vanishes, it can be shown using this analysis that the condition

$$\max_{\delta_x,\delta_y}\det\mathcal{A}=0,$$

defining the boundary, implies $\delta_x=\delta_y=0$.

The OSV instability boundary (35) has been found from the condition that $\mathcal{E}_2$ (34) vanishes. By the same arguments as above, near the boundary $\varepsilon^2\gg\delta_x^2\sim\delta_y^2$ must be satisfied for the sum of $\mathcal{E}_1$ and $\mathcal{E}_2$ to be positive and hence asymptotic corrections to $\mathcal{E}_1$ again do not affect the boundary. With the omitted terms of (68) reintroduced, (34) becomes

$$\mathcal{E}_2=\varepsilon^2\Big((\delta_x^2+\delta_y^2)^{-1}(\varepsilon^2(e_{21}+\mathrm{O}(\varepsilon^2,\alpha))\delta_x^4+(\varepsilon^2e_{22}+\alpha e_4+\mathrm{O}(\varepsilon^4,\alpha^2))\delta_x^2\delta_y^2+\varepsilon^2(e_{23}+\mathrm{O}(\varepsilon^2,\alpha))\delta_y^4\Big).$$

This expression results in the equation for the instability boundary in the form

$$\mathrm{OSV}:\quad \varepsilon^2<-\alpha f_2+\mathrm{O}(\alpha^2). \tag{55}$$

Again, it can be shown that at the boundary $\delta_x=\delta_y=0$.

In Appendix D the ZZ instability boundary is calculated from the condition that

$$\max_{\delta_y}\det\mathcal{A}=0,$$

where $\det\mathcal{A}$ is given by (78). Since on the boundary $\alpha^2\sim\varepsilon^2$, the equation with the omitted terms reintroduced takes the form

$$\max_{\delta_y}\Big(-\delta_y^2(2\varepsilon^2C_3+C_4\xi)(\varepsilon^2b^2\frac{\pi^2}{4k^2}+PC_4\xi)+\mathrm{O}(\delta_y^{14},\alpha^2\delta_y^8,\alpha^6\delta_y^2)\Big)=0. \tag{56}$$

The maximum is attained for

$$\delta_y^2=2\alpha k+\mathrm{O}(\alpha^2)$$

and the condition for the instability is thus

$$\mathrm{ZZ}:\quad \varepsilon^2<f_3\alpha^2+\mathrm{O}(\alpha^3).$$

8 Stability of rolls: numerical results

To examine stability of rolls of wave number k, we solve numerically (with an adapted version of the code of Zheligovsky 1993) the problem (15) for the eigenfunction

$$\mathbf{W}=\mathrm{e}^{\mathrm{i}\delta_x x+\mathrm{i}\delta_y y}\sum_{m=-M}^{m=M}\sum_{n=0}^{n=N}\begin{pmatrix} w_{mn}^1\mathrm{e}^{\mathrm{i}mkx}\cos\pi nz\\ w_{mn}^2\mathrm{e}^{\mathrm{i}mkx}\cos\pi nz\\ w_{mn}^3\mathrm{e}^{\mathrm{i}mkx}\sin\pi nz\\ w_{mn}^4\mathrm{e}^{\mathrm{i}mkx}\sin\pi nz\end{pmatrix}. \tag{57}$$

In computations, the cut off of the series at $N = M = 15$ suffices (the spectrum of the solution in the Fourier space decays by at least 12 orders of magnitude). Location of the maximum of $\mathrm{Re}(\lambda)$ in δ_x and δ_y has been determined with the precision of 10^{-4} (or $2 \cdot 10^{-5}$, if δ_x and δ_y are below 0.01) which allows us to find correctly at least two significant digits of λ.

The dominant eigenvalues of (15),(57) and the values of δ_x and δ_y where the maximum is admitted are shown on fig. 2 for $P = 2$ and $k = 2.15$ (thin vertical line on fig. 1c). In the interval $658.5 \leq R \leq 658.8$, i.e. for a small overcriticality, the Eckhaus mode with $\delta_y = 0$ is dominant, the values of δ_x and λ are close to the ones given in the Table (according to the Table, $\lambda = 0.0136$ and $\delta_x = 0.071$). In the interval $658.9 \leq R \leq 670$ the dominant eigenvalues are complex, they are associated with the OSV eigenmode. The change of the type of the dominant mode implies a discontinuity of δ's. In the interval $658.9 \leq R \leq 661$, $\mathrm{Re}(\lambda)$ depends linearly on $\varepsilon = (R - R_c)^{1/2}$, in agreement with the Table (in fact, for smaller R, where the instability is subdominant, this asymptotics for the eigenvalue of the OSV mode was also confirmed numerically), as predicted in section 6. For higher R the dependence is different, because, as noted in section 7, near the SV and OSV boundaries δ_x and δ_y become asymptotically smaller than ε, while in section 6 we have assumed $\delta_x = \mathrm{O}(\varepsilon)$ and $\delta_y = \mathrm{O}(\varepsilon)$. In agreement with section 7, we observe that $\mathrm{Re}(\lambda)$, δ_x and δ_y vanish at R_{OSV} and R_{SV}, where R_{OSV} and R_{SV} denote the critical values of R for the OSV and SV instabilities. In the interval $(R_{\mathrm{OSV}}, R_{\mathrm{SV}})$, where rolls are stable, the maximal growth rate is zero, admitted for $\delta_x = \delta_y = 0$. For $R > R_{\mathrm{SV}}$ the SV mode is dominant. Near R_{OSV} and R_{SV}, $\mathrm{Re}(\lambda) \sim (R_{\mathrm{OSV}} - R)^2$ and $\mathrm{Re}(\lambda) \sim (R_{\mathrm{SV}} - R)^2$, respectively (while no power law asymptotics has been found for δ_x and δ_y, except for δ_y is almost linear near R_{SV}). Consequently, the SV and OSV boundaries are found by linear extrapolation of $(\mathrm{Re}(\lambda))^{1/2}$ through two computed points close to the boundary. Near the ZZ and SV2 boundaries λ depends on R linearly, and we find the instability boundary by linear interpolation.

The areas of stable rolls found numerically are shown on fig. 1 for several values of P. The difference between the SV, OSV and ZZ boundaries predicted theoretically and found numerically agrees with the estimations of the remainder terms obtained in section 7. For small α and ε, the theoretical and numerical boundaries visually coincide, and the discrepancy remains small on increasing α. The area of stable rolls found numerically is shifted up compared to the one determined analytically, indicating that the contribution of the omitted terms is positive for OSV and ZZ instabilities, and negative for the SV instability.

In view of the good agreement of the analytical and numerical results for these three boundaries, the disagreement for the SV2 boundary is surprising. A possible explanation is that the omitted in (66) terms involving δ_x (which are of no importance for the SV and ZZ instabilities for which $\delta_x = 0$ – e.g., $C\varepsilon^2\delta_x^4\delta_y^2$) can turn out to be relatively strongly negative and come into play already at $\varepsilon = 1$. A more plausible explanation is that for the SV2 mode the employed asymptotic expansions of the operator of linearisation L, its eigenvectors and eigenvalues are valid for much smaller α and ε than for other instabilities, because the values of δ_x and δ_y at the SV2 boundary, maximising the eigenvalue, are relatively large. Note, that the same asymptotic expansion was employed in other analytic studies of the problem, cited in the Introduction.

9 Conclusion

We have presented a complete analytical study of stability of rolls near the onset of convection to perturbations, which are doubly periodic in horizontal directions. In all earlier studies only instabilities of rolls to certain classes of perturbations were shown.

In pursuit of this goal, we have, first, shown that without any loss of generality any instability mode is responsible for either the small-angle, or finite-angle instability. Second, for the small-angle instability modes we have derived inequalities determining regions of stability of rolls. The problem involves four small parameters; while deriving the instability conditions we have considered all asymptotic relations between the small parameters. Finally, we have calculated boundaries for the finite-angle instability; it turns out that consideration of finite-angle instability modes does not modify the region of stability of rolls.

In our analysis only the asymptotically largest terms have been taken into account. A question often arises, whether enough terms of asymptotic expansions have been calculated at various intermediate stages. In (65) orders of the omitted terms in the matrix are given, implying that the omitted terms in expressions (66) and (68), used here to analyse stability, are irrelevant sufficiently close to onset.

This small-angle instability of rolls was studied before, and the SV, OSV and ZZ instability boundaries found here coincide with the earlier results. Our novel results concerning the stability boundaries include the following ones: We have examined the dependence of the SV2 boundary on P for $0.543 < P < 0.782$. For decreasing P, the boundary of the region of stable rolls on the (k, R) plane moves to the left away from the point (k_s, R^s). We have established the asymptotics of the maximum growth rates and the associated eigenmodes (see (46)) considering exhaustively different relations between α and ε. We have derived asymptotic equations describing the regions of the instabilities and estimated remainders in these equations.

The approach that we have followed can be applied to study instabilities of stripe patterns with respect to large-scale perturbations in a generic system, where a large-scale neutral mode exists. Existence of the invariant subspace relies only on the structure of equations of convection, where the linear part preserves wave vectors and nonlinearity is of the second order. Equations ((61) and (63)) defining the entries of matrix $\mathcal{A}$ are general, they remain valid for any other system defined by arbitrary mappings L_j. Stability is analysed by examining the inequalities (23) and (24). This analysis is, perhaps, the most difficult part. It may change significantly for other systems with different asymptotics of the entries of the matrix $\mathcal{A}$, resulting in different asymptotics involved in the inequalities defining instability regions.

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A Calculation of the matrix $\mathcal{A}$

In this Appendix we calculate in the leading order the entries of the matrix $\mathcal{A}$ of the restriction of L on the invariant subspace spanned by $\mathbf{W}_j$, $j = 1,2,3$, and expressions for $\det\mathcal{A}$ and $S(\mathcal{A})\mathrm{tr}\,\mathcal{A} - \det\mathcal{A}$ used to deduce the stability properties of rolls. The matrix, the operator and the basis are expanded in a power series in ε, whose coefficients depend on small parameters α, δ_x and δ_y. Note that by virtue of (21),(22) the action of the mapping $(\delta_x,\delta_y) \to (-\delta_x,-\delta_y)$ amounts to permutation of indices $\mathbf{W}_{2,0} \leftrightarrow \mathbf{W}_{3,0}$. Consequently,

$$\begin{aligned} &A_{12}(\delta_x,\delta_y) = A_{13}(-\delta_x,-\delta_y),\ A_{21}(\delta_x,\delta_y) = A_{31}(-\delta_x,-\delta_y),\\ &A_{23}(\delta_x,\delta_y) = A_{32}(-\delta_x,-\delta_y),\ A_{22}(\delta_x,\delta_y) = A_{33}(-\delta_x,-\delta_y). \end{aligned} \tag{58}$$

Vector fields $\mathbf{W}_{j,0}$ (20)-(22), representing terms of order zero in ε in the series (17), are eigenfunctions of L_0:

$$L_0\mathbf{W}_{j,0} = \lambda_{j,0}\mathbf{W}_{j,0}, \tag{59}$$

hence $A_{jj,0} = \lambda_{j,0}$ and $A_{ij,0} = 0$ for $i \neq j$. The following relations were established in OP2008:

$$\lambda_{1,0} = -P(\delta_x^2+\delta_y^2),$$

$$\lambda_{j,0} = -C_4((k_\pm^2-k^2)^2 + 4\alpha k(k_\pm^2-k^2)) + \mathrm{O}((k_\pm^2-k^2)^3, \alpha^2(k_\pm^2-k^2)),\ j=2,3,$$

$$C_4 = 3P(4gk^2)^{-1}, \quad g = \frac{1}{4}((2\pi^2-k^2)k^{-2}+3P).$$

Here and below in this Appendix, plus is assumed in place of $\pm$ for $j=2$, and minus for $j=3$.

The second and third terms of the series (13) are

$$\begin{aligned} L_1(\mathbf{v},\theta) &= (\mathbf{U}_1^{\mathrm{flow}} \times (\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{U}_1^{\mathrm{flow}}), -(\mathbf{U}_1^{\mathrm{flow}}\cdot\nabla)\theta - (\mathbf{v}\cdot\nabla)\mathbf{U}_1^{\mathrm{temp}}),\\ L_2(\mathbf{v},\theta) &= (\mathbf{U}_2^{\mathrm{flow}} \times (\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{U}_2^{\mathrm{flow}}) + P\theta\mathbf{e}_z, -(\mathbf{U}_2^{\mathrm{flow}}\cdot\nabla)\theta - (\mathbf{v}\cdot\nabla)\mathbf{U}_2^{\mathrm{temp}}). \end{aligned} \tag{60}$$

The ε order entries of the matrix are calculated from the relation

$$L_0\mathbf{W}_{j,1} + L_1\mathbf{W}_{j,0} = \lambda_{j,0}\mathbf{W}_{j,1} + \sum_{i=1}^{3} A_{ij,1}\mathbf{W}_{i,0}. \tag{61}$$

Since the operator L_0 is self-adjoint with respect to the scalar product

$$(\mathbf{w}_1, \mathbf{w}_2) = \mathbf{w}_1^{\text{flow}} \cdot \mathbf{w}_2^{\text{flow}} + PR_c \mathbf{w}_1^{\text{temp}} \cdot \mathbf{w}_2^{\text{temp}}, \tag{62}$$

the scalar product of (61) with $\mathbf{W}_{i,0}$ yields

$$A_{ij,1} = (\mathbf{W}_{i,0}, \mathbf{W}_{i,0})^{-1}(L_1 \mathbf{W}_{j,0}, \mathbf{W}_{i,0}),$$

which gives the $\mathrm{O}(\varepsilon)$ terms of the matrix

$$A_{j1,1} = \pm\tfrac{1}{2}kb\delta_y + bC_2\delta_x\delta_y + \mathrm{O}(\delta^3, \alpha\delta^2),$$

$$A_{1j,1} = (\delta_x^2 + \delta_y^2)^{-1}(-b\delta_x\delta_y\pi^2(2k^2)^{-1} \pm b\pi^2(4k^3)^{-1}\delta_y(3\delta_x^2 - \delta_y^2) + \mathrm{O}(\delta^4, \alpha\delta^3)),$$

where $j = 2, 3$ and $C_2 = (Pk^2 + \pi^2)(P+1)^{-1}(\pi^2 + k^2)^{-1}$. The remaining entries $A_{ij,1}$ vanish. We use the notation $\mathrm{O}(\delta^N) = \mathrm{O}(\sum_{n=0}^{N} \delta_x^n \delta_y^{N-n})$.

Approximations to $\mathbf{W}_{j,1}$ for $j = 2, 3$ are also found from (61). The $\mathrm{O}(\varepsilon^2)$ entries are calculated using the equation

$$L_0\mathbf{W}_{j,2} + L_1\mathbf{W}_{j,1} + L_2\mathbf{W}_{j,0} = \lambda_{j,0}\mathbf{W}_{j,2} + \sum_{i=1}^{3} A_{ij,1}\mathbf{W}_{i,1} + \sum_{i=1}^{3} A_{ij,2}\mathbf{W}_{i,0}; \tag{63}$$

the non-vanishing terms are

$$A_{jj,2} = -C_3 \pm H_1(k\pi^2)^{-1}\delta_x + \mathrm{O}(\delta^2, \alpha\delta),$$
$$A_{ji,2} = -C_3 \pm H_2(k\pi^2)^{-1}\delta_x + \mathrm{O}(\delta^2, \alpha\delta),$$

where $i, j = 2, 3$, $C_3 = Pk^2(P+1)^{-1}a^{-2}$ and $H_1 + H_2 = 4P(27(P+1))^{-1}$ (in what follows only this sum is important).

Finally, entries of the matrix $\mathcal{A}$ are

$$\begin{aligned}
A_{11} &= -P(\delta_x^2 + \delta_y^2) + \mathrm{O}(\varepsilon^2\delta^2), \\
A_{21} &= \frac{1}{2}kb\varepsilon\delta_y + \varepsilon bC_2\delta_x\delta_y + \mathrm{O}(\varepsilon\delta^3, \varepsilon\alpha\delta^2, \varepsilon^3), \\
A_{31} &= -\frac{1}{2}kb\varepsilon\delta_y + \varepsilon bC_2\delta_x\delta_y + \mathrm{O}(\varepsilon\delta^3, \varepsilon\alpha\delta^2, \varepsilon^3), \\
A_{12} &= \varepsilon\Big(-\frac{b\pi^2}{2k^2}\delta_x\delta_y + \frac{b\pi^2}{4k^3}\delta_y(3\delta_x^2 - \delta_y^2)\Big)(\delta_x^2 + \delta_y^2)^{-1} + \mathrm{O}(\varepsilon\delta^2, \varepsilon\alpha\delta, \varepsilon^3), \\
A_{22} &= -\varepsilon^2 C_3 + \varepsilon^2\frac{1}{k\pi^2}H_1\delta_x - C_4\Big((k_+^2 - k^2)^2 + 4\alpha k(k_+^2 - k^2)\Big) \\
&\quad +\mathrm{O}((k_+^2 - k^2)^3, \alpha^2(k_+^2 - k^2), \varepsilon^2\delta^2, \varepsilon^4), \\
A_{32} &= -\varepsilon^2 C_3 + \varepsilon^2\frac{1}{k\pi^2}H_2\delta_x + \mathrm{O}(\varepsilon^2\delta^2, \varepsilon^4), \\
A_{13} &= \varepsilon\Big(-\frac{b\pi^2}{2k^2}\delta_x\delta_y - \frac{b\pi^2}{4k^3}\delta_y(3\delta_x^2 - \delta_y^2)\Big)(\delta_x^2 + \delta_y^2)^{-1} + \mathrm{O}(\varepsilon\delta^2, \varepsilon\alpha\delta, \varepsilon^3), \\
A_{23} &= -\varepsilon^2 C_3 - \varepsilon^2\frac{1}{k\pi^2}H_2\delta_x + \mathrm{O}(\varepsilon^2\delta^2, \varepsilon^4), \\
A_{33} &= -\varepsilon^2 C_3 - \varepsilon^2\frac{1}{k\pi^2}H_1\delta_x - C_4\Big((k_-^2 - k^2)^2 + 4\alpha k(k_-^2 - k^2)\Big) \\
&\quad +\mathrm{O}((k_-^2 - k^2)^3, \alpha^2(k_-^2 - k^2), \varepsilon^2\delta^2, \varepsilon^4).
\end{aligned} \tag{64}$$

In the new basis $\widetilde{\mathbf{W}}_1 = \mathbf{W}_1,\ \widetilde{\mathbf{W}}_2 = \mathbf{W}_2 + \mathbf{W}_3,\ \widetilde{\mathbf{W}}_3 = \mathbf{W}_2 - \mathbf{W}_3$ the matrix of the operator L is

$$
\begin{aligned}
\widetilde{A}_{11} &= -P(\delta_x^2+\delta_y^2) + \mathrm{O}(\varepsilon^2\delta^2),\\
\widetilde{A}_{21} &= \varepsilon b C_2 \delta_x \delta_y + \mathrm{O}(\varepsilon\delta^4, \varepsilon\alpha\delta^2, \varepsilon^3),\\
\widetilde{A}_{31} &= \frac{1}{2} k b \varepsilon \delta_y + \mathrm{O}(\varepsilon\delta^3, \varepsilon^3),\\
\widetilde{A}_{12} &= -\varepsilon b \delta_x \delta_y \frac{\pi^2}{k^2}(\delta_x^2+\delta_y^2)^{-1} + \mathrm{O}(\varepsilon\delta^2, \varepsilon^3),\\
\widetilde{A}_{22} &= -2\varepsilon^2 C_3 - C_4(4k^2\delta_x^2 + \delta_y^4 + 4\alpha k \delta_y^2) + \mathrm{O}(\alpha(\delta_x^2+\delta_y^4), \delta_x^4, \delta_y^6, \varepsilon^2\delta^2, \varepsilon^4),\\
\widetilde{A}_{32} &= -4C_4 k \delta_x(\delta_y^2 + 2\alpha k) + \varepsilon^2 \frac{1}{k\pi^2}(H_1 - H_2)\delta_x + \mathrm{O}(\alpha^2\delta_x, \delta_x^3, \delta_x\delta_y^4, \varepsilon^2\delta^3, \varepsilon^4\delta),\\
\widetilde{A}_{13} &= \varepsilon \frac{b\pi^2}{2k^3}\delta_y(3\delta_x^2 - \delta_y^2)(\delta_x^2+\delta_y^2)^{-1} + \mathrm{O}(\varepsilon\delta^3, \varepsilon\alpha\delta, \varepsilon^3),\\
\widetilde{A}_{23} &= -4C_4 k \delta_x(\delta_y^2 + 2\alpha k) + \varepsilon^2 \frac{1}{k\pi^2}(H_1 + H_2)\delta_x + \mathrm{O}(\alpha^2\delta_x, \delta_x^3, \delta_x\delta_y^4, \varepsilon^2\delta^3, \varepsilon^4\delta),\\
\widetilde{A}_{33} &= -C_4(4k^2\delta_x^2 + \delta_y^4 + 4\alpha k \delta_y^2) + \mathrm{O}(\alpha(\delta_x^2+\delta_y^4), \delta_x^4, \delta_y^6, \varepsilon^2\delta^2, \varepsilon^4).
\end{aligned}
\tag{65}
$$

When calculating (65) with the use of (64), relations (58) were employed to estimate the omitted terms.

From (65) we obtain

$$
\begin{aligned}
&\det\mathcal{A} = d_{01}\delta_x^6 + d_{02}\delta_x^4\delta_y^2 + d_{03}\delta_x^2\delta_y^6 + d_{04}\delta_y^{10}\\
&+\varepsilon^2(\delta_x^2+\delta_y^2)^{-1}(d_{11}\delta_x^6 + d_{12}\delta_x^4\delta_y^2 + d_{13}\delta_x^2\delta_y^4 + d_{14}\delta_y^8)\\
&+\varepsilon^4(\delta_x^2+\delta_y^2)^{-1}(d_{21}\delta_x^2\delta_y^2 + d_{22}\delta_y^4) + \alpha(d_{31}\delta_x^2\delta_y^4 + d_{32}\delta_y^8)\\
&+\alpha^2(d_{41}\delta_x^4 + d_{42}\delta_x^2\delta_y^2 + d_{43}\delta_y^6) + \alpha\varepsilon^2(d_{51}\delta_x^2\delta_y^2(\delta_x^2+\delta_y^2)^{-1} + d_{52}\delta_y^4)\\
&+\mathrm{O}(\delta^2(\delta_x^2+\delta_y^4)^3, \varepsilon^2\delta_x^2\delta^4, \varepsilon^2\delta_y^8, \varepsilon^4\delta^4, \varepsilon^6\delta^2, \alpha\delta_y^4(\delta_x^2+\delta_y^4)^2, \alpha^2\delta^2(\delta_x^2+\delta_y^4)^2, \alpha\varepsilon^2\delta_y^2(\delta_x^2+\delta_y^4)),
\end{aligned}
\tag{66}
$$

where

$$
\begin{aligned}
&d_{01}=-\frac{16P^3}{(P+1)^2},\ d_{02}=d_{01},\ d_{03}=\frac{16P^3}{\pi^2(P+1)^2},\ d_{04}=-\frac{4P^3}{\pi^4(P+1)^2},\\
&d_{11}=-\frac{16P^3}{9\pi^2(P+1)^2},\ d_{12}=\frac{32P(-3P^2+5P+1)}{27\pi^2(P+1)^2},\\
&d_{13}=\frac{16P(-P^2+2P+2)}{9\pi^2(P+1)^2},\ d_{14}=\frac{-8P(P^2+2P+2)}{9\pi^4(P+1)^2},\\
&d_{21}=\frac{224P}{243\pi^4(P+1)},\ d_{22}=-\frac{32P}{81\pi^4(P+1)},\ d_{31}=\frac{64P^3k}{\pi^2(P+1)^2},\\
&d_{32}=-\frac{32P^3k}{\pi^4(P+1)^2},\ d_{41}=\frac{16P^3}{(P+1)^2},\ d_{42}=d_{41},\ d_{43}=-\frac{32P^3}{\pi^2(P+1)^2},\\
&d_{51}=\frac{128Pk}{9\pi^2(P+1)},\ d_{52}=-\frac{32P(P^2+2P+2)k}{9\pi^4(P+1)^2};
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
&\operatorname{tr}\mathcal{A}S(\mathcal{A})-\det\mathcal{A}=e_{01}\delta_x^6+e_{02}\delta_x^4\delta_y^2+e_{03}\delta_x^2\delta_y^4+e_{04}\delta_y^8\\
&+\varepsilon^2(\delta_x^2+\delta_y^2)^{-1}(e_{11}\delta_x^6+e_{12}2\delta_x^4\delta_y^2+e_{13}\delta_x^2\delta_y^4+e_{14}\delta_y^6)+\\
&\varepsilon^4(\delta_x^2+\delta_y^2)^{-1}(e_{21}\delta_x^4+e_{22}\delta_x^2\delta_y^2+e_{23}\delta_y^4)+\alpha^2e_3\delta_x^4+\alpha\varepsilon^2e_4\delta_x^2\delta_y^2(\delta_x^2+\delta_y^2)^{-1}\\
&+\mathrm{O}(\delta^4(\delta_x^2+\delta_y^4)^2,\varepsilon^2\delta^6,\varepsilon^4\delta^4,\varepsilon^6\delta^2,\alpha\delta^4(\delta_x^2+\delta_y^4),\alpha\varepsilon^2\delta_y^2(\delta_x^2+\delta_y^4)),
\end{aligned}
\tag{68}
$$

where

$$
\begin{aligned}
&e_{01}=-\frac{8P^3(P+5)^2}{(P+1)^3},\ e_{02}=-\frac{16P^3(P+5)}{(P+1)^2},\ e_{03}=-\frac{8P^3}{(P+1)},\ e_{04}=-\frac{4P^3}{\pi^2(P+1)},\\
&e_{11}=-\frac{4P^3(P^2+18P+65)}{9\pi^2(P+1)^3},\ e_{12}=\frac{4P(-9P^4-54P^3-99P^2+20P+2)}{27\pi^2(P+1)^3},\\
&e_{13}=-\frac{4P(3P^3+3P^2+22P+26)}{27\pi^2(P+1)^2},\ e_{14}=-\frac{4P^2(P^2+2P+4)}{9\pi^2(P+1)},\\
&e_{21}=-\frac{16P^3(P+5)}{81\pi^4(P+1)^3},\ e_{22}=-\frac{32P(P+3)(3P^2+2P+2)}{273\pi^4(P+1)^3},\\
&e_{23}=-\frac{16P^3}{81\pi^4(P+1)^2},\ e_3=-\frac{512P^3}{(P+1)^3},\ e_4=-\frac{128Pk}{9\pi^2(P+1)}.
\end{aligned}
\tag{69}
$$

B A necessary and sufficient condition for existence of eigenvalues of a 3×3 matrix, which have positive real parts

In this Appendix we show that instead of direct calculation of eigenvalues, in order to study stability of rolls it suffices to check, whether any of the inequalities (23) or

(24) is satisfied for some δ_x and δ_y.

We start by exposition of three lemmas about eigenvalues of a 3×3 matrix.

Lemma 1. Let $\mathcal{A}$ be a 3×3 matrix with real entries. Denote its eigenvalues by λ_i, $i = 1, 2, 3$, and the sum of the second order minors by $S(\mathcal{A})$:

$$S(\mathcal{A}) = A_{11}A_{22} - A_{12}A_{21} + A_{11}A_{33} - A_{13}A_{31} + A_{22}A_{33} - A_{23}A_{32}.$$

Consider the following statements:

$$\textbf{S1:}\ \max_{1\le i\le 3}(\mathrm{Re}\lambda_i) > 0$$

$$\textbf{S2:}\ \det\mathcal{A} > 0$$

$$\textbf{S3:}\ S(\mathcal{A})\mathrm{tr}\,\mathcal{A} - \det\mathcal{A} > 0$$

Then

$$(i)\ \textbf{S2} \Rightarrow \textbf{S1} \quad \text{and} \quad (ii)\ \textbf{S3} \Rightarrow \textbf{S1}.$$

Proof. (i) follows from the identity $\det\mathcal{A} = \lambda_1\lambda_2\lambda_3$ (consider separately two cases: all eigenvalues are real, or two of them are complex conjugate).

Similarly, (ii) follows from the identities

$$S(\mathcal{A}) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \quad \mathrm{tr}\,\mathcal{A} = \lambda_1 + \lambda_2 + \lambda_3$$

(again, consider separately the two cases).

If in the condition of the Lemma the signs ">" are replaced by "$\geq$", the statements remain true. The modified lemma is referred to as Lemma 1'.

Lemma 2. Let $\mathcal{A}(\mathbf{x})$ be a 3×3 matrix with real entries continuously depending on $\mathbf{x} \in \Omega \subset \mathbf{R}^n$, where Ω is a connected domain in $\mathbf{R}^n$. Denote by $\lambda_i(\mathbf{x})$, $i = 1, 2, 3$ the eigenvalues of $\mathcal{A}(\mathbf{x})$. Suppose

$$\exists\ \mathbf{x}_0 \in \Omega \quad \max_{1\le i\le 3}\mathrm{Re}\lambda_i(\mathbf{x}_0) < 0,$$

$$\det\mathcal{A} \neq 0 \quad \forall\mathbf{x} \in \Omega,$$

$$S(\mathcal{A})\mathrm{tr}\,\mathcal{A} - \det\mathcal{A} \neq 0, \quad \forall\mathbf{x} \in \Omega.$$

Then

$$\max_{1\le i\le 3}\mathrm{Re}\lambda_i(\mathbf{x}) < 0 \quad \forall\mathbf{x} \in \Omega.$$

Proof. Suppose there exists $\mathbf{x}_1 \in \Omega$ such that an eigenvalue of $\mathcal{A}(\mathbf{x})$, say, λ_1, has a positive real part. A curve in Ω connects $\mathbf{x}_0$ and $\mathbf{x}_1$. The eigenvalue λ_1 is a continuous function on this curve (because roots of the cubic equation $\det(\mathcal{A} - \lambda I) = 0$ are continuous functions of its coefficients.) Since $\mathrm{Re}\lambda_1$ has different signs at $\mathbf{x}_0$ and $\mathbf{x}_1$, there exists a point $\hat{\mathbf{x}}$ on the curve such that $\mathrm{Re}\lambda_1(\hat{\mathbf{x}}) = 0$. If $\lambda_1(\hat{\mathbf{x}}) = 0$, then

$\det\mathcal{A}(\hat{\mathbf{x}}) = 0$, and if $\lambda_1(\hat{\mathbf{x}})$ is imaginary, then $S(\mathcal{A}(\hat{\mathbf{x}}))\mathrm{tr}\,\mathcal{A}(\hat{\mathbf{x}}) - \det\mathcal{A}(\hat{\mathbf{x}}) = 0$, since in this case

$$S(\mathcal{A})\mathrm{tr}\,\mathcal{A} - \det\mathcal{A} = \mathrm{Re}\lambda_1(2\lambda_3^2 + 4\mathrm{Re}\lambda_1\lambda_3 + 2\lambda_1\bar{\lambda}_1) = 2\mathrm{Re}\lambda_1((\lambda_3 + \mathrm{Re}\lambda_1)^2 + (\mathrm{Im}\lambda_1)^2).$$

Thus a contradiction with the statement of the lemma is obtained and the lemma is proved.

Lemma 3. Let $\mathcal{A}(\mathbf{x})$ and Ω be the same as in the statement of Lemma 2. Assume

$$\begin{gathered} \exists\ \mathbf{x}_0 \in \Omega \quad \max_{1\le i\le 3} \mathrm{Re}\lambda_i(\mathbf{x}_0) < 0, \\ \det\mathcal{A} \neq 0 \quad \forall \mathbf{x} \in \Omega, \\ \exists\ \mathbf{x}_1 \in \Omega \quad S(\mathcal{A}(\mathbf{x}_1))\mathrm{tr}\,\mathcal{A}(\mathbf{x}_1) - \det\mathcal{A}(\mathbf{x}_1) > 0. \end{gathered} \tag{70}$$

Then

$$\exists\ \mathbf{x}_2 \in \Omega \quad \mathrm{Re}\lambda_1(\mathbf{x}_2) > 0,\ \mathrm{Im}\lambda_1(\mathbf{x}_2) \neq 0.$$

Proof. A curve in Ω connects $\mathbf{x}_0$ and $\mathbf{x}_1$. Let ξ be a parameter along this curve, $\xi = 0$ at $\mathbf{x}_0$ and $\xi = 1$ at $\mathbf{x}_1$. There exist ξ_0, $0 < \xi_0 < 1$, and ξ_1, $\xi_0 < \xi_1 < 1$, such that

$$\begin{gathered} S(\mathcal{A}(\xi_0))\mathrm{tr}\,\mathcal{A}(\xi_0) - \det\mathcal{A}(\xi_0) = 0, \\ S(\mathcal{A}(\xi))\mathrm{tr}\,\mathcal{A}(\xi) - \det\mathcal{A}(\xi) > 0 \quad \forall\ \xi \in (\xi_0, \xi_1], \\ S(\mathcal{A}(\xi))\mathrm{tr}\,\mathcal{A}(\xi) - \det\mathcal{A}(\xi) \le 0 \quad \forall\ \xi \in [0, \xi_0]. \end{gathered} \tag{71}$$

Suppose all eigenvalues of $\mathcal{A}(\xi_0)$ are real. By Lemma 1' all of them are non-positive in $[0, \xi_0]$. Due to (70) they do not vanish at ξ_0, hence they are strictly negative at ξ_0. But then (71) can not be satisfied. Hence the assumption that all eigenvalues are real is wrong.

Let λ_1 and $\lambda_2 = \overline{\lambda}_1$ be a pair of complex eigenvalues and λ_3 be real. By continuity, there exists ξ_2, $\xi_2 > \xi_0$, such that $\mathrm{Im}\lambda_1(\xi) \neq 0$ for all $\xi \in [\xi_0, \xi_2]$. The expression

$$S(\mathcal{A})\mathrm{tr}\,\mathcal{A} - \det\mathcal{A} = 2\mathrm{Re}\lambda_1((\lambda_3 + \mathrm{Re}\lambda_1)^2 + (\mathrm{Im}\lambda_1)^2)$$

is positive only if $\mathrm{Re}\lambda_1$ is positive. Consequently, $\mathrm{Re}\lambda_1(\xi_3) > 0$ and $\mathrm{Im}\lambda_1(\xi_3) \neq 0$. The lemma is proved.

The Lemmas are applied to investigate stability of rolls.

Let $\mathcal{A}$ be the matrix calculated in Appendix A. Assume (δ_x, δ_y) is the parameter $\mathbf{x}$ employed in Lemma 2, ε, α and P being fixed. If (23) or (24) is satisfied for some (δ_x, δ_y), Lemma 1 implies existence of an eigenvalue with positive real part.

Suppose (23) and (24) are not satisfied for any δ_x and δ_y. For sufficiently large δ_x ($\delta_x^2 \gg \varepsilon^2$ and $\delta_x^2 \gg \alpha^2$) the matrix has three real negative eigenvalues. Let Ω be $\mathbf{R}^2$ with the origin excluded. Conditions of Lemma 2 are satisfied, hence for any (δ_x, δ_y) all the eigenvalues have negative real parts.

Note that (23) implies that the matrix $\mathcal{A}$ has a real eigenvalue with a positive real part, while (24) does not guarantee that there exist a pair of complex eigenvalues with a positive real part. However, assume in addition that $\det\mathcal{A} < 0$ for all (δ_x, δ_y) (or for all (δ_x, δ_y) in a connected region Ω, where conditions of Lemma 3 are satisfied). Then by Lemma 3 there exists a point (δ_x, δ_y) where $\mathcal{A}$ has a complex eigenvalue with a positive real part.

C A bound for $\alpha < 0$, $P > P_1$, $\alpha^2 \sim \varepsilon^4$

We prove here that under the conditions, stated in the title of the Appendix,

$$D_0 + \varepsilon^2 D_1 + \varepsilon^4 d_{22}\delta_y^4(\delta_x^2 + \delta_y^2)^{-1} + \alpha D_3 + \alpha^2 D_4 + \alpha\varepsilon^2 d_{52}\delta_y^4 \tag{72}$$

is negative for all δ_x and δ_y.

To begin with, note that the terms involving α are asymptotically small and therefore are neglected.

Assume $\delta_y^2 \gg \delta_x^2$. In the leading order (72) is

$$\delta_y^2(d_{02}\delta_x^4 + \delta_x^2(\varepsilon^2 d_{13} + d_{03}\delta_y^4) + \varepsilon^4 d_{22} + \varepsilon^2 d_{14}\delta_y^4 + d_{04}\delta_y^8). \tag{73}$$

This is a quadratic polynomial in δ_x^2, which admits a maximum at

$$\delta_x^2 = -\frac{d_{03}\delta_y^4 + \varepsilon^2 d_{13}}{2d_{02}}. \tag{74}$$

The maximum is

$$-\frac{\delta_y^2}{4d_{02}}\Big(\varepsilon^2(2d_{03}d_{13} - 4d_{02}d_{14})\delta_y^4 + \varepsilon^4(d_{13}^2 - 4d_{02}d_{22})\Big),$$

where both expressions in the brackets are negative for $P > P_1$.

Now assume $\delta_x^2 \sim \delta_y^2$ or $\delta_x^2 \gg \delta_y^2$. In the leading order (72) is equal to

$$(d_{01}\delta_x^6 + d_{02}\delta_x^4\delta_y^2) + \varepsilon^2(d_{11}\delta_x^6 + d_{12}\delta_x^4\delta_y^2 + d_{13}\delta_x^2\delta_y^4)(\delta_x^2 + \delta_y^2)^{-1} + \varepsilon^4 d_{22}\delta_y^4(\delta_x^2 + \delta_y^2)^{-1}. \tag{75}$$

This quadratic polynomial in ε^2 can take positive values only if

$$d_{11}\delta_x^6 + d_{12}\delta_x^4\delta_y^2 + d_{13}\delta_x^2\delta_y^4 > 0. \tag{76}$$

If this is satisfied, the maximum (in ε^2) of (75) is

$$-(d_{11}\delta_x^6 + d_{12}\delta_x^4\delta_y^2 + d_{13}\delta_x^2\delta_y^4)^2(4d_{22}\delta_y^4)^{-1} + d_{01}\delta_x^8 + (d_{01} + d_{02})\delta_x^6\delta_y^2 + d_{02}\delta_x^4\delta_y^4. \tag{77}$$

In view of the inequalities (76), $d_{11} < 0$ and $d_{13} > d_{12}$ for $P > P_1$,

$$d_{13}\delta_x^4\delta_y^2 + d_{13}\delta_x^2\delta_y^4 > d_{11}\delta_x^6 + d_{12}\delta_x^4\delta_y^2 + d_{13}\delta_x^2\delta_y^4.$$

Note that $d_{01} = d_{02}$ and $d_{13}^2 < 4d_{22}d_{02}$ for $P > P_1$; hence (77) is always negative for $P > P_1$.

D The large P limit

In this Appendix we calculate an instability boundary, which is important for $\alpha < 0$ and large P. In the limit of large P the coefficient f_2 in (35) vanishes and instability occurring for $\alpha^2 \sim \varepsilon^2$ may compete with the instability defined by (35).

Suppose $\alpha^2 \sim \varepsilon^2$. Then $\det \mathcal{A} > 0$ for $\alpha < 0$, if either $\delta_x^2 \gg \delta_y^2$ or $\delta_y^2 \gg \delta_x^2$ (due to the presence of the term $\alpha\varepsilon^2 D_5$).

Suppose $\delta_y^2 \gg \delta_x^2$. Represent $\det \mathcal{A}$ (66) (it is simpler to calculate this directly from (64)) as

$$\det \mathcal{A} =$$

$$- \delta_y^2(2\varepsilon^2 C_3 + C_4\xi)(\varepsilon^2 b^2 \frac{\pi^2}{4k^2} + PC_4\xi) \quad (78)$$

$$- 2PC_4k^4\delta_x^4\delta_y^2 + \alpha\varepsilon^2 b^2\pi^2 k\delta_x^2 \quad (79)$$

$$+ \delta_x^2\delta_y^2(-PC_4k^2(\varepsilon^2 C_3 + C_4\xi) + 4P\alpha^2k^2 + \frac{\varepsilon^2 b^2\pi^2}{2}), \quad (80)$$

where

$$\xi = \alpha k\delta_y^2 + \delta_y^4/4.$$

For $\delta_x = 0$ the determinant is given by (78). Considering

$$\varepsilon^2 b^2 \frac{\pi^2}{4k^2} + PC_4\xi$$

as a quadratic polynomial in δ_y^2, we find that (78) is positive (we are interested in large P's, and for them the instability boundary is defined by the second term in (78)) for

$$\varepsilon^2 < f_3\alpha^2, \quad f_3 = \frac{9\pi^2 P^2}{2(P+1)}. \quad (81)$$

Note that

$$\lim_{P\to\infty} f_3 = \infty. \quad (82)$$

For large P and ε not satisfying (81), the contributions to $\det \mathcal{A}$ from (79) and (80) are negative (the proof is omitted). Hence if $\det \mathcal{A} < 0$ for $\delta_x = 0$ and all δ_y, it remains negative for all δ_x and δ_y. Thus (81) is indeed a boundary for stability of rolls.

For $\delta_x^2 \gg \delta_y^2$ the instability boundary is that of the Eckhaus instability

$$\varepsilon^2 < 36\alpha^2\pi^2$$

(or in a more familiar form $R - R^s < 3(R_c(k) - R^s)$), which is below the boundary defined by (81).

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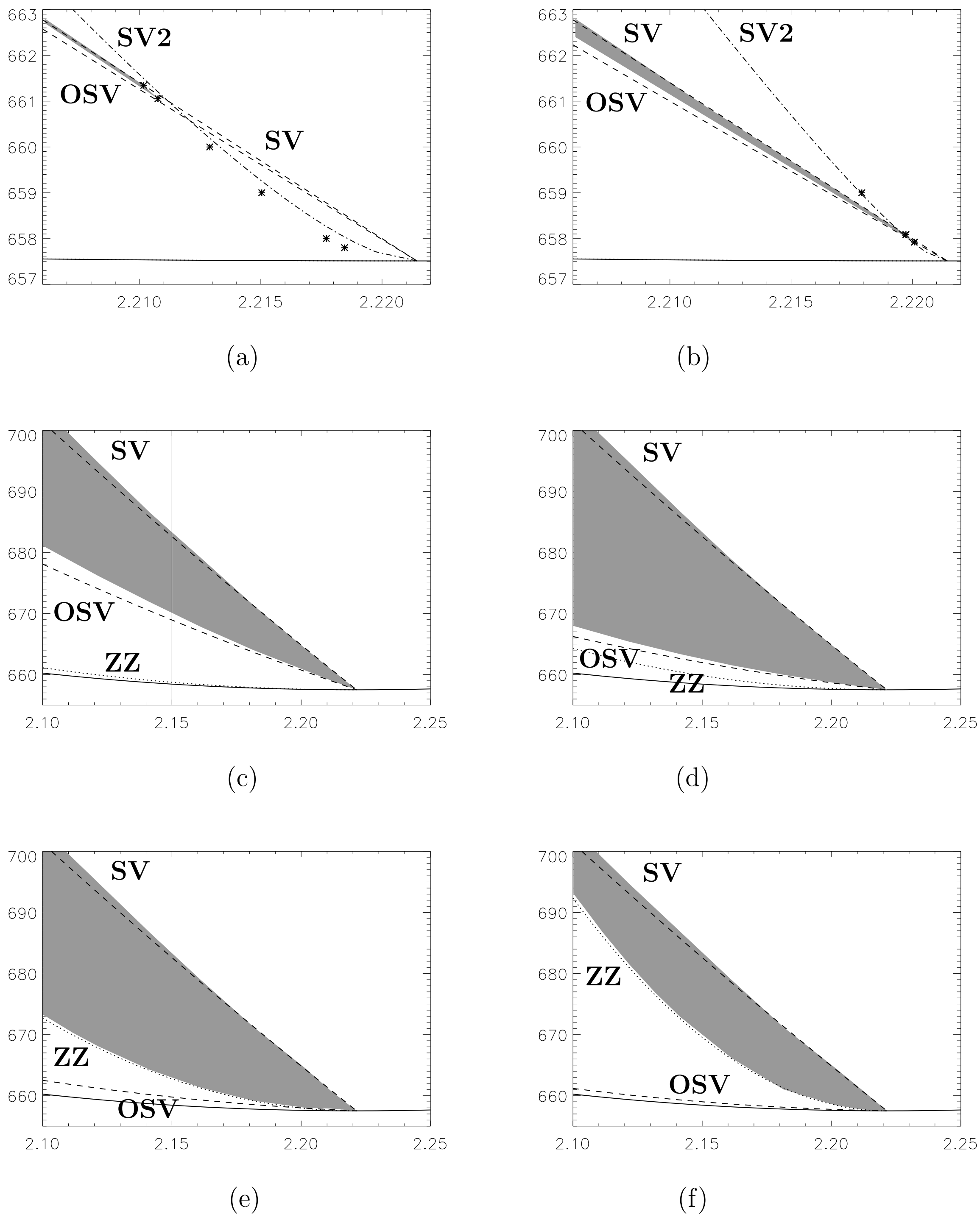

Figure 1. The area of stable rolls (shaded) on the (k, R) plane found numerically (see section 8) and the instability boundaries found analytically for $P = 0.6$ (a) $P = 0.7$ (b), $P = 2$ (c) and $P = 7$ (d), $P = 20$ (e) and $P = 50$ (f). Solid line denotes the onset of convection, dashed lines instability boundaries SV and OSV defined by (32) and (35), dotted line the ZZ boundary (36) and dashed-dotted line the SV2 boundary (41), (42). Stars mark the points where the SV2 boundary is obtained by interpolation. Horizontal axis: k, vertical axis: R.

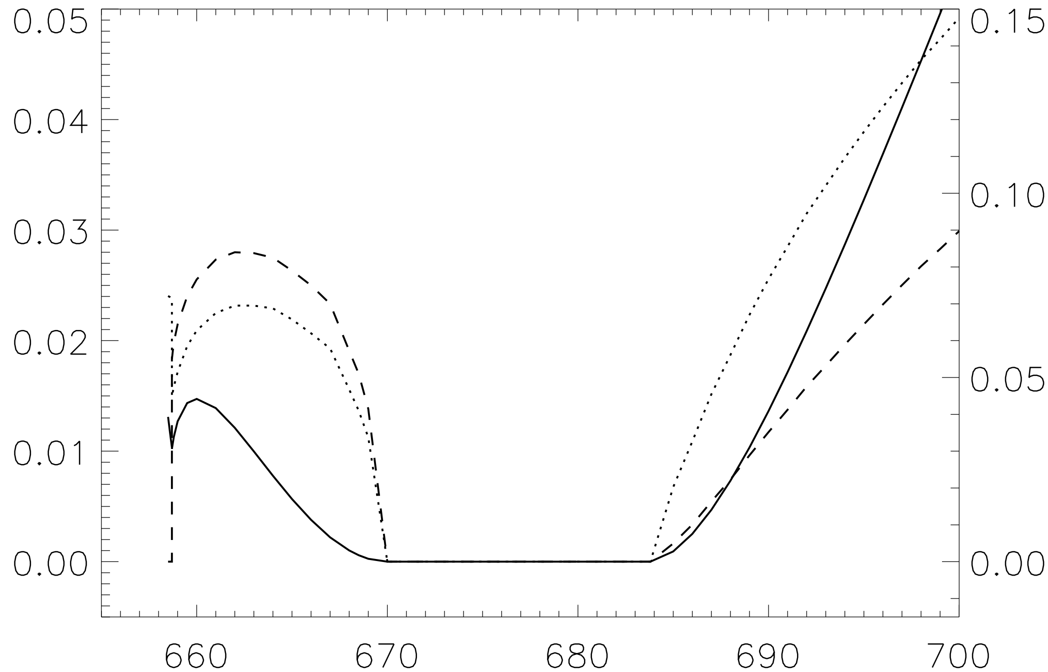

Figure 2. The dominant growth rate (solid line, left vertical axis) and the values of δ_x and δ_y (dotted and dashed lines, respectively, right axis), where the maximum is achieved, versus the Rayleigh number (horizontal axis) for $P = 2$ and $k = 2.15$ (the respective crossection is shown by a thin vertical line on fig. 1c).