3D Dune Dynamics as a Coupled Dynamics of 2D Cross-Sections

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To analyze theoretically the stability of the shape and the migration process of transverse dunes and barchans, we propose a *skeleton model* of 3D dunes described with coupled dynamics of 2D cross-sections. First, 2D cross-sections of a 3D dune parallel to the wind direction are extracted as elements of a skeleton of the 3D dune, hence, the dynamics of each and interaction between them is considered. This model simply describes the essential dynamics of 3D dunes as a system of coupled ordinary differential equations. Using the model we study the stability of the shape of 3D transversal dunes and their deformation to barchans depending on the amount of available sand in the dune field, sand flow in parallel and perpendicular to wind direction.

Sand dunes, which are the largest granular objects on the Earth, move by the wind and exhibit various morphodynamics. As typical shapes of dunes, barchan, transverse dune, linear dune, star dune, dome dune and parabolic dune are known[1, 2]. The steadiness of wind direction and the amount of available sand in each dune field are considered as dominant factors determining these shapes. For example, unidirectional steady wind generates barchans or transverse dunes. The former are crescentic shaped dunes, and are formed in dune fields with small amounts of available sand, whereas transverse dunes, which extend perpendicular to the wind direction, are formed in dune fields with the larger amount of available sand than the barchan-rich field. A characteristic aspect among recent studies of dunes is that quantitative analysis of dune morphodynamics has largely progressed. In particular, water tank experiments and computer models have uncovered the complex processes of dunes[3–8]. They were successful to reproduce formation and migration processes of barchans and other types of dunes under controlled setups. However, theoretical methodology to explain the complex morphodynamics of dunes beyond only reproducing them is yet to be developed. We, here, propose a *skeleton model* of 3D dunes described with coupled dynamics of 2D cross-sections[9], which has a form of coupled ordinary differential equations. Using the model we study the morphodymamics of dunes, particularly the stability of the shape of transverse dunes and its deformation to barchans.

Recently Katsuki and Nishimori has proposed a model for the collision dynamics of two 3D barchans focusing on the dynamics of their central 2D cross sections [8][9]. That is called ABCDE(Aeolian/Aqueous Barchans Collision Dynamical Equations). To introduce the present skeleton model for 3D transverse dunes, we employ similar assumptions to those used in ABCDE.

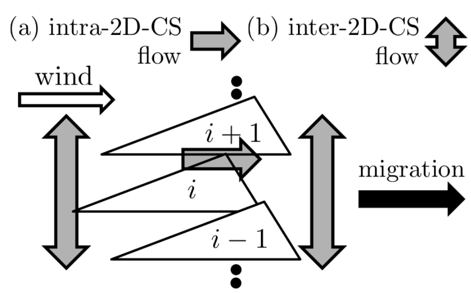

FIG. 1: Outlook of skeleton model. 2D-wind directional cross sections(2D-CSs) are laterally arrayed. They consist of a skeletonized 3D transverse dune or a 3D barchan.

First, laterally arraying 2D wind directional cross-sections (hereafter 2D-CSs) of a 3D transverse dune (or a barchan) are set as the elements of the present skeleton model. Hence, the dynamics of each 2D-CS and the interaction between them is considered (Fig. 1)

As mentioned above, 3D barchans are observed in dune fields with small amount of available sand, thus they are isolated on a hard ground both in wind direction and in lateral direction. Also 3D transverse dunes seen under the same wind condition, extend in lateral direction and are not isolated because of the larger amount of available sand. However, 2D-CS of a transverse dune can be isolated in wind direction from the leeward and the windward 2D-CSs depending on the amount of available sand in the dune field. Here we assume each 2D-CS of a 3D transverse dune is situated on a flat and hard ground and isolated in the wind direction thought directly connected from each other in the lateral direction. In addition, considering the fact that 2D-CSs of transverse dunes and barchans have shape similarity independent of their size as a rough approximation, we assume that the angles of their upwind and downwind slopes (θ and φ, respectively) are constant regardless of their size (Fig. 2(b)). Then, geometrical constants

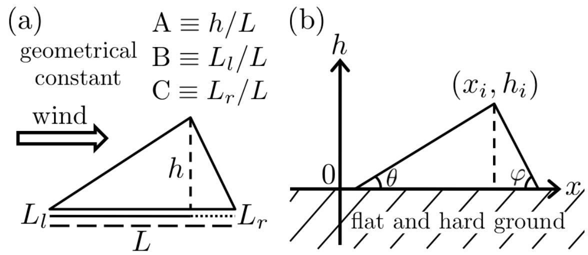

FIG. 2: Shape similarity of the 2D-CSs of a 3D transverse dunes and barchans independent of their size. (a) definition of introduced geometrical constants A, B and C. (b) the wind directional position and the height of ith 2D-CS is the coordinate of its crest (x_i, h_i) is given.

A, B, C are defined as,

$$\mathrm{A} = \frac{h}{L} = \frac{1}{10}, \mathrm{B} = \frac{L_l}{L} = \frac{4}{5}, \mathrm{C} = \frac{L_r}{L} = \frac{1}{5} \tag{1}$$

where h and L are the height and the base length of each 2D-CS, respectively. L_l and L_r are the lengths of the leftward and the rightward parts of L divided by the foot of perpendicular of the crest of 2D-CS, respectively (Fig. 2(a)). Based on above assumptions, the wind directional position and the height of each 2D-CS are uniquely determined if the coordinate (x_i, h_i) $(1 \leq i \leq N)$ of its crest is given (Fig. 2(b)). Note that these setup are taken as a skeleton of not only 3D transverse dunes but also 3D barchans.

Now, we consider the interaction between neighboring 2D-CSs and thenafter the migration process of each. Dune migration occurs by the sand flow along the surface. In our description the sand flow is divided into two types: (a) intra-2D-CS flow (b) inter-2D-CS flow (Fig. 1). In the intra-2D-CS flow, the most relevant quantity for the present modelling is the over-crest sand flow of each 2D-CS, the flux of which is denoted by q. This over-crest flux q is assumed

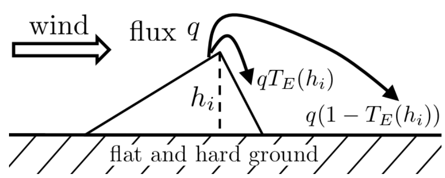

FIG. 3: Intra-2D-CS sand flow. q and $T_E(h_i)$ are over-crest sand flux and sand trapping efficiency, respectively.

constant independent of the height of the 2D-CS, while finite ratio $1-T_E$ $(0 \leq 1-T_E \leq 1)$ of q is assumed to directly escape from the 2D-CS to the leeward hard ground and the remaining ratio T_E is deposited in the downwind slip face of the same 2D-CS (Fig. 3). This ratio T_E is termed as the *sand trapping efficiency*, according to [10]. The higher the height of a 2D-CS, the larger is T_E, whereas the lower the height h the more q directly escapes from the 2D-CS to the leeward hard ground. Therefore T_E has such limits,

$$\lim_{h\to 0} T_E(h) = 0, \lim_{h\to\infty} T_E(h) = 1,$$

Here specific form of $T_E(h)$ is assumed like,

$$T_E(h) = h/(a+h), a = 1.5.$$

Next, the inter-2D-CS sand flow is considered. The inter-2D-CS flux between upwind slopes of neighboring 2D-CSs, i and j, is given as a function of heights and wind directional positions of their crests; $D_{u(i\to j)} = D_u(h_i, h_j, x_i, x_j)$ (eq. (2)). Similarly the inter-2D-CS flux between downwind slopes of neighboring 2D-CS, i and j, is denoted as $D_{d(i\to j)} = D_d(h_i, h_j, x_i, x_j)$ (eq. (3)). Specific forms of them are,

$$D_{u(i\to j)} = \begin{cases} \frac{\nu_u \mathrm{B}}{2\mathrm{A}}\{h_i^2 - [h_j - \frac{\mathrm{A}}{\mathrm{B}}(x_j - x_i)]^2\} & x_j - x_i > 0 \\ \frac{\nu_u \mathrm{B}}{2\mathrm{A}}\{[h_i + \frac{\mathrm{A}}{\mathrm{B}}(x_j - x_i)]^2 - h_j^2\} & x_j - x_i \leq 0 \end{cases} \tag{2}$$

$$D_{d(i\to j)} = \begin{cases} \frac{\nu_d \mathrm{C}}{2\mathrm{A}}\{h_j^2 - [h_i - \frac{\mathrm{A}}{\mathrm{C}}(x_j - x_i)]^2\} & x_j - x_i > 0 \\ \frac{\nu_d \mathrm{C}}{2\mathrm{A}}\{[h_j + \frac{\mathrm{A}}{\mathrm{C}}(x_j - x_i)]^2 - h_i^2\} & x_j - x_i \leq 0. \end{cases} \tag{3}$$

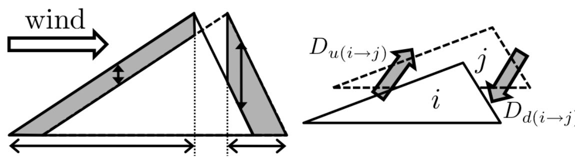

FIG. 4: Schematic explanation for the upwind inter-2D-CS flux $D_{u(i\to j)}$ between two cross sections. The upwind grey area is equivalent to except for the constant ν_u. It corresponds to the area difference between the upwind area of the windward 2D-CS and its overlapped area by the neighboring leeward 2D-CS. In a same way, the downwind inter-2D-CS flux $D_{d(i\to j)}$ between two cross sections is given. The downwind grey area is equivalent to except for the constant ν_d. It corresponds to the area difference between the downwind area of the leeward 2D-CS and its overlapped area by the neighboring windward 2D-CS.

These expressions roughly give the product of the height difference and the overlaped length of upwind and downwind slopes between neighboring 2D-CSs (Fig. 4), where ν_u and ν_d are phenomenological parameters to control the amount of inter-2D-CS sand flow of upwind and downwind slopes, respectively.

Next we consider the migration of each 2D-CS. As mentioned above, the size and the wind directional position of ith 2D-CS is uniquely determined if the coordinate (h_i, x_i) of its crest is given. Therefore to describe the dynamics of (h_i, x_i) $(1 \le i \le N)$ correspond to describe the dynamics of 2D-CSs that gives the skeletonized dynamics of a 3D dune.

Here, Δx_{ui} and Δx_{di} are, respectively, the wind directional displacement of the upwind and the downwind slopes of ith 2D-CS during Δt (Fig. 5), while the change of the coordinate (h_i, x_i) of the 2D-CS crest within the same interval are denoted as Δh_i and Δx_i, respectively. Then, Δh_i and Δx_i are expressed by Δx_{di} and Δx_{ui},

$$\Delta h_i = \mathrm{A}\Delta x_{di} - \mathrm{A}\Delta x_{ui} \tag{4}$$

$$\Delta x_i = \mathrm{B}\Delta x_{di} + \mathrm{C}\Delta x_{ui} \tag{5}$$

where A, B, C are the geometrical constants of (1). Consequently the eroded area per Δt along the upwind slope on

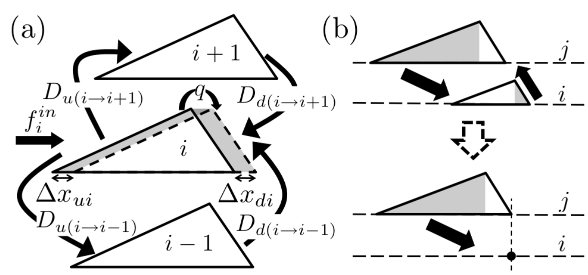

FIG. 5: By the sand flow along dune surface, upwind slope is eroded for Δx_{ui} in the wind direction per Δt, and sand is deposited to downwind slope for Δx_{di} in the wind direction.

ith 2D-CS is expressed as,

$$\frac{2h_i + \Delta h_i}{2}\Delta x_{ui} = \Delta t(q + D_{u(i\to i-1)} + D_{u(i\to i+1)} + f_i^{in}) \tag{6}$$

where q is the over-crest sand flux, $D_{u(i\to i-1)}$ and $D_{u(i\to i+1)}$ are inter-2D-CS flux from ith 2D-CS upwind slope to $(i-1)$th and $(i+1)$th 2D-CSs,respectively. f_i^{in} is incoming flux to ith 2D-CS from the windward hard ground. Similarly the deposited area per Δt along the downwind slope on ith 2D-CS is expressed as,

$$\frac{2h_i + \Delta h_i}{2}\Delta x_{di} = \Delta t(qT_E(h_i) + D_{d(i\to i-1)} + D_{d(i\to i+1)}) \tag{7}$$

where $T_E(h_i)$ is the sand trapping efficiency, $D_{d(i\to i-1)}$ and $D_{d(i\to i+1)}$ are the 2D-CS flux from ith 2D-CS downwind slope to $(i-1)$th and $(i+1)$th 2D-CSs, respectively. Using eqs. (4)-(7) and taking their limits $\Delta t \to 0$ and $\Delta h_i \to 0$, a system of coupled ordinary equations,

$$\frac{dh_i}{dt} = \frac{\mathrm{A}}{h_i}\left(q(T_E(h_i)-1) + \sum_{j=i-1,i+1}(D_{d(i\to j)} - D_{u(i\to j)}) + f_i^{in}\right) \tag{8a}$$

$$\frac{dx_i}{dt} = \frac{1}{h_i}\left(q(\mathrm{B}T_E(h_i)+\mathrm{C}) + \sum_{j=i-1,i+1}(\mathrm{B}D_{d(i\to j)} + \mathrm{C}D_{u(i\to j)}) - f_i^{in}\right) \tag{8b}$$

$$(1 \le i \le N).$$

is obtained, which describes the dynamics of the crests of 2D-CSs, consequently the dynamics of a skeletonized 3D dune.

Because dunes treated here are assumed to migrate on a flat and hard ground, its minimum height is zero. Therefore we add a rule for the vanishment of 2D-CSs, that is, if h_i decreases to $h_i = 0$, then ith 2D-CS is taken as vanished. Note that the wind directional position x of a already vanished 2D-CS is virtually set at the foot of the downwind slope of the neighboring 2D-CS in order to determine the sand flows from the neighboring 2D-CS according to eqs. (2) and (3) (Fig. 5(b)).

Numerical simulation of eqs. (8) is conducted with $N = 1000$ of 2D-CSs. The lateral boundary is set periodic. In the wind direction the boundary condition is set more carefully. As mentioned above, finite ratio $1 - T_E(h_i)$ of the over crest flux q escapes from each 2D-CS. Therefore total amount of escaping sand to the leeward flat ground is,

$$F_{total} = \sum_{i=1}^{N} q(1 - T_E(h_i)),$$

which is set uniformly redistributed to the windward flat ground of system. It means that $f_i^{in} = F_{total}/N$ in eqs. (8) is added to the upwind slope of each 2D-CS. Note that if finite number of 2D-CSs have already vanished they do not catch the redistributed sand. In this sense total amount of sand in the system is conserved as long as all 2D-CSs are kept unvanished, then, total area of 2D-CSs

$$S_{total} = \sum_{i=1}^{N} \frac{h_i^2}{2A}$$

is kept constant too.

In the present model of eqs. (8) accompanied with eqs. (2)(3), three-environmental parameters ν_u, ν_d, q are introduced all of which increase if the wind force over the dune increases. Especially, ν_u, ν_d are related to the increase of sand flow in the lateral direction, while q is related to the sand flow in the wind direction. Here ignoring the correlated increase of these three parameters responding to the increase of wind force, we independently vary them as individual control parameters.

As the initial condition of the numerical simulations, the height of 2D-CS crests are set uniform, i.e., $h_i(0) = H_0$ by varying this height we can control the amount of available sand in the system. Moreover, the initial wind directional position of crests are set wavy with small amplitude of sinuosity, that is $x_i(0) = H_0/20\sin(2\pi i/N)$. We check if the initial amplitude of sinuosity in x_i $(0 \le i \le N)$ grows or not by measuring the quantity

$$Var(t) = \sum_{i=1}^{N}(x_i(t) - x_{i+1}(t))^2.$$

The simulation results are classified into following three types;
I) If $Var(t)/Var(0) < 10^{-5}$ is satisfied at $t = 10^6$ we consider the laterally extending straight transverse dune is stable.
II) If $Var(t)/Var(0) \ge 10^{-5}$ is satisfied at $t = 10^6$ we further perform the calculation after $t = 10^6$ and divide the stability of the transverse dune morphology into two;

II-i) If, at least, one 2D-CS shrinks to become $h_i(t) = 0$ within $10^6 < t \le 10^7$ thereafter the transverse dune will soon deform into the typical shape of barchan (Fig. 6).

II-ii) If the relation $Var(t)/Var(0) \ge 10^{-5}$ is kept hold even at $t = 10^7$, then we take the wavy shape of a transverse dune as temporally stable.

In the first simulation, we fix $\nu_u = 0.1$ and $\nu_d = 0.1$ and vary two parameters: initial dune height H_0 and over-crest sand flux q. The white area in Fig. 7(a) indicate the condition where laterally extending straight transverse dune is stable, while in the black area initial transverse dune is deformed into a barchan. The increase of H_0 enforces the stability of the shape of straight transverse dune and the decrease of H_0 causes the emergence of barchan. This result qualitatively corresponds to the well known fact that barchans are formed in the field with small amount of available sand.

In the next simulation, we fix $\nu_d = 0.1$ and set $H_0 = 30.0$ and vary two parameters, ν_u and q. White and black areas in Fig. 7(b) indicate same as those in Fig. 7(a), whereas initial wavy transverse dune kept wavy in gray area. The larger ν_u, the straight shape of transverse dune is more stabilized. On the other hand the increase of q destabilizes the transverse dune to enforce the deformation to barchan. In this way the balance between the wind directional flow and the lateral flow determine the stable shape of dune.

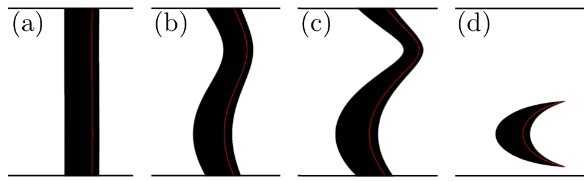

FIG. 6: Time evolution of initial transverse dune to get unstable to deform into the shape of barchan. The center of the position is fixed in the figure. The parameter are fixed as $H_0 = 5.0, \nu_u = 0.1, \nu_d = 0.1, q = 0.1$. (a) $t = 0.0$, (b) $t = 2.4 \times 10^5$, (c) $t = 2.5 \times 10^5$, (d) $t = 2.7 \times 10^5$.

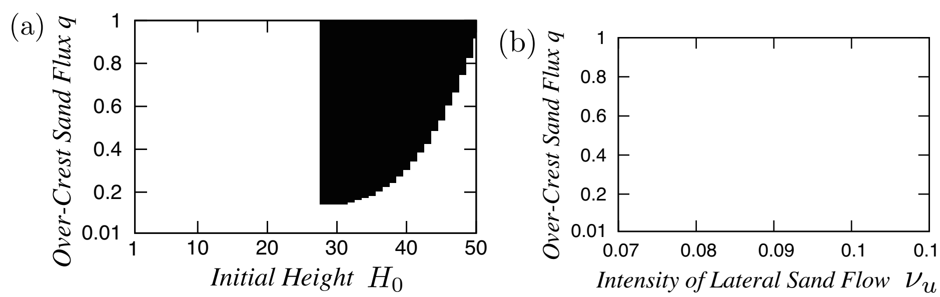

FIG. 7: Results obtained by numerical simulations varying a pair of control parameters (a) H_0 and q, (b) ν_u and q. White area indicate the condition under which straightly extending transverse dune is stable, whereas in the black area initial transverse dune is deformed to the shape of barchan as seen in Fig. 6, and in the gray area, wavy transverse dunes keep their shape for long time. Other parameters are fixed like (a) $\nu_u = 0.1, \nu_d = 0.1$, (b) $\nu_d = 0.1, H_0 = 30.0$.

Finally, we conduct linear stability analysis of two 2D-CSs system which is considered as the simplest skeleton model of a transverse dune. Here the uniformly redistributing rule of escaped sand flow is applied like the above simulations. Thus, as long as two 2D-CSs are kept nonvanished, the total area cross-sections S is kept constant. In this case, (8) consist of four variables (h_1, h_2, x_1, x_2), and h_2 is expressed like $h_2 \equiv \sqrt{2\mathrm{A}S - h_1^2}$ using the constant S and h_1. Defining the wind directional relative position $y = x_2 - x_1$ of two cross-sections, (8) are simplified into two variables (h_1, y) equations.

$$\frac{dh_1}{dt} = \frac{\mathrm{A}}{h_1}\left(q(T_E(h_1) - 1) + D_{d(1\to2)} - D_{u(1\to2)} + f_1^{in}\right) \tag{9a}$$

$$\frac{dy}{dt} = q\mathrm{B}\left(\frac{T_E(h_2)}{h_2} - \frac{T_E(h_1)}{h_1}\right) + (\mathrm{C} - f_1^{in})\left(\frac{1}{h_2} - \frac{1}{h_1}\right) - (\mathrm{B}D_{d(1\to2)} - \mathrm{C}D_{u(1\to2)})\left(\frac{1}{h_2} + \frac{1}{h_1}\right) \tag{9b}$$

A fixed point $(h_1^*, y^*) = (\sqrt{\mathrm{A}S}, 0)$ of (9a),(9b) corresponds to the straight state of transverse dune. We study the stability of this fixed point by a linear analysis.

Figure. 8(a) and (b) show the stability of the fixed point in two set of parameter space: $S - q$ space and $\nu_u - q$ space. In both figures, the fixed point is linearly stable in the white areas, whereas it is unstable in the black areas. These results qualitatively corresponds to those shown in Fig. 7(a) and (b).

In conclusion, we proposed a skeleton model of 3D dune dynamics described as a coupled dynamics of laterally arrayed 2D wind-directional cross sections.

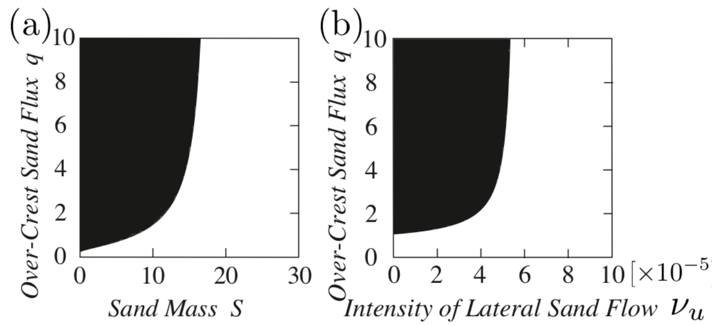

FIG. 8: Results obtained by a linear analysis of the fixed point of (9) varying two pairs of control parameter: (a) (S, q), (b) (ν_u, q), respectively. White area indicate the conditions under which straightly extending transverse dune is stable, whereas in the black area initial transverse dune is deformed to the shape of barchan. Here S is the index of the amount of available sand in the field, thus, these results qualitative correspond to Fig. 7(a) and (b) obtained through numerical simulations. Other parameter are fixed like (a) $\nu_u = 5.0 \times 10^{-5}, \nu_d = 1.0 \times 10^{-4}$, (b) $\nu_d = 1.0 \times 10^{-4}, S = 15$.

By using this model, we studied the stability of the shape of transverse dunes through both numerical simulations and linear analysis and obtained a qualitative correspondence to the previous results obtained through observations of real dunes and more complicated simulation models.

Because of the simplicity of the model, we expect this model supplies us with an effective tool for the theoretical study for the complex dynamics of 3D dunes.

Acknowledgment

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