Mutation-Periodic Quivers, Integrable Maps and Associated Poisson Algebras*

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Abstract

We consider a class of map, recently derived in the context of cluster mutation. In this paper we start with a brief review of the quiver context, but then move onto a discussion of a related Poisson bracket, along with the Poisson algebra of a special family of functions associated with these maps. A bi-Hamiltonian structure is derived and used to construct a sequence of Poisson commuting functions and hence show complete integrability. Canonical coordinates are derived, with the map now being a canonical transformation with a sequence of commuting invariant functions. Compatibility of a pair of these functions gives rise to Liouville's equation and the map plays the role of a Bäcklund transformation.

Keywords: Poisson algebra, bi-Hamiltonian, integrable maps, Bäcklund transformations, Laurent property, cluster algebra, quiver gauge theory.

1 Introduction

Robin Bullough's famous diagram represents a vast array of areas in Mathematics and Mathematical Physics, together with a "neural network" of connections (solid lines when established, dotted when expected) between them. This "Grand Synthesis of Soliton Theory" shows that some remarkable connections between seemingly disparate subjects have come about through the developments of Integrable Systems, which have taken place in the last 40 years or so. Of course, the diagram perpetually evolves, as dotted connections become solid and as new subject areas (with corresponding links) are added to the array. In this paper, I present some connections with subjects that didn't even exist until recently!

*This article is dedicated to the memory of Robin Bullough.

Specifically, the present paper is concerned with *integrable maps* which arise in the context of *cluster mutations* (see Fomin and Zelevinsky (2002b)). This gives a connection to maps with the Laurent property, with the archetypical example being the Somos 4 iteration (see "The On-Line Encyclopedia of Integer Sequences" at Sloane (2009)), which arises in the context of elliptic divisibility sequences in number theory. In Fordy and Marsh (2009) we considered a class of quiver which had a certain periodicity property under "quiver mutation". The corresponding "cluster exchange relations" then give rise to sequences with the Laurent property, which generalise many of the well known examples.

In this paper I first explain some of this background, but the main emphasis will be on some associated Poisson algebras (with respect to an invariant Poisson bracket of log-canonical type). In terms of *canonical* variables we obtain Hamiltonians (invariant under the action of the map) in exponential form. The compatibility of one particular pair of invariant Hamiltonians leads to Liouville's equation for each q_i, with the map now playing the role of a Bäcklund transformation, bringing us back to one of the original ideas in Soliton Theory!

2 The Laurent Property

The Somos 4 sequence is generated by the iteration on the real line

$$x_n x_{n+4} = x_{n+1}x_{n+3} + x_{n+2}^2, \quad \text{with} \quad x_0 = x_1 = x_2 = x_3 = 1, \tag{1}$$

giving

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, \ldots$$

Since we must divide by x_n at each step, it is not obvious that we generate integers from $x_8 = 59$ onwards. Even more, starting with initial conditions $x_0 = s, x_1 = t, x_2 = u, x_3 = v$, we find that each term x_n is a *Laurent polynomial* in these initial values (ie, a polynomial in $s^{\pm 1}$, $t^{\pm 1}$, $u^{\pm 1}$, $v^{\pm 1}$). Integrality of the above numerical sequence then follows by setting $r = s = u = v = 1$. Considering an obvious generalisation of (1) (called the Somos N sequence)

$$x_n x_{n+N} = \sum_{r=1}^{[N/2]} x_{n+r}x_{n+N-r},$$

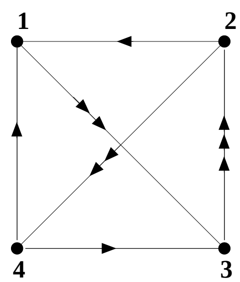
it is found that it too has the Laurent property for $N = 4, 5, 6, 7$, but fails at $N = 8$. *Failure* is rather simple to prove, since non-integer (rational) elements occur fairly soon in the sequence. Ad-hoc proofs exist for various sequences (see Gale (1991) for a proof in the case of Somos 4), and can often be adapted to other sequences. However, a remarkable (but complicated!) proof for a very broad class of iteration was given in Fomin and Zelevinsky (2002a).

At about the same time, *cluster algebras* were introduced in Fomin and Zelevinsky (2002b), and it was shown that any map which arose as a *cluster exchange relation* necessarily had the Laurent property. Cluster algebras are an abstraction of structures which arise in the study of total positivity of matrices and in the canonical basis of a quantum group. However, for this paper we need none of this background. Neither do we need the full definition of a cluster algebra. The most important aspect for us is the association with *quivers* and *quiver mutation*.

2.1 Quiver Mutation

A quiver is a *directed graph*, consisting of N nodes with directed edges between them. There may be several arrows between a given pair of vertices, but for cluster algebras there should be no 1-cycles (an arrow which starts and ends at the same node) or 2-cycles (an arrow from node a to node b, followed by one from node b to node a). A quiver Q, with N nodes, can be identified with the unique skew-symmetric $N \times N$ matrix B_Q with $(B_Q)_{ij}$ given by the number of arrows from i to j minus the number of arrows from j to i.

An important quiver for our discussion is the one corresponding to the Somos 4 sequence (both quiver and matrix in Figure 1).



$$B = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix}$$

Figure 1: The Somos 4 quiver S_4 and its matrix.

Definition 2.1 (Quiver Mutation) *Given a quiver Q we can mutate at any of its nodes. The mutation of Q at node k, denoted by $\mu_k Q$, is constructed (from Q) as follows:*

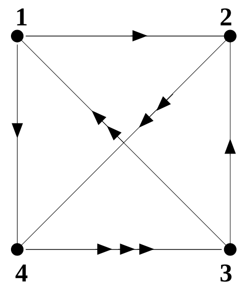
1. *Reverse all arrows which either originate or terminate at node k.*
2. *Suppose that there are p arrows from node i to node k and q arrows from node k to node j (in Q). Add pq arrows going from node i to node j to any arrows already there.*
3. *Remove (both arrows of) any two-cycles created in the previous steps.*

Note that in Step 2, pq is just the number of paths of length 2 between nodes i and j which pass through node k.

Remark 2.2 (Matrix Mutation) *Let B and $\tilde{B}$ be the skew-symmetric matrices corresponding to the quivers Q and $\tilde{Q} = \mu_k Q$. Let b_{ij} and $\tilde{b}_{ij}$ be the* *corresponding matrix entries. Then quiver mutation amounts to the following formula*

$$\tilde{b}_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|) & \text{otherwise.} \end{cases} \tag{2}$$

It is an exercise to show that with these definitions, the Somos 4 quiver and matrix are transformed to those of Figure 2, if we mutate at node 1.



$$\tilde{B} = \begin{pmatrix} 0 & 1 & -2 & 1 \\ -1 & 0 & -1 & 2 \\ 2 & 1 & 0 & -3 \\ -1 & -2 & 3 & 0 \end{pmatrix}$$

Figure 2: Mutation $\tilde{S}_4 = \mu_1 S_4$ of the quiver S_4 at node 1 and its matrix.

2.2 Cluster Exchange Relations

Given a quiver (with N nodes), we attach a variable at each node, labelled $(x_1, \cdots, x_N)$. When we mutate the quiver we change the associated matrix according to formula (2) and, *in addition*, we transform the cluster variables $(x_1, \cdots, x_N) \mapsto (x_1, \cdots, \tilde{x}_\ell, \cdots, x_N)$, where

$$x_\ell \tilde{x}_\ell = \prod_{b_{i\ell}>0} x_i^{b_{i\ell}} + \prod_{b_{i\ell}<0} x_i^{-b_{i\ell}}, \qquad \tilde{x}_i = x_i \text{ for } i \neq \ell. \tag{3}$$

If one of these products is empty (which occurs when all $b_{i\ell}$ have the same sign) then it is replaced by the number 1. This formula is called the (cluster) *exchange relation*. Notice that it just depends upon the ℓ^{th} column of the matrix. Since the matrix is skew-symmetric, the variable x_ℓ **does not** occur on the right side of (3).

After this process we have a new quiver $\tilde{Q}$, with a new matrix $\tilde{B}$. This new quiver has cluster variables $(\tilde{x}_1, \cdots, \tilde{x}_N)$. However, since the exchange relation (3) acts as the identity on all except one variable, we write these new cluster variables as $(x_1, \cdots, \tilde{x}_\ell, \cdots, x_N)$. We can now repeat this process and mutate $\tilde{Q}$ at node p and produce a third quiver $\tilde{\tilde{Q}}$, with cluster variables $(x_1, \cdots, \tilde{x}_\ell, \cdots, \tilde{x}_p, \cdots, x_N)$, with $\tilde{x}_p$ being given by an analogous formula (3).

Remark 2.3 (Involutive Property of the Exchange Relation) *If $p = \ell$, then $\tilde{\tilde{Q}} = Q$, so we insist that $p \neq \ell$.*

Example 2.4 (The Somos 4 Quiver S_4) Placing x_1, x_2, x_3, x_4 respectively at nodes 1 to 4 of quiver S_4 (of Figure 1) gives the *initial cluster*. Along with the *quiver* mutation (leading to $\mu_1 S_4$ of Figure 2), we also have the *exchange relation*

$$x_1\tilde{x}_1 = x_2x_4 + x_3^2. \tag{4}$$

This corresponds to one arrow coming **into** node 1 from each of nodes 2 and 4 with 2 arrows going **out** to node 3.

We can now consider mutations of quiver $\tilde{S}_4 = \mu_1 S_4$. To avoid too many "tildes", let us write $\tilde{x}_1 = x_5$, so quiver $\tilde{S}_4$ has x_5, x_2, x_3, x_4 respectively at nodes 1 to 4. Mutation at node 1 would just take us back to quiver S_4 (as noted in the above remark). We compare the exchange relations we would obtain by mutating at nodes 2 or 3.

Mutation at node 2 would lead to exchange relation

$$x_2\tilde{x}_2 = x_3x_5 + x_4^2, \tag{5}$$

whilst that at node 3 would lead to

$$x_3\tilde{x}_3 = x_2x_5^2 + x_4^3. \tag{6}$$

We see that the right hand sides of formula (4) and (5) are related by a **shift**, whilst formula (6) is entirely *different*. In fact, it can be seen in Figures 1 and 2 that the configuration of arrows at node 2 of quiver $\tilde{S}_4$ is *exactly the same* as that at node 1 of quiver S_4, thus giving the *same* exchange relation. In fact, we have more. The *whole quiver* $\tilde{S}_4$ is obtained from S_4 by just **rotating the arrows**, whilst keeping the nodes fixed. It follows that mutation of quiver $\tilde{S}_4$ *at node* 2 just leads to a further rotation, with node 3 inheriting this same configuration of arrows. If at each step we relabel $\tilde{x}_n$ as x_{n+4}, the n^{th} exchange relation can be written

$$x_nx_{n+4} = x_{n+1}x_{n+3} + x_{n+2}^2. \tag{7}$$

This rotational property of the quiver has lead to an **iteration**, which in this case is just Somos 4.

In Fordy and Marsh (2009) we introduced and studied quivers with this type of rotational property. Consider the $N \times N$ matrix

$$\rho = \begin{pmatrix} 0 & \cdots & \cdots & 1 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}.$$

The above rotation, which we write $S_4 \to \tilde{S}_4 = \rho S_4$, is achieved in the matrix formulation by

$$\tilde{B} = \rho B \rho^{-1},$$

with $N = 4$ in this case.

Consider a quiver $Q = Q(1)$, with N nodes. We consider a sequence of mutations, starting at node 1, followed by node 2, and so on. Mutation at node 1 of a quiver $Q(1)$ will produce a second quiver $Q(2)$. The mutation at node 2 will therefore be of quiver $Q(2)$, giving rise to quiver $Q(3)$ and so on. We define a *period m quiver* as follows.

Definition 2.5 *A quiver Q has period m if it satisfies $Q(m+1) = \rho^m Q(1)$ (with m the minimum such integer). The mutation sequence is depicted by*

$$Q = Q(1) \xrightarrow{\mu_1} Q(2) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{m-1}} Q(m) \xrightarrow{\mu_m} Q(m+1) = \rho^m Q(1), \qquad (8)$$

and called the periodic chain *associated to Q.*

The corresponding matrices would then satisfy $B(m+1) = \rho^m B(1) \rho^{-m}$.

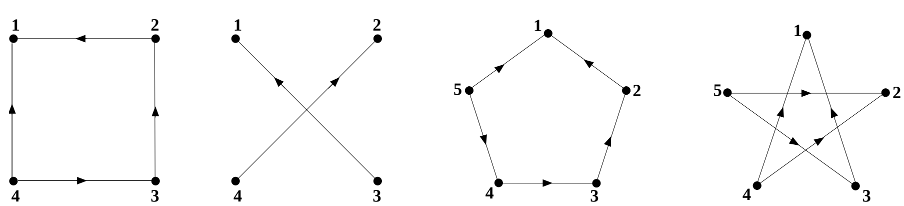
Remark 2.6 (The Sequence of Mutations) *We must perform the* **correct** *sequence of mutations. For instance, if we mutate $\mu_1 S_4$ at node 3, we obtain a quiver which has 5 arrows from node 4 to node 1, which* **cannot** *be permutation equivalent to $Q(1) = S_4$. As we previously saw, the corresponding exchange relation (6) was also different.*

Remark 2.7 (Periodicity and Iterations) **Period 1 quivers** *correspond to iterations on the real line.* **Period m quivers** *correspond to iterations on $\mathbb{R}^m$. The formula (3) consists of only two terms (additively), corresponding to* **incoming** *and* **outgoing** *arrows. Both the Somos 4 and Somos 5 iterations can be built in this way, but* **not** *Somos 6 or 7, which contain 3 terms.*

In Fordy and Marsh (2009) we give a *full* classification of period 1 quivers, a partial classification of period 2 quivers and examples of higher period ones.

2.3 Primitive Quivers

In our classification of period 1 quivers we introduced a special class of quivers, called *primitives*. An important feature of a primitive is that node 1 is a *sink*, so only step 1 of the mutation (Definition 2.1) is needed. We constructed a basis of primitives for each N (see Figure 3 for $N = 4, 5$). The basis consists of $P_N^{(r)}$,



(a) $P_4^{(1)}$ (b) $P_4^{(2)}$ (c) $P_5^{(1)}$ (d) $P_5^{(2)}$

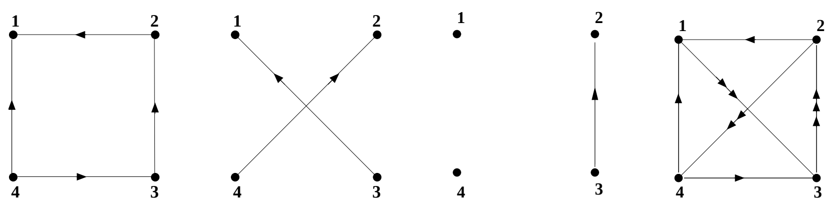
Figure 3: The period 1 primitives for 4 and 5 nodes.

for $1 \leq r \leq N/2$.

In our classification, the primitives are the "atoms" out of which we build the general period 1 quiver for each N. For given N, we can start with an arbitrary linear combination of $P_N^{(r)}$, with integer coefficients

$$Q_N^0 = \sum_{r=1}^{[N/2]} m_r P_N^{(r)}.$$

If all the m_r have the *same* sign, then Q_N^0 already has period 1, but otherwise, we must add "correction terms", which are integer combinations of primitives with $N-2k$ nodes ($1 \leq k \leq [N/2]$). Our classification of period 1 quivers gives the formula for these coefficients in terms of the original coefficients m_r. For the Somos 4 quiver, we have $m_1 = 1, m_2 = -2$ and our formula requires the addition of a further two arrows between nodes 3 and 2 (see Figure 4).



(a) $P_4^{(1)}$ (b) $P_4^{(2)}$ (c) $P_2^{(1)}$ (d) S_4

Figure 4: One of $P_4^{(1)}$ minus two of $P_4^{(2)}$ plus two of $P_2^{(1)}$ gives S_4

2.4 Laurent Property vs Complete Integrability

Each iteration we obtain through our construction is **guaranteed** to have the Laurent property (by the results of Fomin and Zelevinsky (2002b)). However, only special cases are expected to be completely integrable in any sense (see Veselov (1991) for various definitions). For instance, the most general 4 node, period 1 quiver corresponds to the iteration

$$x_n x_{n+4} = x_{n+1}^r x_{n+3}^r + x_{n+2}^s. \tag{9}$$

The iterations with $r = 1, s \in \{0, 1, 2\}$ are analysed by Hone (2007), who shows that these cases are Liouville integrable (even *super-integrable* when $s = 0, 1$).

We first write (9) as a **map** on the 4−dimensional space with coordinates x_0, x_1, x_2, x_3:

$$\varphi(x_0, \ldots, x_3) = \left(x_1, x_2, x_3, \frac{x_1^r x_3^r + x_2^s}{x_0} \right). \tag{10}$$

The log-canonical Poisson bracket $\{x_i, x_j\} = P_{ij} x_i x_j$ (see Gekhtman et al.

(2003) for a general discussion), where

$$P = \begin{pmatrix} 0 & r & s & r(1+s) \\ -r & 0 & r & s \\ -s & -r & 0 & r \\ -r(1+s) & -s & -r & 0 \end{pmatrix}, \tag{11}$$

is *invariant* under the action of the map φ. This means that, if $\tilde{\mathbf{x}} = \varphi(\mathbf{x})$, then $\{\tilde{x}_i, \tilde{x}_j\} = P_{ij}\tilde{x}_i\tilde{x}_j$.

Remark 2.8 *It is an interesting fact that the matrix P is (up to an overall factor) the* **inverse** *of the B matrix for the corresponding quiver. The factor is the Pfaffian $(2+s)r^2 - s^2$, which actually vanishes in the Somos 4 case. Nevertheless, the matrix P, with $r = 1, s = 2$, is still invariant under the map.*

Liouville integrability is defined in the same way as for continuous Hamiltonian systems (see Veselov (1991)). We must first use the Casimir functions (when the Poisson matrix is degenerate) to reduce to the symplectic leaves, whose dimension is $2d$, where d is the number of degrees of freedom. We then require the existence of d, functionally independent Hamiltonians, $h_1, \dots, h_d$, which should be *in involution* (so $\{h_i, h_j\} = 0$, for all i, j). For the discrete case, we have the extra requirement that the functions $h_1, \dots, h_d$ are *invariants* of the map. This means that the map has a system of d commuting *continuous symmetries* (the Hamiltonian flows). In the continuous case we can say that the Hamiltonian system is solvable, *up to quadrature*, but this notion is not carried to the discrete case.

In Hone (2007), it is shown that for cases $r = 1, s = 0, 1$, there are 3 independent, invariant functions, out of which it is possible to construct two Poisson commuting functions. Such systems (with additional first integrals) are known as *super-integrable*. In the case $r = 1, s = 2$ (Somos 4) the Poisson bracket is degenerate, with two Casimir functions, which are *not invariant* under the map. However, the action of the map on these Casimirs is by a $2-$dimensional integrable map (a special case of the symmetric QRT map (see Quispel et al. (1988))).

It is not known whether the map (9) is Liouville integrable for other values of r and s, but some of the standard integrability tests (such as *algebraic entropy* (see Bellon and Viallet (1999))) indicate **non-integrability**.

Remark 2.9 *Isolating and analysing the integrable cases is one of the most interesting outstanding problems.*

3 The $P_N^{(1)}$ Iteration as a Map

The following iteration corresponds to the period 1 primitive $P_N^{(1)}$ with N nodes:

$$x_n x_{n+N} = x_{n+1}x_{n+N-1} + 1, \tag{12}$$

with initial conditions $x_i = a_i$ for $0 \le i \le N-1$. In Fordy and Marsh (2009) it was shown that there exists a special sequence of functions

$$J_n = \frac{x_n + x_{n+2}}{x_{n+1}}, \quad \text{satisfying} \quad J_{n+N-1} = J_n. \tag{13}$$

With the given initial conditions, we have $\{J_i = c_i : 0 \le i \le N-2\}$, together with the periodicity condition, which can also be written as $J_n = c_n$ with $c_{n+N-1} = c_n$.

Theorem 3.1 (Linearisation) *If the sequence $\{x_n\}$ is given by the iteration (12), with initial conditions $\{x_i = a_i : 0 \le i \le N-1\}$, then it also satisfies*

$$x_n + x_{n+2(N-1)} = S_N\, x_{n+N-1}, \tag{14}$$

where S_N is a function of $c_0, \cdots, c_{N-2}$, which is symmetric under cyclic permutations.

Here we restrict to the case of even N. The first few S_N take the form

$$\begin{aligned} &S_2 = c_0; \quad S_4 = c_0c_1c_2 - c_0 - c_1 - c_2; \\ &S_6 = c_0c_1c_2c_3c_4 - c_0c_1c_2 - c_1c_2c_3 - c_2c_3c_4 - c_3c_4c_0 - c_4c_0c_1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad + c_0 + c_1 + c_2 + c_3 + c_4. \end{aligned}$$

The function S_N is an *invariant function* of the nonlinear map (12), so this linearisation depends upon the particular initial conditions.

3.1 The Log-Canonical Poisson Bracket

As in the 4^{th} order case (10), write the N^{th} order iteration (12) as a map of the space with coordinates $(x_0, \ldots, x_{N-1})$, given by

$$\varphi(x_0, \ldots, x_{N-1}) = \left(x_1, \ldots, x_{N-1}, \frac{x_1 x_{N-1} + 1}{x_0} \right). \tag{15}$$

Again we seek an invariant Poisson bracket of log-canonical form:

$$\{x_i, x_j\} = P_{ij} x_i x_j, \quad 0 \le i < j \le N-1, \tag{16}$$

for some constants P_{ij}. We seek the value of these constants for which this Poisson bracket is invariant under the action of the map φ. Writing $\tilde{\mathbf{x}} = \varphi(\mathbf{x})$, we require

$$\{\tilde{x}_i, \tilde{x}_j\} = P_{ij} \tilde{x}_i \tilde{x}_j.$$

The shift structure of the map (15) implies a banded structure, with $P_{i+1\,j+1} = P_{ij}$, so the undetermined constants are P_{0j}, $j = 1, \ldots, N-1$. The precise form of $\tilde{x}_{N-1}$ puts strong constraints on these, which can be determined up to an overall multiplicative constant.

Lemma 3.2 *For a nontrivial Poisson bracket of the form (16) to be invariant under the map (15), we require N to be* **even***, in which case the coefficients take the form*

$$P_{ij} = \begin{cases} 1 & \text{when } i < j \text{ and } i+j \text{ is odd,} \\ 0 & \text{when } i < j \text{ and } i+j \text{ is even.} \end{cases}$$

This Poisson bracket is non-degenerate.

Remark 3.3 *Again, it is an interesting fact that the matrix P is (up to an overall factor) the* **inverse** *of the B matrix for the corresponding quiver.*

3.2 The Poisson Algebra of Functions J_m

The independent functions $J_0, \dots, J_{N-2}$, written in terms of the coordinates $(x_0, \dots, x_{N-1})$, are given by

$$J_m = \frac{x_m + x_{m+2}}{x_{m+1}}, \quad m = 0, \dots, N-3, \quad J_{N-2} = \frac{x_0 x_{N-2} + x_1 x_{N-1} + 1}{x_0 x_{N-1}}. \tag{17}$$

Under the action of the map φ they satisfy the cyclic conditions

$$J_n \circ \varphi = J_{n+1}, \;\; n = 0, \dots, N-3 \quad \text{and} \;\; J_{N-2} \circ \varphi = J_0. \tag{18}$$

We just need to calculate the $N-3$ brackets $\{J_0, J_n\}$, $n = 1, \dots, N-3$, since all others follow through the above relations. These are easily calculated to be

$$\begin{aligned} &\{J_0, J_1\} = 2J_0J_1 - 2, \quad \{J_0, J_{2m-1}\} = 2J_0J_{2m-1}, \quad 2 \le m \le M-1, \\ &\{J_0, J_{2m}\} = -2J_0J_{2m}, \quad 1 \le m \le M-2, \end{aligned} \tag{19}$$

where $N = 2M$. The cyclic action of φ then implies

$$\begin{aligned} &\{J_m, J_{m+1}\} = 2J_mJ_{m+1} - 2, \quad 1 \le m \le N-3, \\ &\{J_m, J_n\} = 2(-1)^{m+n-1} J_m J_n \quad \text{for} \quad 1 \le m \le n-2 \le N-4, \\ &\{J_0, J_{N-2}\} = -2J_0J_1 + 2, \end{aligned} \tag{20}$$

where the relation for $\{J_0, J_{N-2}\}$ was obtained from that of $\{J_{N-3}, J_{N-2}\}$ through the action of φ.

By taking cyclic sums of any function of the J_n we can build functions which are invariant under the action of φ. Since the Poisson bracket (16) (with P_{ij} as given in Lemma 3.2) is non-degenerate (on a $2M-$dimensional space), our task is to select M invariant functions which are *in involution*. It is, in fact, easier to work with the Poisson bracket relations (19) and (20), which define a Poisson bracket on the $(2M-1)$ dimensional $J-$space. The corresponding Poisson matrix P is the sum of two homogeneous parts: $P = P_2 + P_0$, each of which is itself a Poisson matrix. These therefore define a compatible pair of Poisson brackets:

Definition 3.4 (Compatible Poisson Brackets) *The matrices P_0, P_2 and $P = P_2 + P_0$ define compatible Poisson brackets*

$$\{f,g\}_i = \nabla f P_i \nabla g, \quad i = 0,2 \quad \textit{and} \quad \{f,g\}_P = \nabla f (P_2 + P_0) \nabla g.$$

We use these brackets to define a *bi-Hamiltonian ladder* (see Magri (1978)), starting with the Casimir function of P_0 and ending with that of P_2:

$$(P_2 + P_0)(\nabla h_M - \nabla h_{M-1} + \nabla h_{M-2} - \cdots + (-1)^{M+1} \nabla h_1) = 0, \tag{21}$$

where h_k is a homogeneous polynomial of degree $2k - 1$. The homogeneity property of P_0, P_2, h_k leads to equation (21) decoupling into a sequence of $M+1$ homogeneous equations (the bi-Hamiltonian ladder):

$$P_0 \nabla h_1 = 0, \;\; P_0 \nabla h_k = P_2 \nabla h_{k-1}, \quad \text{for} \quad 2 \leq k \leq M, \quad \text{and} \quad P_2 \nabla h_M = 0. \tag{22}$$

The functions h_1, h_M are easy to find from the form of the Poisson matrices:

$$h_1 = \sum_{k=1}^{2M-1} J_k, \quad h_M = \prod_{k=1}^{2M-1} J_k. \tag{23}$$

The remaining functions, $h_2, \ldots, h_{M-1}$, are obtained by solving the "central" sequence of equations (22). Since P_0 has a Casimir function, we need to check that the equations are compatible, in that $\nabla h_1 P_2 \nabla h_{k-1} = 0$. We use the following result

Lemma 3.5 (Bi-Hamiltonian Relations) *With the Poisson brackets given by Definition 3.4, the functions $h_1, \ldots, h_M$ satisfy*

$$\{h_i, h_j\}_0 = \{h_i, h_{j-1}\}_2 \quad \textit{and} \quad \{h_i, h_j\}_2 = \{h_{i+1}, h_j\}_0.$$

Proof: The ladder relations (22) imply

$$\{h_i, h_j\}_0 = \nabla h_i P_0 \nabla h_j = \nabla h_i P_2 \nabla h_{j-1} = \{h_i, h_{j-1}\}_2,$$

and

$$\begin{aligned} \{h_i, h_j\}_2 = \nabla h_i P_2 \nabla h_j &= -\nabla h_j P_2 \nabla h_i \\ &= -\nabla h_j P_0 \nabla h_{i+1} = \nabla h_{i+1} P_0 \nabla h_j = \{h_{i+1}, h_j\}_0. \end{aligned}$$

Lemma 3.6 (Compatibility of Equations (22)) *Equations (22) are compatible.*

Proof: The compatibility condition $\nabla h_1 P_2 \nabla h_{k-1} = 0$ is just $\{h_1, h_{k-1}\}_2 = 0$. The first equation is just

$$\{h_1, h_1\}_2 = 0,$$

which is obviously satisfied, so it is possible to solve for h_2. Now suppose we have functions $h_1, \ldots, h_{k-1}$. The compatibility condition is

$$\{h_1, h_{k-1}\}_2 = \{h_2, h_{k-1}\}_0 = \{h_2, h_{k-2}\}_2 = \cdots = \{h_s, h_s\}_\ell = 0,$$

for some s, ℓ.

To solve Equations (22) write the equations in terms of the coordinates $(J_0, \ldots, J_{2M-3}, z_1)$ (with $z_1 = h_1$), after which P_0 has a complete row (and column) of zeros, with the non-zero part being invertible. The above compatibility means that the final entry in the column vector $P_2 \nabla h_{k-1}$ is zero. These calculations are straightforward and give rise to a sequence of functions of $(J_0, \ldots, J_{2M-3}, z_1)$. These are only defined up to an additive function of z_1, which can be discarded. Replacing z_1 by h_1, we obtain the desired functions of $(J_0, \ldots, J_{2M-3}, J_{2M-2})$. We then have the following theorem of Magri (1978):

Theorem 3.7 (Complete Integrability) *The functions $h_1, \ldots, h_M$ are in involution with respect to both of the above Poisson brackets*

$$\{h_i, h_j\}_0 = \{h_i, h_j\}_2 = 0, \quad \textit{and hence} \quad \{h_i, h_j\}_P = 0.$$

It then follows from Liouville's theorem that the functions $h_1, \ldots, h_M$ define a completely integrable Hamiltonian system.

Proof: Without loss of generality, choose $i < j$. Then

$$\{h_i, h_j\}_0 = \{h_i, h_{j-1}\}_2 = \{h_{i+1}, h_{j-1}\}_0 = \cdots = \{h_k, h_k\}_\ell = 0,$$

for some k, ℓ. Similarly

$$\{h_i, h_j\}_2 = \{h_{i+1}, h_j\}_0 = \{h_{i+1}, h_{j-1}\}_2 = \cdots = \{h_k, h_k\}_\ell = 0,$$

for some k, ℓ.

The Casimir Function

Formula (21) just states that the function

$$\mathcal{C} = h_M - h_{M-1} + h_{M-2} - \cdots + (-1)^{M+1} h_1 \tag{24}$$

is the Casimir of the Poisson matrix P, so (with respect to $\{\ ,\ \}_P$) commutes with each J_i and hence with *all* functions of J_i (not just with $h_1, \ldots, h_M$).

Example 3.8 (The Case $N = 4$) Here we have 3 basic functions J_0, J_1, J_2. With, $M = 2$, we have h_1 and h_2, given by (23).

Example 3.9 (The Case $N = 6$) Here we have 5 basic functions $J_0, \ldots, J_4$. With, $M = 3$ we have h_1 and h_3, given by (23), and

$$h_2 = J_0 J_1 J_2 + J_1 J_2 J_3 + J_2 J_3 J_4 + J_3 J_4 J_0 + J_4 J_0 J_1.$$

Example 3.10 (The Case $N = 8$) Here we have 7 basic functions $J_0, \dots, J_6$. With $M = 4$, we have h_1 and h_4, given by (23), and

$$h_2 = \sum_{i=0}^{6} J_i J_{i+1}(J_{i+2} + J_{i+4}), \qquad h_3 = \sum_{i=0}^{6} J_i J_{i+1} J_{i+2} J_{i+3} J_{i+4},$$

the indices here being taken modulo 6.

Remark 3.11 *It can be seen from the list following Theorem 3.1, that S_4, S_6 (replacing c_i by J_i) are just $\mathcal{C}$ of the above examples. We can use the linear difference equation (14) to* **define** *S_N, repeatedly using the formula $x_n x_{n+N} = x_{n+1}x_{n+N-1} + 1$ to rewrite this as a function of $x_0, \dots, x_{N-1}$.*

Conjecture: The Casimir function for general N can also be written as

$$\mathcal{C} = \frac{x_0 + x_{2(N-1)}}{x_{N-1}} \quad \text{written in terms of } x_0, \dots, x_{N-1}. \tag{25}$$

4 The Maps in Canonical Coordinates

The Poisson bracket (16), with the P_{ij} being given by Lemma 3.2, naturally separates the odd and even numbered variables, from which we construct respectively canonical variables p_i and q_i as follows:

$$q_i = \log(x_{2(i-1)}), \;\; i = 1, \dots, M, \tag{26}$$

$$p_1 = \frac{1}{2}\log(x_1 x_{N-1}), \quad p_i = \frac{1}{2}\log\left(\frac{x_{2i-1}}{x_{2i-3}}\right), \;\; i = 2, \dots, M, \tag{27}$$

where $N = 2M$. Defining

$$\pi_r = \sum_{i=1}^{r} p_i - \sum_{i=r+1}^{M} p_i, \quad 0 \le r \le M-1, \quad \pi_M = \sum_{i=1}^{M} p_i,$$

(so $\pi_i = \log(x_{2i-1})$) the inverse of this transformation is written

$$x_{2r} = e^{q_{r+1}}, \quad x_{2r+1} = e^{\pi_{r+1}}, \quad 0 \le r \le M-1, \tag{28}$$

and the functions J_k take the form

$$\begin{aligned} J_{2r} &= e^{-\pi_{r+1}}(e^{q_{r+1}} + e^{q_{r+2}}), \;\; 0 \le r \le M-2, \\ J_{2r+1} &= e^{-q_{r+2}}(e^{\pi_{r+1}} + e^{\pi_{r+2}}), \;\; 0 \le r \le M-2, \\ J_{2M-2} &= e^{-q_1-\pi_M}(e^{q_1+q_M} + e^{\pi_1+\pi_M} + 1). \end{aligned} \tag{29}$$

The map φ (see (15)) is canonical, now having the form

$$\begin{aligned} &\tilde{q}_r = \pi_r,\ 1 \le r \le M \\ &\tilde{p}_1 = \tfrac{1}{2}(q_2 - q_1 + Log(1 + e^{2p_1})), \quad \tilde{p}_M = \tfrac{1}{2}(-q_1 - q_M + \log(1 + e^{2p_1})), \\ &\tilde{p}_r = \tfrac{1}{2}(q_{r+1} - q_r),\ \ 2 \le r \le M-1. \end{aligned} \tag{30}$$

The variables π_r, transform as

$$\tilde{\pi}_r = q_{r+1},\ \ 1 \le r \le M-1, \quad \tilde{\pi}_M = -q_1 + \log(1 + e^{(\pi_1 + \pi_M)}). \tag{31}$$

The functions (29) inherit the cyclic behaviour (18) under this map.

Now consider the function

$$\begin{aligned} \mathcal{C} &= \sum_{i=1}^{M-1} e^{-\pi_i}(e^{-q_i} + e^{-q_{i+1}}) + e^{-\pi_M}(e^{q_1} + e^{-q_M}) + e^{\pi_M - q_1} \\ &= \sum_{i=1}^{M-1} e^{-q_{i+1}}(e^{-\pi_i} + e^{-\pi_{i+1}}) + e^{-q_1}(e^{-\pi_1} + e^{\pi_M}) + e^{q_1 - \pi_M}. \end{aligned} \tag{32}$$

The second line is just a re-ordering of the first, but useful.

Lemma 4.1 (Symmetry under the map (30)) *Under the map (30), the function $\mathcal{C}$ is invariant: $\tilde{\mathcal{C}} = \mathcal{C}$.*

Proof

Using (31) it is easy to show that

$$\begin{aligned} &e^{-\pi_i}(e^{-q_i} + e^{-q_{i+1}}) \to e^{-q_{i+1}}(e^{-\pi_i} + e^{-\pi_{i+1}}), \\ &e^{\pi_M - q_1} \to e^{-q_1}(e^{-\pi_1} + e^{\pi_M}), \quad e^{-\pi_M}(e^{q_1} + e^{-q_M}) \to e^{q_1 - \pi_M}, \end{aligned}$$

so the first line of (32) transforms to the second, giving the result.

Theorem 4.2 (Casimir Function) *The function $\mathcal{C}$ is a Casimir function for the Poisson algebra of functions J_i.*

Proof

First note that

$$\left\{J_0, e^{-\pi_i}(e^{-q_i} + e^{-q_{i+1}})\right\} = \left\{J_0, e^{-\pi_M}(e^{q_1} + e^{-q_M})\right\} = \left\{J_0, e^{\pi_M - q_1}\right\} = 0,$$

so $\{J_0, \mathcal{C}\} = 0$. Since $\mathcal{C}$ is an invariant function under the map (30), this implies that $\{J_r, \mathcal{C}\} = 0$, for all r, giving the result.

Remark 4.3 *We now have* 3 *expressions for the Casimir function of the J algebra ((24), (25) and (32)), which coincide on all known explicit examples. However, I have no proof that these are the same.*

5 The Bäcklund Transformation for Liouville's Equation

Here we consider the Hamiltonian flows generated by the Casimir $\mathcal{C}$ and the first Hamiltonian h_1. Suppose these flows are respectively parameterised by x and t, so we have

$$f_x = \{f, \mathcal{C}\}, \quad f_t = \{f, h_1\}, \quad \text{for any function} \quad f(q_1, \ldots, p_M).$$

Since these Hamiltonians Poisson commute, their respective flows commute, so can be considered as coordinate curves on the level surface given by $\mathcal{C} = c_1, h_1 = c_2$. Consider the second order partial derivative

$$q_{ixt} = \{\{q_i, \mathcal{C}\}, h_1\} = \{\{q_i, h_1\}, \mathcal{C}\}.$$

To calculate this in general, we need the formula

$$\{q_i, \pi_j\} = \begin{cases} 1 & \text{if } i \leq j, \\ -1 & \text{if } i \geq j+1. \end{cases}$$

First consider q_1. From the definitions (29), we have

$$\begin{aligned} &\{q_1, J_{2r}\} = -J_{2r}, \quad \{q_1, J_{2r+1}\} = J_{2r+1}, \quad 0 \leq r \leq M-2 \\ &\{q_1, J_{2M-2}\} = -J_{2M-2} + 2e^{\pi_1 - q_1}. \end{aligned} \tag{33}$$

Since $\mathcal{C}$ commutes with all J_k,

$$\{\{q_1, h_1\}, \mathcal{C}\} = 2\{e^{\pi_1 - q_1}, \mathcal{C}\}.$$

We have

$$\begin{aligned} &\{e^{\pi_1 - q_1}, e^{-\pi_1}(e^{-q_1} + e^{-q_2})\} = 2e^{-2q_1}, \\ &\{e^{\pi_1 - q_1}, e^{-\pi_i}(e^{-q_i} + e^{-q_{i+1}})\} = 0, \text{ for } i \neq 1, \\ &\{e^{\pi_1 - q_1}, e^{-\pi_M}(e^{q_1} + e^{-q_M})\} = 0, \quad \{e^{\pi_1 - q_1}, e^{\pi_M - q_1}\} = 0, \end{aligned} \tag{34}$$

so

$$q_{1xt} = \{\{q_1, h_1\}, \mathcal{C}\} = 4e^{-2q_1}.$$

Now act with φ on this equation (recalling the formulae (30) and (31)) to obtain Liouville's equation for each of q_i, π_i:

$$q_{ixt} = 4e^{-2q_i}, \qquad \pi_{ixt} = 4e^{-2\pi_i}, \quad i = 1, \ldots, M. \tag{35}$$

The Bäcklund transformation for this equation is well known (see Rogers and Schief (2002)), but here we show how to construct it from our canonical transformation (30) and (31).

Again, first consider q_1 and $\tilde{q}_1 = \pi_1$. Looking at the formulae (34), we see that $q_1 - \pi_1$ commutes with all but one term in the expression (32) for $\mathcal{C}$. The remaining term gives

$$\{q_1 - \pi_1, \mathcal{C}\} = \{q_1 - \pi_1, e^{-\pi_1}(e^{-q_1} + e^{-q_2})\} = -2e^{-q_1-\pi_1}.$$

We now use (33), together with their consequence under the map

$$\{\pi_1, J_0\} = J_0 - 2e^{q_1-\pi_1}, \quad \{\pi_1, J_{2r}\} = J_{2r}, \text{ for } r \neq 0,$$

$$\{\pi_1, J_{2r+1}\} = -J_{2r+1}, \quad \{\pi_1, J_{2M-2}\} = J_{2M-2},$$

so

$$\{q_1 + \pi_1, h_1\} = 2(e^{\pi_1-q_1} - e^{q_1-\pi_1}).$$

In summary, we have shown

$$q_{1x} - \tilde{q}_{1x} = -2e^{-q_1-\tilde{q}_1}, \qquad q_{1t} + \tilde{q}_{1t} = 2(e^{\tilde{q}_1-q_1} - e^{q_1-\tilde{q}_1}), \tag{36}$$

which is the (self-)Bäcklund transformation for Liouville's equation (35) (for $i = 1$).

Again, act with φ on these equations to obtain

$$q_{ix} - \pi_{ix} = -2e^{-q_i-\pi_i}, \qquad q_{it} + \pi_{it} = 2(e^{\pi_i-q_i} - e^{q_i-\pi_i}), \quad i = 1, \ldots, M,$$

$$\pi_{ix} - q_{i+1\,x} = -2e^{-\pi_i-q_{i+1}}, \quad i = 1, \ldots, M-1,$$

$$\pi_{it} + q_{i+1\,t} = 2(e^{q_{i+1}-\pi_i} - e^{\pi_i-q_{i+1}}), \quad i = 1, \ldots, M-1.$$

We can act again with φ, but the calculation is slightly more complicated, since it now involves $\tilde{\pi}_M$ (see (31)). We get a relationship involving derivatives of 3 variables (π_M, q_1 and π_1), but can use (36) to eliminate derivatives of $\pi_1 = \tilde{q}_1$, to obtain

$$\pi_{Mx} + q_{1x} = -2(e^{q_1-\pi_M} - e^{\pi_M-q_1}), \qquad \pi_{Mt} - q_{1t} = 2e^{-\pi_M-q_1}.$$

Notice that x, t seem to have reversed their roles at this step. However, the next action of φ (again requiring more complicated manipulations) brings us full circle to the original formulae (36) for q_1, π_1.

6 Conclusions

In Fordy and Marsh (2009) a new class of quiver, with a certain periodicity property, was introduced and partially classified. The corresponding cluster mutation relations give rise to iterations with the Laurent property. An important open question is the classification of the subclass of such iterations which define Liouville integrable maps. The main content of this paper is the study of one particular family of such maps. The question of integrability for the general class is considered in Fordy and Hone (2010).

In Fordy and Marsh (2009) we noted a surprising connection between our examples of periodic quivers and those which arise in the context of *quiver gauge theories* (see Hanany et al. (2005)). Unfortunately, I had no space to describe this here, but an explanation of this is also an important open question.

This brings us back, finally, to Robin Bullough's famous diagram. Some new boxes and connections are needed to incorporate the subject of this paper, but that is the nature of this diagram, which will grow indefinitely and become more and more complex as new discoveries are made.

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