New explicit exact solutions for the Liénard equation and its applications

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Abstract

In this letter, new exact explicit solutions are obtained for the Liénard equation, and the applications of the results to the generalized Pochhammer-Chree equation, the Kundu equation and the generalized long-short wave resonance equations are presented.

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1. Introduction

Nonlinear partial differential equations (NLPDEs) describe various nonlinear phenomena in natural and applied sciences such as fluid dynamics, plasma physics, solid state physics, optical fibers, acoustics, mechanics, biology and mathematical finance. It is of significant importance to construct exact solutions of NLPDEs from both theoretical and practical points of view. Up to now, many powerful methods for solving NLPDEs have been proposed, such as the inverse scattering method[1], Bäcklund and Darboux transform[2]-[3], Hirota's bilinear method[4], truncated painlevé expansion method[5]-[10], homogeneous balance method[11], variational iteration method[12], homotopy perturbation method[13], tanh-function method[14], Jacobian elliptic function expansion method[15]-[19], Fan sub-equation method[20]-[22], auxiliary equation method[23]-[25], F-expansion method[26]-[28]and so on.

The last five methods mentioned above belong to a class of method called subsidiary ordinary differential equation method(sub-ODE method for short). The sub-ODE method which were often used the Riccati equation, Jacobian elliptic equation, projective Riccati equation, etc. In this letter,

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we choose the Liénard equation

$$a''(\xi) + l\,a(\xi) + m\,a^3(\xi) + n\,a^5(\xi) = 0,\ \ lmn \neq 0, \tag{1}$$

as the subsidiary ordinary differential equation. By means of some proper transformations, a number of NLPDEs with strong nonlinear terms can be reduced to Eq.(1), thus seeking explicit exact solutions of these nonlinear equations can be attributed to solve (1). Therefore, to search for exact solutions of the Liénard equation (1) is a very important job and it has attracted much attention. For example, Behera and Khare[29] has shown that the exact solution of Eq.(1) can be expressed in terms of the Weierstrass function. Dey et al.[30] investigated Eq.(1) and established the exact solution of a one-parameter family of generalized Liénard equation with pth order nonlinearity by mapping it to the field equation of the ϕ^6-field theory. By means of different methods, Kong[31], Zhang[32]-[33] and Feng[34]-[36] have given some explicit exact solitary wave solutions of Eq.(1). In Refs.[32]-[36], Zhang and Feng derived three kinds of solitary wave solutions of Eq.(1) as follows:

If $l < 0, m > 0, n \leq 0$ or $l < 0, m \leq 0, n > 0$, Eq.(1) possesses the solitary wave solution,

$$a_1(\xi) = \pm \left[\frac{4\sqrt{\dfrac{3l^2}{3m^2 - 16nl}}\,\mathrm{sech}^2\sqrt{-l}\,\xi}{2 + \left(-1 + \dfrac{\sqrt{3}\,m}{\sqrt{3m^2 - 16nl}}\right)\mathrm{sech}^2\sqrt{-l}\,\xi} \right]^{\frac{1}{2}}. \tag{2}$$

If $l < 0, m > 0$ and $3m^2 - 16nl = 0$, Eq.(1) admits exact solutions,

$$a_2(\xi) = \pm \left[-\frac{2\,l}{m}\,(1 + \tanh(\sqrt{-l}\,\xi)\,) \right]^{\frac{1}{2}},\ \ a_3(\xi) = \pm \left[-\frac{2\,l}{m}\,(1 - \tanh(\sqrt{-l}\,\xi)\,) \right]^{\frac{1}{2}}. \tag{3}$$

The various methods used in [29]-[36] are very useful and the applications of the solutions of the Liénard equation to some important NLPDEs are quite perfect. However, it is natural to ask whether Eq.(1) can support other new exact solutions. The present letter is motivated by the desire to improve the work made in [31]-[36] by introducing more solutions of Eq.(1) including all the solutions given in [31]-[36] but also other formal solutions.

The rest of this letter is organized as follows. In Section 2, we find some new exact solutions for the Liénard equation (1). In Section 3, we use these special solutions to solve the generalized Pochhammer-Chree equation, the Kundu equation and the generalized long-short wave resonance equations. And we conclude the letter in the last section.

2. New exact solutions of the Liénard equation

Generally speaking, it is difficult to give the general solution of Eq.(1). In what follows, we will consider some special cases. Based on Refs.[31]-[36], we can have the following solutions of Eq.(1),

$$a_{1\pm}(\xi) = \pm\left[\frac{-4\,l}{m + \epsilon\sqrt{m^2 - 16nl/3}\,\cosh(2\sqrt{-l}\,\xi)}\right]^{\frac{1}{2}}, \quad m^2 - 16nl/3 > 0,\ l < 0, \tag{4a}$$

$$a_{2\pm}(\xi) = \pm\left[\frac{-4\,l}{m + \epsilon\sqrt{16nl/3 - m^2}\,\sinh(2\sqrt{-l}\,\xi)}\right]^{\frac{1}{2}}, \quad m^2 - 16nl/3 < 0,\ l < 0, \tag{4b}$$

$$a_{3\pm}(\xi) = \pm\left[-\frac{2l}{m}\left(1 + \epsilon\,\tanh(\sqrt{-l}\,\xi)\right)\right]^{\frac{1}{2}}, \quad m^2 - 16nl/3 = 0,\ m > 0,\ l < 0,\ n < 0, \tag{4c}$$

$$a_{4\pm}(\xi) = \pm\left[-\frac{2l}{m}\left(1 + \epsilon\,\coth(\sqrt{-l}\,\xi)\right)\right]^{\frac{1}{2}}, \quad m^2 - 16nl/3 = 0,\ m > 0,\ l < 0,\ n < 0, \tag{4d}$$

$$a_{5\pm}(\xi) = \pm\left[\frac{-4\,l}{m + \epsilon\sqrt{m^2 - 16nl/3}\,\cos(2\sqrt{l}\,\xi)}\right]^{\frac{1}{2}}, \quad m^2 - 16nl/3 > 0,\ l > 0, \tag{4e}$$

where $\epsilon = \pm 1$. It is easily seen that $a_{3\pm}(\xi)$ reproduces two solutions given in Eq.(3). There is a tiny symbolic error in the solution $a_1(\xi)$(the coefficient of $\operatorname{sech}^2\sqrt{-l}\,\xi$ in the numerator of fraction (2) should be $-4\sqrt{\frac{3l^2}{3m^2-16nl}}$). It is easily proved that the correct solution $a_1(\xi)$ and the solution $a_{1\pm}(\xi)$ with $\epsilon = 1$ are actually the same and only different in the form. And the other solutions $a_{2\pm}(\xi)$, $a_{4\pm}(\xi)$ and $a_{5\pm}(\xi)$ are firstly reported here.

To our best knowledge, the periodic wave solutions expressed in terms of Jacobian elliptic function to Eq.(1) have not been considered in existed literature. Now we assume $\mathrm{JacobiSN}(\xi, r) = \mathrm{sn}(\xi)$, $\mathrm{JacobiCN}(\xi, r) = \mathrm{cn}(\xi)$ and $\mathrm{JacobiDN}(\xi, r) = \mathrm{dn}(\xi)$, and r is the modulus of Jacobian elliptic functions($0 \le r \le 1$). With the aid of symbolic computation software such as MAPLE, after direct computations, we find three kinds of elliptic periodic wave solutions of Eq.(1) when the parameter coefficients l, m, n satisfy certain conditions,

$$a_{6\pm}(\xi) = \pm\left[-\frac{3m}{8n}\left(1 + \epsilon\,\mathrm{sn}\left(\frac{\sqrt{3}\,m}{4r\sqrt{-n}}\,\xi\right)\right)\right]^{\frac{1}{2}}, \quad l = \frac{3m^2(5r^2 - 1)}{64n\,r^2},\ m > 0,\ n < 0, \tag{5a}$$

$$a_{7\pm}(\xi) = \pm\left[-\frac{3m}{8n}\left(1 + \epsilon\,\mathrm{cn}\left(\frac{\sqrt{3}\,m}{4r\sqrt{n}}\,\xi\right)\right)\right]^{\frac{1}{2}}, \quad l = \frac{3m^2(4r^2 + 1)}{64n\,r^2},\ m < 0,\ n > 0, \tag{5b}$$

$$a_{8\pm}(\xi) = \pm\left[-\frac{3m}{8n}\left(1 + \epsilon\,\mathrm{dn}\left(\frac{\sqrt{3}\,m}{4\sqrt{n}}\,\xi\right)\right)\right]^{\frac{1}{2}}, \quad l = \frac{3m^2(r^2 + 4)}{64\,n},\ m < 0,\ n > 0. \tag{5c}$$

To our knowledge, the solutions $a_{6\pm}(\xi)$, $a_{7\pm}(\xi)$ and $a_{8\pm}(\xi)$ are firstly presented here.

It is well known that there are many other Jacobian elliptic functions which can be generated by sn(ξ), cn(ξ) and dn(ξ). For the sake of simplicity, the solutions in terms of ns(ξ), nd(ξ), nc(ξ), sc(ξ), cs(ξ), sd(ξ), ds(ξ), cd(ξ), dc(ξ) are not considered here.

3. Applications

Example 1. The generalized Pochhammer-Chree (PC) equation can be written as

$$u_{tt} - u_{ttxx} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0, \tag{6}$$

which describes the propagation of longitudinal deformation waves in an elastic rod[37]. Zhang[32] and Feng[35] have given some explicit solitary wave solutions of Eq.(6) by means of the method of solving algebraic equations. Li and Zhang[38] studied the bifurcation problem of travelling wave solutions for Eq.(6) by using the bifurcation theory of planar dynamical systems.

In order to solve Eq.(6), its solutions may be supposed as:

$$u(x,t) = u(\xi), \;\; \xi = x - v\,t, \tag{7}$$

where v is a real constant. Substituting ansatz (7) into Eq.(6) yields,

$$v^2 u''(\xi) - v^2 u^{(4)}(\xi) - (a_1 u + a_3 u^3 + a_5 u^5)_{\xi\xi} = 0, \tag{8}$$

Integrating Eq.(8) twice and setting the integration constant to zero, we obtain

$$u''(\xi) + \frac{a_1 - v^2}{v^2} u(\xi) + \frac{a_3}{v^2} u^3(\xi) + \frac{a_5}{v^2} u^5(\xi) = 0. \tag{9}$$

Up to now, by means of the ansatz (7), we reduce the generalized PC equation (6) to the Liénard equation (1) for the case $l = \frac{a_1 - v^2}{v^2}$, $m = \frac{a_3}{v^2}$ and $n = \frac{a_5}{v^2}$. Substituting the solutions (4a)-(4e) and the solutions (5a)-(5c) of Eq.(1) into (7), we can obtain a series of exact travelling wave solutions to Eq.(6) (where $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$).

When $v^2 - a_1 > 0$ and $3a_3^2 - 16a_5(a_1 - v^2) > 0$, Eq.(6) has bell-shape solitary wave solution,

$$u_{1\pm}(x,t) = \pm \left[\frac{4(v^2 - a_1)}{a_3 + \epsilon_1 \sqrt{a_3^2 - 16a_5(a_1 - v^2)/3}\, \cosh(\frac{2\sqrt{v^2 - a_1}}{v}\,\xi)} \right]^{\frac{1}{2}}.$$

When $v^2 - a_1 > 0$ and $3a_3^2 - 16a_5(a_1 - v^2) < 0$, Eq.(6) has the singular solitary wave solution,

$$u_{2\pm}(x,t) = \pm \left[\frac{4(v^2 - a_1)}{a_3 + \epsilon_1 \sqrt{16a_5(a_1 - v^2)/3 - a_3^2}\, \sinh(\frac{2\sqrt{v^2 - a_1}}{v}\,\xi)} \right]^{\frac{1}{2}}.$$

When $a_3 > 0$, $a_5 < 0$, $v^2 - a_1 > 0$ and $3a_3^2 - 16a_5(a_1 - v^2) = 0$, Eq.(6) has two kink-shape solitary wave solutions,

$$u_{3\pm}(x,t) = \pm\left[\frac{2(v^2-a_1)}{a_3}\left(1 + \epsilon_1 \tanh\left(\frac{\sqrt{v^2-a_1}}{v}\,\xi\right)\right)\right]^{\frac{1}{2}},$$

$$u_{4\pm}(x,t) = \pm\left[\frac{2(v^2-a_1)}{a_3}\left(1 + \epsilon_1 \coth\left(\frac{\sqrt{v^2-a_1}}{v}\,\xi\right)\right)\right]^{\frac{1}{2}}.$$

When $v^2 - a_1 < 0$ and $3a_3^2 - 16a_5(a_1 - v^2) > 0$, Eq.(6) has the trigonometric function solution,

$$u_{5\pm}(x,t) = \pm\left[\frac{4(v^2-a_1)}{a_3 + \epsilon_1\sqrt{a_3^2 - 16a_5(a_1-v^2)/3}\,\cos(\frac{2\sqrt{a_1-v^2}}{v}\,\xi)}\right]^{\frac{1}{2}}.$$

When $a_5 < 0$ and $a_3 > 0$, Eq.(6) has the Jacobian sine function solution,

$$u_{6\pm}(x,t) = \pm\frac{1}{2}\left[-\frac{3a_3}{2\,a_5}\left(1 + \epsilon_1\,\mathrm{sn}\left(\frac{\sqrt{3}\,a_3}{4r\,v\,\sqrt{-a_5}}\,\xi\right)\right)\right]^{\frac{1}{2}},$$

where $v = \epsilon_2\sqrt{a_5\,(64\,a_5 r^2 a_1 - 15\,{a_3}^2 r^2 + 3\,{a_3}^2)}/(8\,r\,a_5)$.

When $a_5 > 0$ and $a_3 < 0$, Eq.(6) has two periodic wave solutions. One is

$$u_{7\pm}(x,t) = \pm\frac{1}{2}\left[-\frac{3a_3}{2\,a_5}\left(1 + \epsilon_1\,\mathrm{cn}\left(\frac{\sqrt{3}\,a_3}{4r\,v\,\sqrt{a_5}}\,\xi\right)\right)\right]^{\frac{1}{2}},$$

where $v = \epsilon_2\sqrt{a_5\,(64\,a_5 r^2 a_1 - 12\,{a_3}^2 r^2 - 3\,{a_3}^2)}/(8\,r a_5)$. And another one is

$$u_{8\pm}(x,t) = \pm\frac{1}{2}\left[-\frac{3a_3}{2\,a_5}\left(1 + \epsilon_1\,\mathrm{dn}\left(\frac{\sqrt{3}\,a_3}{4v\,\sqrt{a_5}}\,\xi\right)\right)\right]^{\frac{1}{2}},$$

where $v = \epsilon_2\sqrt{a_5\,(64\,a_5 a_1 - 3\,{a_3}^2 r^2 - 12\,{a_3}^2)}/(8\,a_5)$.

Among the above solutions, only $u_{1\pm}(x,t)$ with $\epsilon_1 = 1$ and $u_{3\pm}(x,t)$ reproduce the results given in Refs.[32]-[35], and the other solutions have not been found before.

Example 2. Next we consider the Kundu equation,

$$iu_t + u_{xx} + \beta\,|u|^2 u + \delta\,|u|^4 u + i\alpha\,(|u|^2 u)_x + i\,s\,(|u|^2)_x\,u = 0, \tag{10}$$

where β, δ, α, s are real constants. Eq.(10) was derived by Kundu[39] in the study of integrability and it is an important special case of the generalized complex Ginzburg-Laudau equation[40]. Meanwhile, Eq.(10) and its special cases arise in various physical and mechanical applications, such as plasma physics, nonlinear fluid mechanics, nonlinear optics and quantum physics. Feng[34] derived the explicit exact solitary wave solutions of Eq.(10) by using the algebraic curve method. Zhang *et al.*[41] studied the orbital stability of solitary waves for Eq.(10) by means of spectral analysis.

Assume that Eq.(10) has solutions of the the form

$$u(x,t) \,=\, \phi(\xi)\,\mathrm{e}^{i(\psi(\xi)-\omega\,t)}, \quad \xi = x \,-\, v\,t, \tag{11}$$

where ω and v are constants to be determined. Substituting Eq.(11) into Eq.(10) and then separating the real part and imaginary part yields,

$$(\omega \,+\, v\psi'(\xi))\phi(\xi) \,+\, \phi''(\xi) - \phi(\xi)\psi'^2(\xi) \,-\, \alpha\,\phi^3(\xi)\psi'(\xi) \,+\, \beta\,\phi^3(\xi) \,+\, \delta\,\phi^5(\xi) \,=\, 0, \tag{12a}$$

$$-\,v\,\phi'(\xi) \,+\, 2\,\phi'(\xi)\psi'(\xi) \,+\phi(\xi)\psi''(\xi) + (3\alpha + 2\,s)\,\phi^2(\xi)\phi'(\xi) \,=\, 0. \tag{12b}$$

Letting

$$\psi'(\xi) \,=\, A \,+\, B\,\phi^2(\xi). \tag{13}$$

Substituting Eq.(13) into Eq.(12b) and setting the coefficients of $\phi'(\xi)$, $\phi^2(\xi)\phi'(\xi)$ to zero, we have $A = v/2$, $B = -(3\alpha + 2s)/4$. Then Eq.(13) becomes,

$$\psi'(\xi) \,=\, \frac{v}{2} \,-\, \frac{3\alpha + 2s}{4}\,\phi^2(\xi). \tag{14}$$

Substituting Eq.(14) into Eq.(12a) yields the Liénard equation of the form,

$$\phi''(\xi) \,+\, l\,\phi(\xi) \,+\, m\,\phi^3(\xi) \,+\, n\,\phi^5(\xi) \,=\, 0, \tag{15}$$

where l, m, n are given by

$$l \,=\, \omega + \frac{v^2}{4}, \quad m \,=\, \beta - \frac{\alpha\,v}{2}, \quad n \,=\, \delta + \frac{(\alpha - 2s)(3\alpha + 2s)}{16}.$$

By the transformations (11) and (14), the exact solutions of Eq.(10) can be obtained by using the solutions of Eq.(1) given in Section 2. In the following solutions, $\psi(\xi)$ is given by Eq.(14), $\Delta_1 = (2\beta - \alpha v)^2 - (4\omega + v^2)(16\delta + (\alpha - 2s)(3\alpha + 2s))/3$.

When $\Delta_1 > 0$, and $v^2 + 4\omega < 0$, Eq.(10) has the solitary wave solution,

$$u_1(x,t) \,=\, \phi(x \,-\, v\,t)\,\mathrm{e}^{i(\psi(x-\,v\,t)\,-\omega\,t)},$$

$$\phi(\xi) \,=\, \pm\left[\frac{-2(4\omega + v^2)}{2\beta - \alpha v + \epsilon\sqrt{\Delta_1}\,\cosh(\sqrt{-(v^2 + 4\omega)}\;\xi)}\right]^{\frac{1}{2}}.$$

When $\Delta_1 < 0$, and $v^2 + 4\omega < 0$, Eq.(10) has the singular solitary wave solution,

$$u_2(x,t) \,=\, \phi(x \,-\, v\,t)\,\mathrm{e}^{i(\psi(x-\,v\,t)\,-\omega\,t)},$$

$$\phi(\xi) \,=\, \pm\left[\frac{-2(4\omega + v^2)}{2\beta - \alpha v + \epsilon\sqrt{-\Delta_1}\,\sinh(\sqrt{-(v^2 + 4\omega)}\;\xi)}\right]^{\frac{1}{2}}.$$

When $v^2 + 4\omega < 0$ and $\alpha\, v - 2\beta < 0$, Eq.(10) has two kink-shape solitary wave solutions,

$$u_3(x,t) \;=\; \phi(x -\; v\,t)\,\mathrm{e}^{i(\psi(x- v\,t)\,-\omega\,t)},$$

$$\phi(\xi) = \pm\left[\frac{v^2+4\omega}{\alpha\,v-2\beta}\left(1\;+\;\epsilon\,\tanh(\frac{\sqrt{-(4\omega+v^2)}}{2}\,\xi)\right)\right]^{\frac{1}{2}},$$

$$u_4(x,t) \;=\; \phi(x -\; v\,t)\,\mathrm{e}^{i(\psi(x- v\,t)\,-\omega\,t)},$$

$$\phi(\xi) = \pm\left[\frac{v^2+4\omega}{\alpha\,v-2\beta}\left(1\;+\;\epsilon\,\coth(\frac{\sqrt{-(4\omega+v^2)}}{2}\,\xi)\right)\right]^{\frac{1}{2}},$$

where ω is determined by $\Delta_1 = 0$.

When $\Delta_1 > 0$ and $v^2 + 4\omega > 0$, Eq.(10) has the periodic solution of trigonometric function,

$$u_5(x,t) \;=\; \phi(x -\; v\,t)\,\mathrm{e}^{i(\psi(x- v\,t)\,-\omega\,t)},$$

$$\phi(\xi) = \pm\left[\frac{-2(4\omega+v^2)}{2\beta-\alpha v+\epsilon\sqrt{\Delta_1}\,\cos(\sqrt{v^2+4\omega}\;\xi)}\right]^{\frac{1}{2}}.$$

When $4s\alpha + 4s^2 - 3\alpha^2 - 16\delta > 0$, $2\beta - \alpha\, v > 0$, Eq.(10) has the Jacobian elliptic sine function solution,

$$u_6(x,t) \;=\; \phi(x -\; v\,t)\,\mathrm{e}^{i(\psi(x- v\,t)\,-\omega\,t)},$$

$$\phi(\xi) = \pm\left[\frac{3(2\beta-\alpha\,v)}{4s\alpha+4s^2-3\alpha^2-16\delta}\left(1\;+\;\epsilon\,\mathrm{sn}(\frac{\sqrt{3}(2\beta-\alpha\,v)}{2r\,\sqrt{4s\alpha+4s^2-3\alpha^2-16\delta}}\,\xi)\right)\right]^{\frac{1}{2}},$$

where ω is determined by $r^2(v^2+4\omega)(16\delta+(3\,\alpha+2\,s)(\alpha-2\,s)) - 3(\beta - v\alpha/2)^2(5\,r^2-1) = 0$.

When $4s\alpha + 4s^2 - 3\alpha^2 - 16\delta < 0$, $2\beta - \alpha\, v < 0$, Eq.(10) has two Jacobian elliptic function solutions. One is

$$u_7(x,t) \;=\; \phi(x -\; v\,t)\,\mathrm{e}^{i(\psi(x- v\,t)\,-\omega\,t)},$$

$$\phi(\xi) = \pm\left[\frac{3(2\beta-\alpha\,v)}{4s\alpha+4s^2-3\alpha^2-16\delta}\left(1\;+\;\epsilon\,\mathrm{cn}(\frac{\sqrt{3}(2\beta-\alpha\,v)}{2r\,\sqrt{3\alpha^2+16\delta-4s\alpha-4s^2}}\,\xi)\right)\right]^{\frac{1}{2}},$$

where ω is determined by $r^2\,(v^2+4\omega)(16\delta+(3\,\alpha+2\,s)(\alpha-2\,s)) - 3\,(\beta-v\alpha/2)^2(4\,r^2+1) = 0$. And another one is

$$u_8(x,t) \;=\; \phi(x -\; v\,t)\,\mathrm{e}^{i(\psi(x- v\,t)\,-\omega\,t)},$$

$$\phi(\xi) = \pm\left[\frac{3(2\beta-\alpha\,v)}{4s\alpha+4s^2-3\alpha^2-16\delta}\left(1\;+\;\epsilon\,\mathrm{dn}(\frac{\sqrt{3}(2\beta-\alpha\,v)}{2\,\sqrt{3\alpha^2+16\delta-4s\alpha-4s^2}}\,\xi)\right)\right]^{\frac{1}{2}},$$

where ω is determined by $(v^2+4\omega)(16\delta+(3\,\alpha+2\,s)(\alpha-2\,s)) - 3\,(\beta-v\alpha/2)^2(r^2+4) = 0$.

The solutions $u_1(x,t)$ with $\epsilon = 1$, $u_3(x,t)$, $u_4(x,t)$ are same as the results reported in [34]. Other solutions have not been reported in [34]. In addition, the Kundu equation (10) contains several important nonlinear models when taking different choices for the parameters α, β, δ and s. For example, if $s = 0$, Eq.(10) reduces to the derivative Schrödinger equation[39]

$$iu_t + u_{xx} + \beta |u|^2 u + \delta |u|^4 u + i\alpha (|u|^2 u)_x = 0; \tag{16}$$

if $\delta = 2\sigma^2$, $\alpha = -2\sigma$, $s = 4\sigma$, then Eq.(10) becomes the Gerdjikov-Ivanov equation[42],

$$iu_t + u_{xx} + \beta |u|^2 u + 2\sigma^2 |u|^4 u + 2i\sigma u^2 \bar{u}_x = 0. \tag{17}$$

Obviously, the explicit exact solutions of Eq.(16) and Eq.(17) can be derived from the above solutions.

Example 3. Finally we consider the generalized long-short wave resonance equations with strong nonlinear term,

$$\begin{aligned} i S_t + S_{xx} &= \alpha LS + \gamma |S|^2 S + \delta |S|^4 S, \\ L_t + \beta |S|^2_x &= 0, \end{aligned} \tag{18}$$

where S is the envelope of the short wave, and L is the amplitude of the long wave and is real. The parameters α, β, γ and δ are arbitrary real constants. Recently, Shang[43] obtained several kinds of explicit exact solutions of Eq.(18).

In order to seek the exact solutions of Eq.(18), we introduce the following transformation,

$$S(x,t) = \phi(x,t)\, \mathrm{e}^{i\,(k\,x + \omega\, t + \xi_0)}, \tag{19}$$

where $\phi(x,t)$ is a real-valued function, and k and ω are constants to be determined, ξ_0 is an arbitrary constant. Substituting Eq.(19) into Eq.(18) and then separating the real and imaginary parts yields,

$$\phi_{xx} - (\omega + k^2)\,\phi - \alpha\, L\, \phi - \gamma\, \phi^3 - \delta\, \phi^5 = 0, \tag{20a}$$

$$\phi_t + 2\, k\, \phi_x = 0, \tag{20b}$$

$$L_t + 2\, \beta\, \phi \phi_x = 0. \tag{20c}$$

In view of Eq.(20b) we suppose

$$\phi(x,t) = \phi(\xi) = \phi(x - 2k\, t + \xi_1), \tag{21}$$

where ξ_1 is an arbitrary constant. Therefore we also assume

$$L(x,t) = \psi(\xi) = \psi(x - 2k\, t + \xi_1). \tag{22}$$

Substituting Eq.(21) into Eq.(20c) yields,

$$\psi(\xi) \,=\, \frac{\beta\,\phi^2(\xi)}{2\,k} \,+\, C, \tag{23}$$

where C is an integration constant.

Substituting Eqs.(21)-(23) into Eq.(20a), we have,

$$\phi''(\xi) \,+\, l\,\phi(\xi) \,+\, m\,\phi^3(\xi) \,+\, n\,\phi^5(\xi) \,=\, 0, \tag{24}$$

where the parameters l, m, n are given by

$$l \,=\, -\,(\omega + k^2 + \alpha\,C), \quad m \,=\, -(\gamma + \frac{\alpha\,\beta}{2\,k}), \quad n \,=\, -\,\delta. \tag{25}$$

Similar to **Example 1**, by means of the transformations (19), (21)-(23), we can also reduce the generalized long-short wave resonance equations (18) to the Liénard equation (1). Together with Eq.(21) and Eq.(23), substituting the solutions of the Lienard equation given in Section 2 into Eq.(19) and Eq.(22) yields abundant periodic wave solutions of the generalized long-short wave resonance equations (18). In the following eight sets of solutions, $\Delta_2 = (\gamma + \frac{\alpha\,\beta}{2\,k})^2 - 16\delta\,(\omega + k^2 + \alpha\,C)/3$, $\epsilon = \pm 1$, and $\xi = x - 2\,k\,t + \xi_1$ with k being nonzero arbitrary constant.

When $\Delta_2 > 0$ and $\omega + k^2 + \alpha\,C > 0$, Eqs.(18) has a set of bell-shape solitary wave solutions,

$$L_1(x,t) \,=\, \frac{4\,\beta\,(\omega + k^2 + \alpha\,C)}{-2k\,\gamma - \alpha\,\beta \,+\, 2k\,\epsilon\,\sqrt{\Delta_2}\,\cosh(2\sqrt{\omega + k^2 + \alpha\,C}\,\xi)} \,+\, C,$$

$$S_1(x,t) \,=\pm \left[\frac{4\,(\omega + k^2 + \alpha\,C)}{-(\gamma + \frac{\alpha\,\beta}{2\,k}) \,+\, \epsilon\sqrt{\Delta_2}\,\cosh(2\sqrt{\omega + k^2 + \alpha\,C}\,\xi)}\right]^{\frac{1}{2}} \mathrm{e}^{i\,(k\,x + \omega\,t + \xi_0)}.$$

When $\Delta_2 < 0$, $\omega + k^2 + \alpha\,C > 0$ Eqs.(18) has a set of singular solitary wave solutions,

$$L_2(x,t) \,=\, \frac{4\,\beta\,(\omega + k^2 + \alpha\,C)}{-2k\,\gamma - \alpha\,\beta \,+\, 2k\,\epsilon\,\sqrt{-\Delta_2}\,\sinh(2\sqrt{\omega + k^2 + \alpha\,C}\,\xi)} \,+\, C,$$

$$S_2(x,t) \,=\pm \left[\frac{4\,(\omega + k^2 + \alpha\,C)}{-(\gamma + \frac{\alpha\,\beta}{2\,k}) \,+\, \epsilon\sqrt{-\Delta_2}\,\sinh(2\sqrt{\omega + k^2 + \alpha\,C}\,\xi)}\right]^{\frac{1}{2}} \mathrm{e}^{i\,(k\,x + \omega\,t + \xi_0)}.$$

When $\Delta_2 = 0$, $\omega + k^2 + \alpha C > 0$, and $2k\gamma + \alpha\beta < 0$, Eqs.(18) has two sets of kink-shape solitary wave solutions,

$$L_3(x,t) \,=\, -\frac{2\beta\,(\omega + k^2 + \alpha\,C)}{2k\gamma + \alpha\beta}\,\Big(1 \,+\, \epsilon\,\tanh(\sqrt{\omega + k^2 + \alpha\,C}\,\xi\,)\Big) \,+\, C,$$

$$S_3(x,t) \,=\pm \left[-\frac{4k(\omega + k^2 + \alpha\,C)}{2k\gamma + \alpha\beta}\,\Big(1 \,+\, \epsilon\,\tanh(\sqrt{\omega + k^2 + \alpha\,C}\,\xi\,)\Big)\right]^{\frac{1}{2}} \mathrm{e}^{i\,(k\,x + \omega\,t + \xi_0)},$$

$$L_4(x,t) \;=\; -\frac{2\beta\,(\omega+k^2+\alpha\,C)}{2k\gamma+\alpha\beta}\;\Big(1\,+\,\epsilon\,\coth(\sqrt{\omega+k^2+\alpha\,C}\,\xi\,)\Big)\,+\,C,$$

$$S_4(x,t) \;=\pm\,\left[-\frac{4k(\omega+k^2+\alpha\,C)}{2k\gamma+\alpha\beta}\;\Big(1\,+\,\epsilon\,\coth(\sqrt{\omega+k^2+\alpha\,C}\,\xi\,)\Big)\right]^{\frac{1}{2}}\;\mathrm{e}^{i\,(k\,x+\omega\,t+\xi_0)},$$

When $\Delta_2>0$ and $\omega+k^2+\alpha\,C<0$, Eqs.(18) has a set of trigonometric function solutions,

$$L_5(x,t) \;=\; \frac{4\,\beta\,(\omega+k^2+\alpha\,C)}{-2k\,\gamma-\alpha\,\beta\,+\,2k\,\epsilon\,\sqrt{\Delta_2}\,\cos(2\sqrt{-(\omega+k^2+\alpha\,C)}\,\xi)}\,+\,C,$$

$$S_5(x,t) \;=\pm\,\left[\frac{4\,(\omega+k^2+\alpha\,C)}{-(\gamma+\dfrac{\alpha\,\beta}{2\,k})\,+\,\epsilon\sqrt{\Delta_2}\,\cos(2\sqrt{-(\omega+k^2+\alpha\,C)}\,\xi)}\right]^{\frac{1}{2}}\;\mathrm{e}^{i\,(k\,x+\omega\,t+\xi_0)}.$$

When $\delta>0$ and $k(\alpha\beta+2k\gamma)<0$, Eqs.(18) has a set of Jacobian elliptic sine function solutions,

$$L_6(x,t) \;=\; -\frac{3\beta\,(\alpha\beta+2k\,\gamma)}{32k^2\,\delta}\;\left(1\,+\,\epsilon\,\mathrm{sn}(-\frac{\sqrt{3}\,(\alpha\beta+2k\gamma)}{8k\,r\,\sqrt{\delta}}\,\xi\,)\right)\,+\,C,$$

$$S_6(x,t) \;=\pm\,\left[-\frac{3(\alpha\beta+2k\,\gamma)}{16k\,\delta}\;\left(1\,+\,\epsilon\,\mathrm{sn}(-\frac{\sqrt{3}\,(\alpha\beta+2k\gamma)}{8k\,r\,\sqrt{\delta}}\,\xi\,)\right)\right]^{\frac{1}{2}}\;\mathrm{e}^{i\,(k\,x+\omega\,t+\xi_0)},$$

where ω is determined by $64r^2\delta\,(\omega+k^2+\alpha\,C)-3(5r^2-1)\,(\gamma+\dfrac{\alpha\beta}{2k})^2=0$.

When $\delta<0$, $k(\alpha\beta+2k\gamma)>0$, Eqs.(18) has two sets of Jacobian elliptic function solutions. One is

$$L_7(x,t) \;=\; -\frac{3\beta\,(\alpha\beta+2k\,\gamma)}{32k^2\,\delta}\;\left(1\,+\,\epsilon\,\mathrm{cn}(-\frac{\sqrt{3}\,(\alpha\beta+2k\gamma)}{8k\,\sqrt{-\delta}}\,\xi\,)\right)\,+\,C,$$

$$S_7(x,t) \;=\pm\,\left[-\frac{3(\alpha\beta+2k\,\gamma)}{16k\,\delta}\;\left(1\,+\,\epsilon\,\mathrm{cn}(-\frac{\sqrt{3}\,(\alpha\beta+2k\gamma)}{8k\,\sqrt{-\delta}}\,\xi\,)\right)\right]^{\frac{1}{2}}\;\mathrm{e}^{i\,(k\,x+\omega\,t+\xi_0)},$$

where ω is determined by $64r^2\delta\,(\omega+k^2+\alpha\,C)-3(4r^2+1)\,(\gamma+\dfrac{\alpha\beta}{2k})^2=0$. And another one is

$$L_8(x,t) \;=\; -\frac{3\beta\,(\alpha\beta+2k\,\gamma)}{32k^2\,\delta}\;\left(1\,+\,\epsilon\,\mathrm{dn}(-\frac{\sqrt{3}\,(\alpha\beta+2k\gamma)}{8k\,\sqrt{-\delta}}\,\xi\,)\right)\,+\,C,$$

$$S_8(x,t) \;=\pm\,\left[-\frac{3(\alpha\beta+2k\,\gamma)}{16k\,\delta}\;\left(1\,+\,\epsilon\,\mathrm{dn}(-\frac{\sqrt{3}\,(\alpha\beta+2k\gamma)}{8k\,\sqrt{-\delta}}\,\xi\,)\right)\right]^{\frac{1}{2}}\;\mathrm{e}^{i\,(k\,x+\omega\,t+\xi_0)},$$

where ω is determined by $64\delta\,(\omega+k^2+\alpha\,C)-3(4+r^2)\,(\gamma+\dfrac{\alpha\beta}{2k})^2=0$.

With the aid of Maple, we have checked all solutions by putting them back into the original Equation.

4. Conclusions

The Liénard equation is used to describe fluid-mechanical and nonlinear elastic mechanical phenomena. Moreover, a number of NLPDEs with strong nonlinear terms can be reduced to the Liénard equation by some proper transformations. Therefore, To search for new special solutions of the Liénard equation is a very important job. In this letter, we obtain eight kinds of explicit exact solutions for the Liénard equation, which include solitary wave solutions, periodic wave solutions in terms of trigonometric function and Jacobian elliptic function. By means of these solutions, we obtain a variety of explicit exact solutions for the generalized PC equation, the Kundu equation and the generalized long-short wave resonance equations. These solutions may be important explain some physical phenomena. The method presented here is also applicable to solve other nonlinear equations with strong nonlinear terms. For example, the Ablowitz equation[44],

$$i\,u_{tt} \;=\; u_{xx} \;-\; 4\,i\,u^2\,\bar{u}_x \;+\; 8\,|u|^4\,u;$$

the third-order generalized NLS equation(also called RKL model)[45],

$$iu_z + u_{tt} + 2|u|^2u + i\alpha u_{ttt} + i\beta(|u|^2u)_t + i\gamma(|u|^4u)_t + \delta|u|^4u \;=\; 0,$$

and the nonlinear equations which were considered in Ref.[32] and Ref.[34].

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