Statistical Approach to Quantum Chaotic Ratchets

Itzhack Dana

Minerva Center and Department of Physics,

Bar-Ilan University, Ramat-Gan 52900, Israel

Abstract

The quantum ratchet effect in fully chaotic systems is approached by studying, for the first time, *statistical* properties of the ratchet current over well-defined sets of initial states. Natural initial states in a semiclassical regime are those that are *phase-space uniform* with the *maximal possible* resolution of one Planck cell. General arguments in this regime, for quantum-resonance values of a scaled Planck constant $\hbar$, predict that the distribution of the current over all such states is a zero-mean Gaussian with variance $\sim D\hbar^2/(2\pi^2)$, where D is the chaotic-diffusion coefficient. This prediction is well supported by extensive numerical evidence. The average strength of the effect, measured by the variance above, is *significantly larger* than that for the usual momentum states and other states. Such strong effects should be experimentally observable.

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I. INTRODUCTION

Understanding quantum transport in generic Hamiltonian systems, which are classically nonintegrable and exhibit chaos, is a problem of both fundamental and practical importance. The study of simple model systems have led to the discovery of a variety of quantum-transport phenomena [1–18], several of which have been observed in atom-optics experiments [2, 3, 16–18] and allow to control the quantum motion of cold atoms or Bose-Einstein condensates in different ways. Recently, classical and quantum Hamiltonian *"ratchets"* have started to attract a considerable interest, both theoretically [10–15] and experimentally [16–18]. Ratchets are usually conceived as spatially periodic systems with noise and dissipation in which a directed current of particles can emerge from an unbiased (zero-mean) external force due to some spatial/temporal asymmetry [19]. In classical Hamiltonian ratchets [10], dissipation is absent and noise is replaced by deterministic chaos. A basic general result for Hamiltonian dynamics under an unbiased force is that the average current of an initially *uniform* ensemble of particles in phase space is zero [10]. As a consequence, a completely chaotic system carries essentially no ratchet current. On the other hand, the corresponding quantized system can feature significant ratchet effects [11–18]. An important problem is to understand the nature of these full-chaos quantum effects in a semiclassical regime, in particular how precisely they vanish, as expected, in the classical limit. All the studies of quantum chaotic ratchets until now have mainly focused on the impact of several kinds of asymmetries on the quantum directed current from a *fixed* initial state. It is, however, well established that the current is *sensitive* to the initial state [10, 12, 13, 15, 17, 18] and this sensitivity is expected to be especially high in a semiclassical full-chaos regime, reflecting the exponential sensitivity of chaotic motion to initial conditions. Thus, to get a comprehensive understanding of the quantum ratchet effect, it is necessary to adopt a more *global* approach, *not* limited to a single initial state.

In this paper, the semiclassical full-chaos regime of quantum ratchets is approached by studying, for the first time, *statistical* properties of the current over sets of initial states with well-defined natural characteristics. The systems considered are generalizations of the paradigmatic kicked Harper models (KHMs) [1, 4–9, 14, 15], with Hamiltonian

$$\hat{H} = L\cos(\hat{p}) + KV(\hat{x})\sum_{t=-\infty}^{\infty}\delta(t'-t), \qquad (1)$$

where L and K are parameters, $\hat{x}$ and $\hat{p}$ are scaled position and momentum operators, $V(\hat{x})$ is a general 2π-periodic potential, t' is time, and t is the integer time labeling the kicks. Generalized KHMs such as (1) describe several realistic systems [6, 7, 9, 15], in particular they are exactly related [6, 7] to kicked harmonic oscillators, which are experimentally realizable by atom-optics methods [3], and to kicked charges in a magnetic field [7]. Recently [14], the systems (1) were shown to exhibit generically a significant and robust quantum momentum current (ratchet acceleration) under full-chaos conditions. The initial state was chosen, as in other works, as a zero-momentum state. In our statistical approach, we identify natural initial states for the semiclassical regime as those that are analogous as much as possible to a phase-space uniform ensemble, for which classical ratchet effects are totally absent. These are states which are uniform in *phase space* with the *maximal possible* resolution of one Planck cell. Such uniformity is not featured by a momentum state which is uniform in position but is infinitely localized in momentum.

Assuming quantum-resonance values of a scaled Planck constant $\hbar = [\hat{x}, \hat{p}]/i$ in the semiclassical regime, we derive an estimate for the distribution of the quantum momentum current I over maximally uniform initial states: This distribution is a Gaussian with mean $\langle I \rangle = 0$ and variance $(\Delta I)^2 = \langle I^2 \rangle \sim D\hbar^2/(2\pi^2)$, where D is the chaotic-diffusion coefficient. A good agreement is found between this estimate and extensive numerical results using an exact formula for I which we also derive. Examples are shown in Fig. 1 and will be discussed further in Sec. IV. The average strength of the effect, measured by the variance above, is found to be *significantly larger* than that for the usual momentum states and other states exhibiting also zero-mean Gaussian current distributions (see the insets of Fig. 1 and Sec. IV). Our results should be experimentally observable to some extent using states approximating the maximally uniform states.

The paper is organized as follows. In Sec. II, we define maximally uniform states in phase space. The main result, i.e., an estimate of the momentum-current distribution over these states in a semiclassical full-chaos regime, is derived in Sec. III. Numerical evidence for this result is provided in Sec. IV, where we also study the momentum-current distributions of states which approximate the maximally uniform states; momentum states are the crudest approximating states. Conclusions are presented in Sec. V, where we briefly

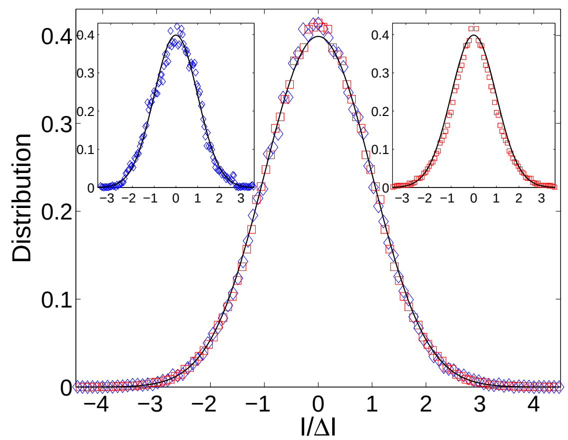

FIG. 1: (Color online) Distributions of the normalized quantum momentum current $I/\Delta I$ over maximally uniform states for $\hbar = 2\pi/121$ in two extreme cases of fully chaotic systems (1) with $K = 15$: The symmetric case "S" with $V(x) = \cos(x)$ and $L = K$ (red squares, $\Delta I = 0.086$) and the strongly asymmetric case "A" with $V(x) = \cos(x) + \sin(2x)$ and $L = K/2$ (blue diamonds, $\Delta I = 0.214$); the latter case was studied in Ref. [14] for a zero-momentum initial state. The origin of ratchet currents in case S is explained in Sec. IV. The insets show the distribution of $I/\Delta I$ over momentum states in case A with $K = 20$ (left inset, $\Delta I = 0.043$) and over low-order approximations of the maximally uniform states in case S with $K = 15$ (right inset, $\Delta I = 0.026$), see Sec. IV for more details. The solid line in all plots is a zero-mean Gaussian with variance 1.

mention possible experimental realizations of the strong quantum-ratchet effects predicted. Detailed derivations of some exact results are given in the Appendix.

II. MAXIMALLY UNIFORM STATES IN PHASE SPACE

Maximally uniform states in phase space are defined on the basis of the translation operators $\hat{T}_x(a) = \exp(i\hat{p}a/\hbar)$ and $\hat{T}_p(b) = \exp(-i\hat{x}b/\hbar)$ shifting $\hat{x}$ and $\hat{p}$ by a and b, respectively. A state $|\psi\rangle$ is uniform on the phase-space lattice with unit cell formed by a and b if it is invariant under application of $\hat{T}_x(a)$ and $\hat{T}_p(b)$ up to constant phase factors: $\hat{T}_x(a)\,|\psi\rangle = \exp(i\alpha)\,|\psi\rangle$ and $\hat{T}_p(b)\,|\psi\rangle = \exp(-i\beta)\,|\psi\rangle$. This means that $|\psi\rangle$ is a simultaneous eigenstate of $\hat{T}_x(a)$ and $\hat{T}_p(b)$, so that these operators must commute. Using $\hat{T}_x(a)\hat{T}_p(b) = \exp(-iab/\hbar)\hat{T}_p(b)\hat{T}_x(a)$, we see that $[\hat{T}_x(a), \hat{T}_p(b)] = 0$ only if ab is a multiple of $h = 2\pi\hbar$. Maximally uniform states $|\psi\rangle$ correspond to the smallest unit cell, i.e., the Planck cell with area $ab = h$. In this case, one can easily check that the position representation of the eigenstates $|\psi\rangle$ is explicitly given by [20]

$$\langle x\,|\psi_{\mathbf{w}}\rangle = \frac{1}{\sqrt{b}}\sum_{n=-\infty}^{\infty} e^{2\pi i n w_2/b}\delta(x - w_1 - na), \tag{2}$$

where $\mathbf{w} = (w_1, w_2)$ range in the Planck cell, $0 \le w_1 < a$, $0 \le w_2 < b$, and specify the phases α and β above: $\alpha = w_2 a/\hbar$ and $\beta = w_1 b/\hbar$. The states (2) for all $\mathbf{w}$ form a complete and orthonormal set [20]. Simple choices of (a, b) can be made by observing that the one-period evolution operator for (1), $\hat{U} = \exp[-L\cos(\hat{p})/\hbar]\exp[-KV(\hat{x})/\hbar]$, is 2π-periodic in both $(\hat{x}, \hat{p})$. Thus, the torus T^2: $0 \le x, p < 2\pi$ is a reduced phase space for the system. For simplicity, we shall assume from now on that there are precisely an integer number N of Planck cells within T^2, choosing $a = 2\pi/N$ and $b = 2\pi$. Then, $\hbar = ab/(2\pi) = 2\pi/N$, so that the semiclassical regime $\hbar \ll 1$ corresponds to $N \gg 1$. As it is well known [4, 8], the values $2\pi/N$ of $\hbar$ are those for which a classical-quantum correspondence can be most easily established for systems describable on a phase-space torus. These values correspond to the main quantum resonances in the semiclassical regime.

III. SEMICLASSICAL ESTIMATE OF THE MOMENTUM-CURRENT DISTRIBUTION

The momentum-current operator $\hat{I}$ can be formally defined in the Heisenberg picture as $\hat{I} = \lim_{t\to\infty}\hat{U}^{-t}\,\hat{p}\,\hat{U}^{t}/t$, for integer time t. Then, the momentum current $I(\mathbf{w})$ for initial state (2) is the expectation value of $\hat{I}$ in (2). More precisely, we show in the Appendix that in the basis of states (2) $\hat{I}$ is essentially represented by $\langle\psi_{\mathbf{w}}|\hat{I}|\psi_{\mathbf{w}'}\rangle = I(\mathbf{w})\delta(w_1 - w_1')\delta(w_2 - w_2')$, where $I(\mathbf{w})$ is given by the explicit exact formula:

$$I(\mathbf{w}) = -\hbar\sum_{j=1}^{N}|\phi_j(0;\mathbf{w})|^2\frac{\partial E_j(\mathbf{w})}{\partial w_1}. \tag{3}$$

Here $\phi_j(0;\mathbf{w})$, $j = 1, \ldots, N$, are coefficients appearing in expressions connecting states (2) with the N quasienergy (Floquet) eigenstates $|\Psi_{j,\mathbf{w}}\rangle$ of the evolution operator $\hat{U}$ for $\hbar = 2\pi/N$ and $E_j(\mathbf{w})$ are the corresponding quasienergy levels.

We now derive from general arguments the following estimate for the distribution $\Gamma(I)$ of $I(\mathbf{w})$ over $\mathbf{w}$ in a semiclassical full-chaos regime:

$$\Gamma(I) \sim \frac{1}{\sqrt{2\pi}\Delta I} \exp\left[-\frac{I^2}{2(\Delta I)^2}\right], \quad (\Delta I)^2 \sim \frac{2D}{N^2} = \frac{D\hbar^2}{2\pi^2}, \tag{4}$$

where D is the chaotic-diffusion coefficient. To derive (4), we first identify natural classical analogs of the states (2). To this end, let us calculate the momentum representation $\langle p \,|\psi_{\mathbf{w}}\rangle = \int_{-\infty}^{\infty} \exp(-ipx/\hbar)\langle x \,|\psi_{\mathbf{w}}\rangle \, dx$ of $|\psi_{\mathbf{w}}\rangle$. One has, up to an irrelevant constant factor,

$$\langle p \,|\psi_{\mathbf{w}}\rangle = \sum_{n=-\infty}^{\infty} e^{-2\pi i n w_1/a} \delta(p - w_2 - nb). \tag{5}$$

It is clear from the delta combs (2) and (5) that $|\psi_{\mathbf{w}}\rangle$ is associated with the phase-space lattice $(x, p) = \mathbf{w} + \mathbf{z}(\mathbf{n})$, where $\mathbf{z}(\mathbf{n}) = (n_1 a, n_2 b) = (2\pi n_1/N, 2\pi n_2)$ for all integers $\mathbf{n} = (n_1, n_2)$. In fact, one can easily show that the Husimi distribution of (2) is peaked on every point of the lattice $\mathbf{w} + \mathbf{z}(\mathbf{n})$. This lattice, viewed as an initial phase-space ensemble, is thus a classical analogue of $|\psi_{\mathbf{w}}\rangle$. Next, consider the classical one-period map M for the systems (1): $p_{t+1} = p_t + Kf(x_t)$, $x_{t+1} = x_t - L\sin(p_{t+1})$, where $f(x) = -dV/dx$. For initial conditions $\mathbf{z}_0 = (x_0, p_0)$, the classical momentum current in t iterations is $I_{\mathrm{c},t}(\mathbf{z}_0) = \Delta p_t(\mathbf{z}_0)/t$, where $\Delta p_t(\mathbf{z}_0) = p_t - p_0$. Since the map M is clearly 2π-periodic in both x and p, one can restrict $\mathbf{z}_0$ to the phase-space torus T^2: $0 \le x, p < 2\pi$. Accordingly, the initial ensemble $\mathbf{w} + \mathbf{z}(\mathbf{n})$ with $\mathbf{z}(\mathbf{n}) = (2\pi n_1/N, 2\pi n_2)$ can be restricted to a finite lattice of N points in T^2 with $n_1 = 0, \ldots, N-1$ and $n_2 = 0$. The classical analog of the quantum current $I(\mathbf{w})$ is the average $\bar{I}_{\mathrm{c},t}(\mathbf{w})$ of $I_{\mathrm{c},t}(\mathbf{z}_0) = \Delta p_t(\mathbf{z}_0)/t$, with $\mathbf{z}_0 = \mathbf{w} + \mathbf{z}(\mathbf{n})$, over this finite lattice:

$$\bar{I}_{\mathrm{c},t}(\mathbf{w}) = \frac{1}{Nt} \sum_{\mathbf{n}} \Delta p_t[\mathbf{w} + \mathbf{z}(\mathbf{n})], \tag{6}$$

for some time t to be specified below. Now, under strong-chaos conditions (large K and L) and for sufficiently large t, each of the N quantities $\Delta p_t[\mathbf{w} + \mathbf{z}(\mathbf{n})]$ in (6) is expected to behave diffusively, i.e., to be distributed over $\mathbf{w}$ approximately as a Gaussian with mean $\langle \Delta p_t \rangle = 0$ and variance $\left\langle (\Delta p_t)^2 \right\rangle \approx 2Dt$. Since these N quantities are associated with N different initial points $\mathbf{w} + \mathbf{z}(\mathbf{n})$ in the chaotic region, they should behave as independent (uncorrelated) random variables. It then follows from the central limit theorem that for large enough N the average current (6) is distributed over $\mathbf{w}$ as a Gaussian with $\left\langle \bar{I}_{\mathrm{c},t} \right\rangle = 0$ and $\left\langle \bar{I}_{\mathrm{c},t}^2 \right\rangle \approx N \left\langle (\Delta p_t)^2 \right\rangle / (Nt)^2 \approx 2D/(Nt)$. This shows how $\left\langle \bar{I}_{\mathrm{c},t}^2 \right\rangle$ decays to zero as $t \to \infty$, when chaotic orbits explore ergodically all the phase space. Since there are precisely N Planck cells in T^2, a typical such orbit will explore phase space, after a time $t \sim N$, up to the maximal quantum resolution of one Planck cell. Then, the distribution of $I(\mathbf{w})$ over $\mathbf{w}$ is expected to be approximately the same as that of the classical currents (6) for $t = N$, i.e., a zero-mean Gaussian with variance $(\Delta I)^2 = \langle I^2 \rangle \sim 2D/N^2$; this is Eq. (4).

IV. NUMERICAL EVIDENCE AND APPROXIMATING STATES

In this section, we provide numerical evidence for the semiclassical estimate (4) using the exact formula (3) and study the momentum-current distributions for states which approximate the maximally uniform states; at the lowest order of approximation, the approximating states are just momentum states. First, the distribution $\Gamma(I)$ was calculated using (3) for several potentials $V(x)$ and many large values of K and L in (1). A good agreement was generally found between $\Gamma(I)$ and a zero-mean Gaussian distribution for sufficiently small $\hbar$. As representative examples, Fig. 1 shows distributions of $I/\Delta I$ in two extreme cases "S" and "A" described in the caption. For the assumed quantum-resonance values $2\pi/N$ of $\hbar$, the origin of ratchet currents in the symmetric case S is the same as that already established in recent theoretical [13] and experimental [17, 18] works on quantum-resonance ratchets: This is a relative asymmetry caused by the non-coincidence of the symmetry centers of a symmetric potential with those of a symmetric initial state. The potential $V(x) = \cos(x)$ in case S has symmetry centers at $x = 0, \pi$ while the state (2) has them at $x = w_1,\ w_1 + a/2$. Thus, for generic values of w_1, $I(\mathbf{w}) \neq 0$.

Natural approximations of the maximally uniform states, denoted in what follows by $\left|\psi_{\mathbf{w}}^{(B)}\right\rangle$ for integer B, correspond to truncations of the infinite sum in (5):

$$\langle p \left|\psi_{\mathbf{w}}^{(B)}\right\rangle = \sum_{n=-B}^{B} e^{-2\pi i n w_1/a} \delta(p - w_2 - nb). \qquad (7)$$

The states (7) are superpositions of the $2B + 1$ momentum states $|p = w_2 + nb\rangle$, $|n| \leq B$, and should be experimentally realizable (see next section). In particular, $\left|\psi_{\mathbf{w}}^{(0)}\right\rangle$ are just momentum states with $p = w_2$. We denote by $I_B(\mathbf{w})$ the momentum current for initial state (7) and by $(\Delta I_B)^2$ the corresponding variance over $\mathbf{w}$. The left inset of Fig. 1 shows the distribution of $I_B(\mathbf{w})/\Delta I_B$ over $\mathbf{w}$ for $B = 0$ (momentum states) in case A while the right inset shows it for $B = 2$ in case S; the currents for momentum states vanish in case S. The fact that these distributions are again approximately zero-mean Gaussians could be expected from the simple relation (16) between $I_B(\mathbf{w})$ and $I(\mathbf{w})$ derived in the Appendix. Actually, we show in the Appendix that this relation leads to the exact result

$$(\Delta I_B)^2 = \left\langle I_B^2 \right\rangle \leq (\Delta I)^2 = \left\langle I^2 \right\rangle . \tag{8}$$

The quantities ΔI and ΔI_B were extensively studied as functions of several parameters. Consider the naturally normalized variance $R \equiv N^2(\Delta I)^2/(2D_{\rm ql})$, where $D_{\rm ql} = K^2 \int_0^{2\pi}[f(x)]^2 dx/2$ is the "quasilinear" value of the diffusion coefficient D, obtained from the KHM map M above by neglecting all the force-force correlations $C_t = \langle f(x_0)f(x_t)\rangle$, $t \neq 0$; for sufficiently strong chaos, D is very close to $D_{\rm ql}$. The semiclassical estimate for the variance in Eq. (4) would imply that $R \approx D/D_{\rm ql}$. Indeed, Fig. 2 shows a reasonably good agreement between R and $D/D_{\rm ql}$ versus K in both cases S and A. Discrepancies arise mainly around peaks of $D/D_{\rm ql}$, especially the peak near $K \approx 6.5$ in case S, due to a small accelerator-mode island. Thus, for general large K with $R \approx 1$, ΔI increases almost linearly with K like $\sqrt{D_{\rm ql}}$.

Fig. 3 shows loglog plots of ΔI versus N in cases S and A. The results agree very well with the N^{-1} behavior predicted by (4). Fig. 4 shows plots of $\Delta I_B/\Delta I$ versus B in the two cases. We see that ΔI_B is always smaller than ΔI, in accordance with the exact inequality (8), and approaches monotonically ΔI as the order B of approximation increases. For small B, ΔI_B is significantly smaller than ΔI and attains its minimal value at $B = 0$, corresponding to momentum states ($\Delta I_0 = 0$ in case S). Within the limited domain of N we could study numerically, ΔI_B appears to decay with N like N^{μ}, where μ ranges between -0.9 to -1.1 with an error not smaller than ± 0.03.

V. CONCLUSIONS

In conclusion, we have presented a first study of the semiclassical full-chaos regime of the quantum ratchet effect using a novel statistical approach which is most required in view of the sensitivity of the effect to the initial state. For maximally uniform states $|\psi_{\mathbf{w}}\rangle$ in phase

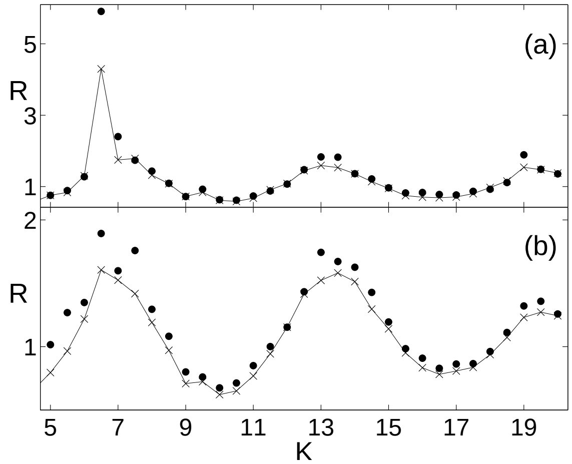

FIG. 2: Filled circles: The quantity $R = N^2(\Delta I)^2/(2D_{\rm ql})$ versus K for $N = 2\pi/\hbar = 121$ in cases S [(a), $D_{\rm ql} = K^2/4$] and A [(b), $D_{\rm ql} = 5K^2/4$] defined in the caption of Fig. 1. Crosses joined by a line: $D/D_{\rm ql}$ versus K.

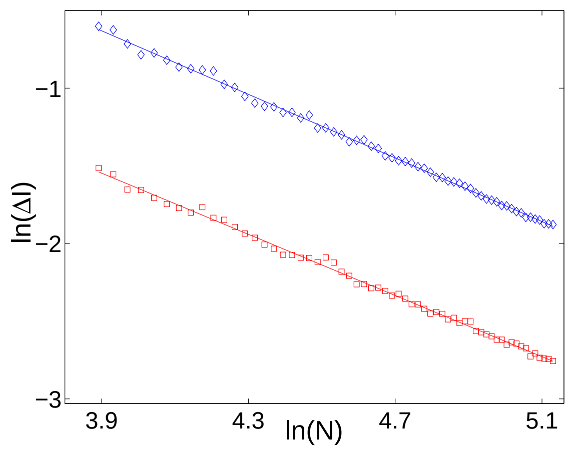

FIG. 3: (Color online) Loglog plots of ΔI versus $N = 2\pi/\hbar$ ($49 \leq N \leq 169$) for $K = 15$ in case S (red squares, with linear fit having slope $\mu = -0.98 \pm 0.01$) and in case A (blue diamonds, $\mu = -1.02 \pm 0.01$).

space, used here for the first time as natural initial states, the momentum-current distribution (4) exhibits clear fingerprints of classical chaotic diffusion, a genuine quantum-chaos phenomenon. The simple formula for the variance in Eq. (4) involves just D and $\hbar^2$ and is thus interestingly similar to the well-known one for the localization length ξ in the kicked rotor [21], $\xi \sim D/\hbar^2$ (in our notation). This variance was shown to be significantly larger than that for momentum states ($B = 0$ in Fig. 4), which were standardly used in previous

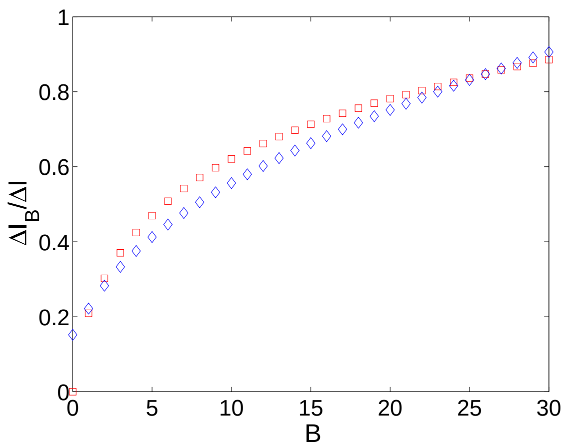

FIG. 4: (Color online) $\Delta I_B/\Delta I$ versus B for $N = 121$ and $K = 15$ in case S (red squares) and in case A (blue diamonds).

works within the ordinary (single-state) approach. The states $|\psi_{\mathbf{w}}\rangle$ then appear to give the *strongest* quantum ratchet effect known until now.

One can approximate $|\psi_{\mathbf{w}}\rangle$ to order B by the states (7) which are superpositions of $2B+1$ momentum states and whose variance increases with B (see Fig. 4). Superpositions of two momentum states were recently used in experimental realizations of quantum-resonance ratchets [17, 18]. The states (7) can be experimentally prepared for at least $B \lesssim 10$ [22]. Also, the systems (1) are exactly related [6, 7] to kicked harmonic oscillators which are experimentally realizable [3]. Thus, the current distributions and the strong quantum ratchet effects predicted in this work should be observable to some extent in the laboratory. More detailed aspects of our statistical approach and its extension to other systems and parameter regimes will be considered in future studies.

ACKNOWLEDGMENTS

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APPENDIX

We derive here formula (3) and the inequality (8). We start with a summary of results from Ref. [8]. Since the evolution operator $\hat{U} = \exp[-L\cos(\hat{p})/\hbar]\exp[-KV(\hat{x})/\hbar]$ is 2π-

periodic in $(\hat{x}, \hat{p})$, it commutes with both $\hat{T}_x(2\pi) = \hat{T}_x^N(a)$ $(a = 2\pi/N)$ and $\hat{T}_p(b)$ $(b = 2\pi)$. Therefore, one can find simultaneous eigenstates of $\hat{U}$, $\hat{T}_x^N(a)$, and $\hat{T}_p(b)$. The general eigenstates of $\hat{T}_x^N(a)$ and $\hat{T}_p(b)$ are given in terms of (2) by

$$|\Psi_{j,\mathbf{w}}\rangle = \sum_{m=0}^{N-1} \phi_j(m;\mathbf{w})\,|\psi_{w_1,w_2+ma}\rangle\,. \tag{9}$$

Here $\phi_j(m;\mathbf{w})$, $j = 1,\ldots,N$, form N independent vectors of coefficients, $\mathbf{V}_j(\mathbf{w}) = \{\phi_j(m;\mathbf{w})\}_{m=0}^{N-1}$, which are determined from the eigenvalue equation $\hat{U}\,|\Psi_{j,\mathbf{w}}\rangle = \exp[-iE_j(\mathbf{w})]\,|\Psi_{j,\mathbf{w}}\rangle$, where $E_j(\mathbf{w})$ are the quasienergies. It is clear from Eq. (9) that $\mathbf{V}_j(\mathbf{w})$ are the representation of $|\Psi_{j,\mathbf{w}}\rangle$ in the N-basis $|\psi_{w_1,w_2+ma}\rangle$, $m = 0,...,N-1$. In this basis, $\hat{U}$ is represented by an $N\times N$ unitary matrix $\hat{\mathbf{M}}(\mathbf{w})$ with known elements [8]. Thus, $\hat{\mathbf{M}}(\mathbf{w})\mathbf{V}_j(\mathbf{w}) = \exp[-iE_j(\mathbf{w})]\mathbf{V}_j(\mathbf{w})$. This completes the summary of relevant results from Ref. [8].

Let us now calculate the matrix element $\left\langle\psi_{\mathbf{w}}|\hat{I}|\psi_{\mathbf{w}'}\right\rangle$, where $\hat{I} = \lim_{t\to\infty}\hat{U}^{-t}\,\hat{p}\,\hat{U}^t/t$ is the momentum-current operator. First, we have

$$\left\langle\psi_{\mathbf{w}}|\hat{U}^{-t}\,\hat{p}\,\hat{U}^t/t|\psi_{\mathbf{w}'}\right\rangle = \langle\psi_{\mathbf{w},t}|\hat{p}|\psi_{\mathbf{w}',t}\rangle\,/t, \tag{10}$$

where $|\psi_{\mathbf{w},t}\rangle = \hat{U}^t\,|\psi_{\mathbf{w}}\rangle$. Using the completeness of the eigenvectors $\mathbf{V}_j(\mathbf{w})$ of $\hat{\mathbf{M}}(\mathbf{w})$, i.e., $\sum_{j=1}^N \phi_j^*(m;\mathbf{w})\phi_j(m';\mathbf{w}) = \delta_{m,m'}$, we can invert Eq. (9) to get $|\psi_{\mathbf{w}}\rangle = \sum_{j=1}^N \phi_j^*(0;\mathbf{w})\,|\Psi_{j,\mathbf{w}}\rangle$. One then has

$$\langle\psi_{\mathbf{w},t}\,|\hat{p}|\,\psi_{\mathbf{w}',t}\rangle = \sum_{j,j'=1}^{N} \phi_{j'}(0;\mathbf{w})\phi_j^*(0;\mathbf{w}')\left\langle\hat{U}^t\Psi_{j',\mathbf{w}}\,|\hat{p}|\,\hat{U}^t\Psi_{j,\mathbf{w}'}\right\rangle. \tag{11}$$

To determine the asymptotic behavior of Eq. (11) for large t, we use the equation $\hat{U}^t(\hat{x},\hat{p})\,|\Psi_{j,\mathbf{w}}\rangle = \exp[-itE_j(\mathbf{w})]\,|\Psi_{j,\mathbf{w}}\rangle$, the expansion (9), and the fact that $\hat{p} = -i\hbar d/dx$ is represented by $-i\hbar d/dw_1$ in the basis (2) [20], due to the delta comb in x. Using also the orthonormality of (2), $\langle\psi_{\mathbf{w}}|\psi_{\mathbf{w}'}\rangle = \delta(w_1 - w_1')\delta(w_2 - w_2')$ [20], and of $\mathbf{V}_j(\mathbf{w})$, we find that the dominant terms in $\left\langle\hat{U}^t\Psi_{j',\mathbf{w}}\,|\hat{p}|\,\hat{U}^t\Psi_{j,\mathbf{w}'}\right\rangle$ for large t give the asymptotic behavior

$$\left\langle\hat{U}^t\Psi_{j',\mathbf{w}}\,|\hat{p}|\,\hat{U}^t\Psi_{j,\mathbf{w}'}\right\rangle \sim -t\,\hbar\frac{\partial E_j(\mathbf{w})}{\partial w_1}\delta_{j,j'}\delta(w_1 - w_1')\delta(w_2 - w_2'), \quad t \gg 1, \tag{12}$$

where we assume for simplicity that $0 \le w_2, w_2' < a$. After inserting (12) in (11) and dividing by t, we get from (10) in the limit $t\to\infty$: $\left\langle\psi_{\mathbf{w}}|\hat{I}|\psi_{\mathbf{w}'}\right\rangle = I(\mathbf{w})\delta(w_1-w_1')\delta(w_2-w_2')$, where $I(\mathbf{w})$ is given by formula (3).

To derive the inequality (8), we first notice that the states (5) and (7) can be easily related as follows:

$$\left|\psi_{\mathbf{w}}^{(B)}\right\rangle = \int_0^a dw_1' g_B(w_1' - w_1) \left|\psi_{w_1',w_2}\right\rangle, \tag{13}$$

where

$$g_B(w_1) = \frac{1}{\sqrt{(2B+1)a}} \sum_{n=-B}^{B} \exp(2\pi i n w_1/a). \tag{14}$$

The factor before the sum in (14) assures the normalization

$$\int_0^a dw_1 g_B^2(w_1) = 1. \tag{15}$$

Eq. (15) implies, because of $\langle\psi_{\mathbf{w}}|\psi_{\mathbf{w}'}\rangle = \delta(w_1 - w_1')\delta(w_2 - w_2')$ and $\left\langle\psi_{\mathbf{w}}|\hat{I}|\psi_{\mathbf{w}'}\right\rangle = I(\mathbf{w})\delta(w_1 - w_1')\delta(w_2 - w_2')$ (see above), that the states (13) satisfy the orthonormality relation $\left\langle\psi_{w_1,w_2}^{(B)}|\psi_{w_1,w_2'}^{(B)}\right\rangle = \delta(w_2 - w_2')$ and $\left\langle\psi_{w_1,w_2}^{(B)}|\hat{I}|\psi_{w_1,w_2'}^{(B)}\right\rangle = I_B(\mathbf{w})\delta(w_2 - w_2')$, where

$$I_B(\mathbf{w}) = \int_0^a dw_1' g_B^2(w_1' - w_1) I(w_1', w_2). \tag{16}$$

Then, (16) is clearly the momentum current for initial state (13).

Now, using (15), (16), and the Cauchy-Schwarz inequality

$$\left|\int_0^a dw_1' F(w_1') G(w_1')\right|^2 \leq \int_0^a dw_1' \left|F(w_1')\right|^2 \int_0^a dw_1' \left|G(w_1')\right|^2$$

with the identifications $F(w_1') = g_B(w_1' - w_1)I(w_1', w_2)$ and $G(w_1') = g_B(w_1' - w_1)$, we get:

$$I_B^2(\mathbf{w}) \leq \int_0^a dw_1' g_B^2(w_1' - w_1) I^2(w_1', w_2). \tag{17}$$

Using (17) and, again, (15) in the definition $\langle I_B^2\rangle = \int_0^a dw_1 \int_0^b dw_2 I_B^2(\mathbf{w})/h$, with $\langle I^2\rangle$ similarly defined, we finally obtain that $\langle I_B^2\rangle \leq \langle I^2\rangle$. This is inequality (8), where $(\Delta I_B)^2 = \langle I_B^2\rangle$ since $\langle I_B\rangle = 0$, as easily implied by (16) and $\langle I\rangle = 0$.

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