

Emergence from Symmetry: A New Type of Cellular Automata

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Abstract

In the realm of cellular automata (CA), Conway's Game of Life (Life) has gained the most fondness, due to its striking simplicity of rules but an impressive diversity of behavior. Based on it, a large family of models were investigated, e.g. Seeds, Replicator, Larger than Life, etc. They all inherit key ideas from Life, determining a cell's state in the next generation by counting the number of its current living neighbors. In this paper, a different perspective of constructing the CA models is proposed. The kernel, Local Symmetric Distribution Principle, interrelates some fundamental concepts in physics, which maybe raise a wide interest. With a rich palette of configurations, this model also hints its capability of universal computation. Moreover, it possesses a general tendency to evolve towards certain symmetrical directions. Some illustrative examples are given and physical interpretations are discussed in depth. To measure the evolution's fluctuating between order and chaos, three parameters are introduced, i.e. order parameter, complexity index and entropy. In addition, we focus on some particular simulations and giving a brief list of open problems as well.

1 Introduction

The conception of cellular automaton (CA) stems from John von Neumann's research on self-replicating artificial systems with computational universality [22]. However, it was John Horton Conway's fascinating Game of Life (Life) [15, 8] which simplified von Neumann's ideas that truly opened up a whole new field of science. It vividly illustrates the emergence of complex behavior from simple rules and thus possesses profound philosophical connotations. Albeit useless for computation in practice, Life has been proven theoretically as powerful as a universal Turing machine [8]. Since then, CA has widely spread its voice outside computer laboratories. A plethora of models are developed to simulate physical or social systems, such as lattice gas automata in molecular dynamics [16, 14, 19], the Asymmetric Simple Exclusion Process on traffic problems [26] and Langton's ant—a particular species of artificial life [17]. For those who are interested, Ref. [11] provides a detailed discussion of various CA models. Besides, comments on CA together with other extremely simplified models in physics can be found in [5]. A thorough survey on the aspects concerning mathematical physics refers to [27].

To trace the development of physical thoughts in this field, we would like to mention two books. In the late 1970s, Konrad Zuse conceived an essay entitled *Calculating Space* [34], in which he advocated that physical laws are discrete by nature and that the entire history of our universe is just the output of a giant deterministic CA. Although crazy-sounding, it indeed suggested a possible replacement of traditional depicts in our textbooks, and signified the outset of *digital physics*. Thirty-three years later, Stephen Wolfram, a pioneer of researches on the complexity of CA [30, 31], published a notable book named *A New Kind of Science* [32], in which he extensively and systematically argued the discoveries on CA are not isolated facts but closely interrelated with all kinds of disciplines.

No doubt that the invention of Life marked a watershed in the history of CA. It has developed a lasting cult and has many variations (see [12] and references therein). In the compact notation used by Golly¹, Life is denoted as B3/S23, where “B” stands for “birth” and “S” for “survival”. In a nutshell, Life is mainly generalized through three ways. (i) Simply change the number of living neighbors that determines dead cells' birth and living cells' survival. Some examples include Replicator (B1357/S1357), Seeds (B2/S) and Plow World (B378/S012345678). (ii) Assign more states to a cell

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¹ Golly is a cross-platform open-source simulation system for Life and some other interesting CA models. It is available at <http://golly.sourceforge.net/>.

or modify the geometry of the universe. One of the most famous is a three-state (live, ghost, death) automaton called Brian’s Brains. (iii) Extend the distance of Moore neighborhood beyond one, such as Large than Life [13] and Kaleidoscope of Life [1]. In the latter, a cell’s state is determined by whether there are exactly 4 living cells at Moore-distance 1 or 2. Since Life is at the root of these models, they all share certain sameness inevitably. To explore something utterly different from Life, novel methods to construct CA models are needed.

We will begin in Sec. 2 with introducing the transition rules, giving some examples and outlining its features. Then, physical interpretations of the automaton are viewed from different perspectives (Sec. 3). It should be noted that those discussions can not be limited to this model only. In Sec. 4, three evolution parameters are defined and discussed. Despite that insufficient efforts are spent on these parameters (no specific assertions are made), they may exemplify a heuristic approach to mathematical investigations. After taking an especial look at several independent topics in Sec. 5, the paper finishes with a brief summary and a reemphasis on Zuse’s paradigm-shifting ideas (Sec. 6).

2 Game of Symmetry

2.1 Rules

As the same to Life, we adopt Moore neighbors to construct this CA model; that is, a neighborhood consists of nine cells. Each cell has two possible states: dead and alive. For the ease of formulation, sometimes we may use empty and occupied, or 0 and 1 instead if convenient. For intuition, we also use particles to indicate living cells, generating and annihilating to describe the evolution process of birth and death, and so forth. The rules we will set down are called *Local Symmetric Distribution Principle* (LSDP). Among 512 possible patterns for a neighborhood, sixteen are considered as *meta-configurations* (Fig. 1), the distribution of whose Moore neighbors is symmetrical with respect to the center particle. Actually, the first five comprise all the rest via a special superposition. More precisely, the set of those configurations forms a bounded semilattice \mathcal{S} . In other words, it is a commutative monoid with the presentation

$$\mathcal{S} = \langle a, b, c, d \mid a^2 = a, b^2 = b, c^2 = c, d^2 = d \rangle. \quad (1)$$

It is assumed that each neighborhood spontaneously tries to reach one of the meta-configurations, i.e. all cells are updated simultaneously, conforming to the following transitions:

1. A dead cell comes back to life when its neighbors are symmetrically distributed.
2. A living cell with symmetrically distributed neighbors survives to the next generation.
3. A living cell with asymmetrically distributed neighbors transitions distinctively in two cases: if activating one dead neighbor would get to a meta-configuration, it just introduces such a desired change; otherwise, it becomes a dead cell.

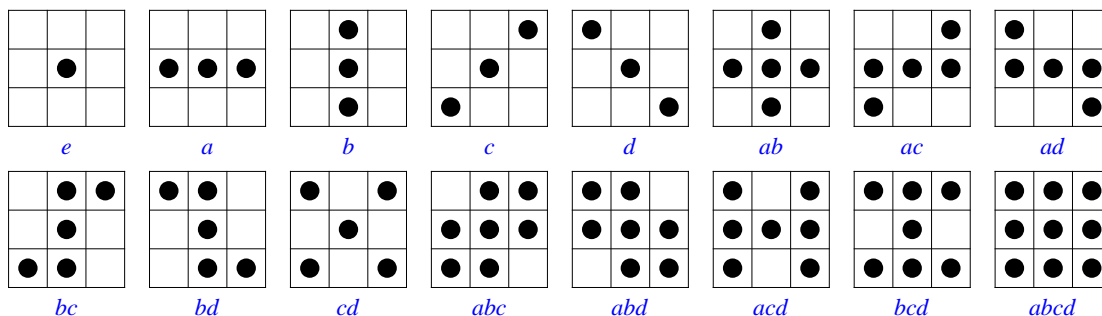


FIG. 1. Sixteen meta-configurations

This model is also a zero-player game. The evolution is determined by applying the above rules repeatedly to each cell in the former generation. Now a cell’s state depends on the distribution of its neighbors rather than the sum of their states (express the rules in Life using 0/1 notation), therefore it is no longer a totalistic automaton. As a consequence, different landscapes in this “universe” can be expected. A further understanding of LSDP will be discussed in the next section, and now we shall see some useful configurations to get familiar with the rules.

2.2 Examples of Patterns

The simplest static patterns are *still lives*, including an isolated particle, i.e. the first configuration in Fig. 1. It seems that all still lives are collections of isolated particles (Fig. 2). Experimentations show that the largest number of noninteracting particles that can be placed in 7×7 grid is 10. Remarkably, the first two patterns in Fig. 2 can be used as different lattices in the simulations of physical systems. *Oscillators*, the majority of which are period 2, can be assembled easily from several basic modules that will be classified later. It is worth special notice that the octagonal configuration in Fig. 3 serves as a base of various particle emitters. Of course, there exist many other configurations that emit particle beams (particles propagating along a line). For instance, the toad configuration²—an oscillator in Life—is also an oscillator in our model. Simply adding one particle in the appropriate cells will just obtain an emitter.

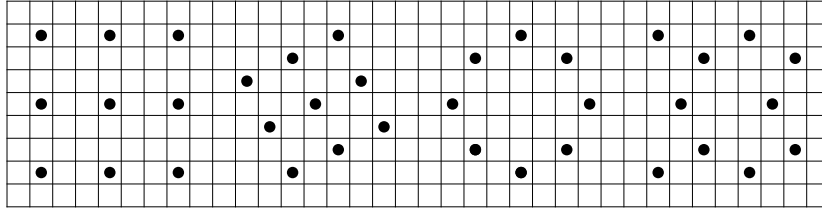


FIG. 2. Some patterns of still lives in 7×7 grid

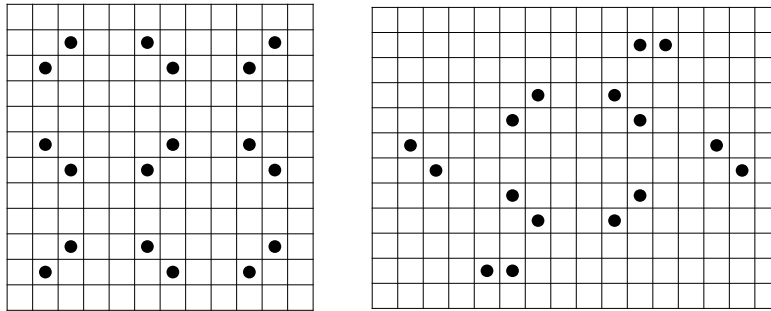


FIG. 3. Two constructions of particle emitters

Other funny patterns include *gliders* or *spaceships*, a frequent occurrence of which in Life makes it Turing complete. This kind of patterns is the most valuable and usually at the heart of a universal constructor. Collision-based computation (see Ref. [2]) is so widely studied that perhaps it has set apart a pure research field called *glider dynamics*. Two examples are shown in Fig. 4. They both have a period of 2 and travel across the grid diagonally at the same velocity.

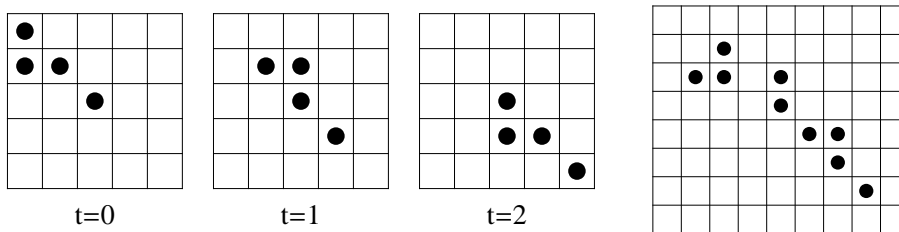


FIG. 4. The simplest glider's evolution (left) and a lightweight spaceship (right)

Patterns called *diverters* refer to those configurations that can change their evolution direction in finite time. An initial state like that in Fig. 5, only differing in the number of horizontally distributed particles (denoted by n , $n \geq 3$), will eventually extend itself in vertical direction after $n - 1$ generations. In sharp contrast, this series of configurations in Life nearly have nothing in common during evolution, reflecting a highly dependence on the absolute number of neighbors' population rather than global

²See http://en.wikipedia.org/wiki/Conway's_Game_of_Life.

distribution. Therefore, this model may not be so sensitive to an external disturbance. It is meaningful to reveal how global effects emerge from local rules.

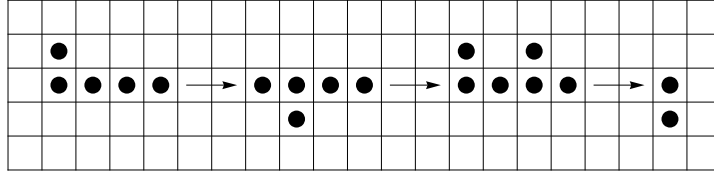


FIG. 5. Evolution of a simple diverter ($n = 4$)

2.3 Features

The most striking feature of this automaton is the much easier implementation of infinite growth (usually at a slow rate). It needs only two particles, directly adjacent to each other horizontally, vertically or diagonally, to form such a pattern. For this advantage, more convenience is provided to simulate interactions between relatively independent systems and thence more powerful calculating machines could be expected. But on the other hand, it becomes more impossible to predict the long-term output of the evolution, not merely because of chaotic factors but also the high chances that two ostensibly disconnected systems can amalgamate with each other after an extremely long time.

The key to Life's fascination lies in its achieving an astonishing diversity of behavior, fluctuating between randomness and order. However, most interesting patterns in the Life lexicon, e.g. *Gosper's glider gun*, *puffer train*, and *Methuselahs* are all obtained through brute-force computer experiments. Game of Symmetry can add some design into this method to construct more complicated objects by virtue of its numerous and flexible subassemblies. Some have been alluded before and we shall see more in Sec. 5.1.

Another feature is the general evolution tendency to become symmetrical. Once it happens, the symmetry may increase in richness but cannot be lost unless a nearby subpattern comes close enough to bring about an unrecoverable interference. This characteristic is also possessed by Life and some other models. However, our automaton seems to have a specific ability to recognize how the configuration lacks symmetry and automatically evolves towards the direction in which symmetry is broken most. There is no difficulty in understanding that such a property have already been embodied in the transition rules. This point can be demonstrated by diverters and interactions between parallel particle beams (Fig. 6). Although the universe has exhibited a favor of uniformity, sometimes it can become totally disordered when such a persistence is impossible. As the famous warning puts, *nature does not always share our tastes about a beautiful theory* [25], we can not expect the universe always operates according to our preferences.

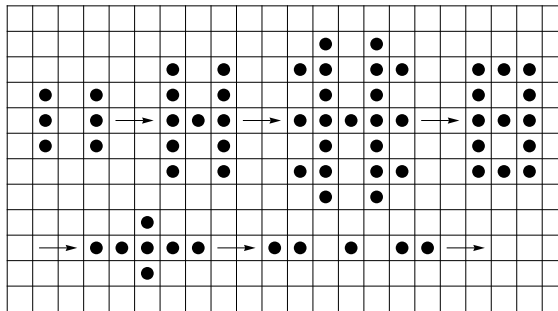


FIG. 6. Interactions between two parallel particle beams that illustrate how the system's evolution direction changes to reach a symmetrical state

2.4 About SuperSymmetry

All configurations in this paper have been tested on the program SuperSymmetry (it has no much relation to the supersymmetry in string theory). To realize some similar features to Golly, such as an

infinite universe, extremely fast generation and unbounded zooming out for astronomical patterns, it is still under development.

3 Physical Interpretation

Symmetry, conveying much more meanings than geometrical aesthetics, involves many fundamental concepts of modern science. In particle physics, symmetries of the laws almost determine the properties of particles found in nature. Noether's renowned theorem states that, roughly speaking, for every symmetrical transformation, there is a corresponding conserved quantity. Philip Warren Anderson had made a succinct comment in his thought-provoking article: *it is only slightly overstating the case to say that physics is the study of symmetry* [3].

As already noted, this automaton has an intrinsic property of conserving symmetry in the initial configuration, which results from transition rules. Nevertheless, the universe still lacks spatial isotropy, although the meta-configurations are selected regardless of such a difference. Even from the mathematical angle, LSDP is somewhat special. It should be regarded as another type of symmetry that cannot be classified into any existing categories. Recall one of the principles in quantum mechanics, which posits that particles of the same kind are utterly indistinguishable. To a degree, LSDP is close to Bose-Einstein statistics, allowing two or more identical particles to occupy the same state. However, such an analogy is superficial; perhaps it associates with a hidden facet of the universe.

Moreover, systems governed by LSDP tend to evolve towards a pattern in which all particles resemble each other from a local perspective, as if a force is exerted. This maybe result in the order creation from chaos. According to the contentions in Refs. [24, 18], self-organization in a dynamical system is characterized by a final state which is more ordered than the initial; and the notion of emergence means that the whole is greater than the sum of the parts. Referring to the examples and features in Sec. 2, we can clearly see that our model is just an excellent conceptual laboratory for illustrating these abstract notions (or many others).

Normally, CA models use local rules to determine how a cell transitions, i.e. it only considers the nearby neighbors. When designing an algorithm, we never need to make calculations over all cells. Whereas, in the real-world, especially at the microscopic level, strong-coupling systems are common, such as quantum entanglement and superconductivity. Therefore, it appears that our model fails to simulate the whole universe. However, if we suppose the automaton evolves extremely fast, e.g. 10^{-43} s (Planck time) per generation, the local rules will certainly produce the same outcomes as nonlocal ones. In this sense, Zuse's suggestion can be quite reasonable, provided that the cell is also at the Planck scale (10^{-35} m). Remarkably, Gerardus 't Hooft had expressed similar ideas in [23]. Based on a cogent argument, 't Hooft said: *I think Conway's game of life is the perfect example of a toy universe. I like to think that the universe we are in is something like this.*

Nowadays, most physicists would agree on the statement that *information is physical* (Rolf Landauer's aphorism). Nevertheless, when it comes to the idea that *everything is information*³, controversies are ubiquitous. Notwithstanding that there is not yet a theory with information as its core having reached the status of physical law, it is much better to leave adequate room for such a belief. We hope that it can blossom in years to teach us how to think computationally about nature and bring us fruitful insights on physics. Just as the research on automata has implied, the discretization of both space and time may be the best clue guiding us to the theory of everything⁴.

4 Evolution Parameters

4.1 Symmetry-evolution Diagram

For the sake of simplicity, we shall discuss some preliminary terms first. For a living cell, its eight neighbors forms a *local environment* of the center particle, thus a separate system at most has 256 different local environments. We say a particle symmetrical if its environment belongs to one of the sixteen meta-configurations; otherwise, it is considered to be asymmetrical. To depict the process of order-disorder transitions, it is convenient to use n_1, n_2 to denote the number of symmetrical and

³John Wheeler also made another pithy summary of his insights: *It from bit* [20]. Building upon the researches into black hole thermodynamics and the holographic principle (see [10] for reference), Jacob Bekenstein extended Wheeler's original ideas in [7].

⁴In loop quantum gravity (LQG), space-time has a granular structure and each node in a spin network determines an elementary grain of space [28, 4]. A particular perspective to combine digital physics with LQG is discussed in [33].

asymmetrical particles respectively and then define the *order parameter* by the expression

$$\xi = \frac{n_1 - n_2}{n_1 + n_2}. \quad (2)$$

Order parameter is a term from the theory of phase transition and critical phenomena. However, it has completely different meanings here. Obviously, $\xi = 1$ relates a still-live pattern; and a dynamically evolving configuration may get closer and closer to 1 but can never reach. It is worthy of attention that $\xi = -1$ doesn't mean the corresponding configuration will disappear or break down soon. On the contrary, such a pattern as a whole can be quite stable (see Fig. 4).

In addition, *symmetry-evolution diagrams* are used for graphical representations. In Fig. 7, we plot the particle number n against evolution time t . If positive, n represents symmetrical particles; otherwise, it indicates asymmetrical ones. This convention does not hold in Table 1, where $n = n_1 + n_2$. In some cases, this kind of diagrams can be extremely irregular, which may provides a possible usage of the particle numbers as pseudo-random numbers.

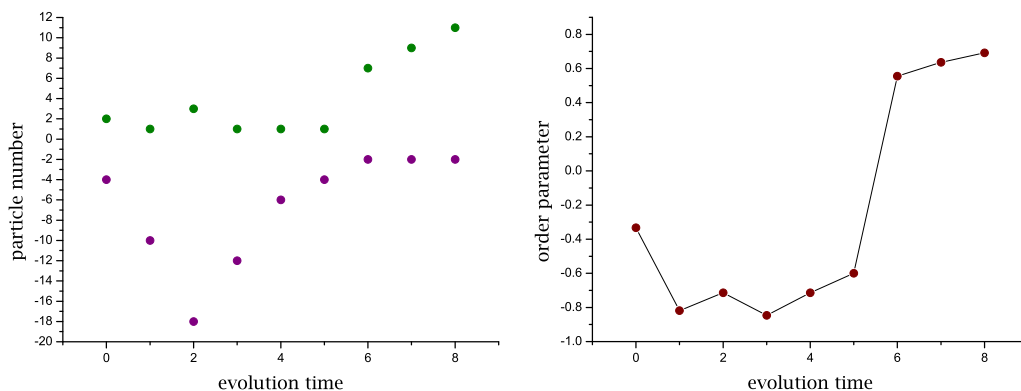


FIG. 7. Particle number of the beam configuration in Fig. 6 fluctuates irregularly before it reaches convergence to a quite stable state (left). The corresponding order parameter curve is also given (right).

4.2 Complexity Index

Since a pattern can easily grow to infinity, the comparison of two configurations' complexity cannot merely rely on their absolute population number. *Complexity index* $\phi(t)$ is defined as a function referring to the total number of different local environments in a generation. For example, the complexity index of the simplest glider (Fig. 4) is invariant during its evolution, i.e. $\phi(t) = 4, \forall t > 0$. It is evident that all still-live patterns have the same complexity index of 1, thus they belong to the same *complexity class*. A larger index usually corresponds to more complex behavior. However, this term invites the problem of how to deal with separated parts that coexist but have no impacts on each other. Different means to do this will yield different values. In principle, it depends on whether you would like to regard those parts as mutually independent closed systems or just subsystems of a larger one.

4.3 Entropy

We introduce the concept of *entropy* for each generation by adopting Shannon's definition of classical information entropy:

$$\mathcal{H} = - \sum_i p_i \log_2 p_i, \quad (3)$$

where $p_i, i = 1, 2, \dots$, are proportional distributions of particles' local environments. Meanwhile, the symbol \mathcal{P} is used to denote the *maximum probability* in this distribution. We still take the pattern in Fig. 6 as an example to illustrate the calculations. From Table 1, we can conclude that entropy decreases as the configuration goes to a more ordered specification in the next generation. By contrast, the thermal entropy of any isolated systems must increase, or remains constant at best. However, this remark does not connote that our model forbids any continuous increment of entropy. In a subtle way,

entropy relates both particle numbers and the complexity index, but it is much more conducive to make global evaluations. Due to its far-reaching implications, discussions about the concept of entropy can also be found in many surveys on CA. Ref. [6] reveals a simple linear relation between entropy and the largest Lyapunov exponent in the lattice gas model, which may be enlightening.

TABLE 1. Evolution parameters of the configuration in Fig. 6

t	n	ϕ	\mathcal{H}	\mathcal{P}	t	n	ϕ	\mathcal{H}	\mathcal{P}
0	6	3	1.5850	0.3333	4	7	7	2.8074	0.1429
1	11	9	3.0958	0.1818	5	5	5	2.3219	0.2000
2	21	21	4.3923	0.0476	6	9	3	0.9864	0.7777
3	13	11	3.3927	0.1538	7	11	3	0.8659	0.8182

5 Further Exploration

5.1 Particle Collision

Particle-like structures in CA do not gain sufficient attention among researchers⁵. Perhaps this is due to the lack of an ideal model. Fortunately, our automaton offers a dozen of patterns facilitating such an investigation. Here we only consider the interactions between two particle beams that travel towards each other. To simplify our description, the number of particles in each beam is restricted to 2 and configurations are set so that they will immediately start interactions in the next generation. Fig. 8 shows a few examples. Elaborate arrangements of (a), (c), (d), (f) and (l) can constitute many beautiful oscillators. The configuration in (g) provides us a neat example to illustrate the conservation of momentum and the generation of two gliders. In contrast, (h) acts quite strangely: it creates isolate particles along a diagonal, i.e. the symmetry axis. Besides, (b), (e) and (k) also display gliders' creation from collisions between particle beams, in which (b) only generates the simplest gliders while (k) can also produce the lightweight spaceship in Fig. 4. All these are resources of great utility, which can be used as basic modules when designing complicated projects on this automaton. Some of them may play crucial roles in universal computation, such as transporting data, self-replicating and generating signal sequences. However, it should be pointed out that the evolution may be slightly different in certain patterns when beams consist of more particles than two.

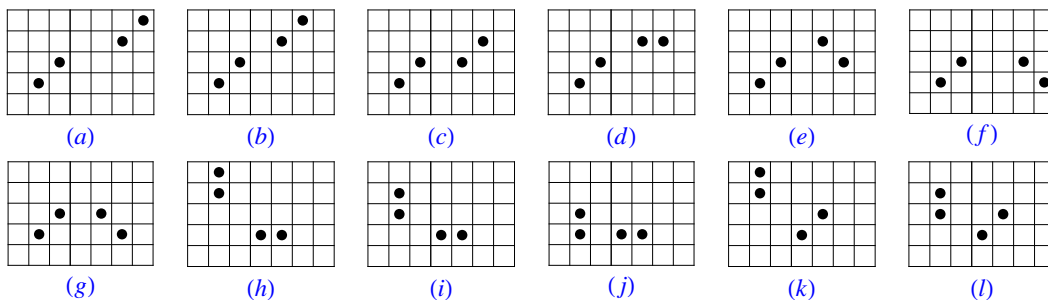


FIG. 8. Collisions between two simplified beams

5.2 Evolution from Randomness

Quantum field theory tells us that the vacuum is not empty and that all fields undergo quantum fluctuations, leading to particle pair production and annihilation all the time. Albeit counter-intuitive, this kind of picture has profoundly enhanced our understanding of nature. Thus, the evolution results from random initial configurations also deserve a hard effort. They are essential to evaluate the thesis whether physical laws can emerge from random processes, just as John Wheeler puts it: *Law without Law* [20]. Besides, they are helpful to give a comprehensive impression of this model's hallmark.

⁵We refer to [9] for a discussion of such structures in one-dimensional automaton.

If we only consider a cell's nearest neighbors, taking no account of those at Moore distance 2 (neighbors of $d \geq 3$ doesn't contribute to its fate in the next generation), the probability of a cell's being alive will be $\frac{31}{512} \approx 0.06$. It is a small value, but the strong interactions between adjacent particles may endow them a much longer life, as manifested in the glider's evolution. Therefore, we should feel confidence in the richness emerging from random specifications.

5.3 Open Problems

It is often hard to cover many aspects in a single manuscript. The following problems are selected with the criteria of its significance and the fascination to pique widespread interest:

- Chances are that this CA model is also Turing complete, but it lacks rigorous proofs at present.
- Does there exist a spaceship translating themselves across the grid horizontally or vertically?
- Does there exist a gun that can repeatedly shoot out gliders?
- Do there exist solitons that remain intact after an interaction with others?
- Can we carry out more experimentation of physics in this universe, such as percolation, quantum computing and lattice scattering?
- Discuss its mathematical structures in a strict form, e.g. connections to the theory of dynamical systems.
- Explore possible applications to signal encoding, random sequence generating, or even artificial intelligence.
- Generalize this model to higher dimensions, or stochastic automata.

6 Conclusion

In this paper, we have constructed a new type of automata and discussed its links to some physical concepts. Much to our delight, the model also achieves a bewildering variety of patterns, especially dozens of easily controlled modules. Comparisons with Life are emphasized, and challenging problems are listed, but more simulations are not strived for. As has been revealed, the mathematical constructor is of simplicity, but no related discussions are provided.

In closing, we would like to quote Zuse's perceptive predictions made forty years ago: *Incorporation of the concepts of information and the automaton theory in physical observations will become even more critical, as even more use is made of whole numbers, discrete states and the like* [34]. It is somewhat like Edwin Abbott's *Flatland*, which disseminated a crazy idea of many dimensions *thereby contributing to the enlargement of the imagination*. A century later, string theorists have made great advances in their quest to the Holy Grail of physics [29, 21]. One of its underpinnings is just the assumption of 11 dimensions.

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