Integrable Euler top and nonholonomic Chaplygin ball

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Abstract

We discuss the Poisson structures, Lax matrices, r-matrices, bi-hamiltonian structures, the variables of separation and other attributes of the modern theory of dynamical systems in application to the integrable Euler top and to the nonholonomic Chaplygin ball.

1 Introduction

The main aim of this paper is to prove that the integrable Euler top and the nonholonomic Chaplygin ball are very similar dynamical systems like birds of a feather flock together. Thus, on example of these twins, we want to show how all the machinery developed for integrable systems can be carried to the theory of solvable nonholonomic systems.

The integrable Euler case of rigid body motion with the fixed center of mass (the Euler top) is relatively simple in the sense that its equations of motion do not linearize on Abelian surfaces, but on the elliptic curves. Of course, this does not make the Euler top entirely trivial [14]. A classical description of the Euler top can be found in any textbook on classical mechanics, see, for instance, [1, 2, 6]. In contrast with this standard approach we will consider the Euler top on the whole phase space $so(3) \ltimes \mathbb{R}^3$ instead of only $so(3)$.

The nonholonomic Chaplygin ball [10] is that of a dynamically balanced 3-dimensional ball that rolls on a horizontal table without slipping or sliding. 'Dynamically balanced' means that the geometric center coincides with the center of mass. However, the mass distribution is not assumed to be homogeneous. The inertia matrix can be any symmetric positive definite three by three matrix. The no slip, no slide condition is a non-holonomic constraint on the velocities. The ball is allowed to rotate about its vertical axis. There is a large body of literature dedicated to the Chaplygin ball, including the study of its generalizations. See [5, 8, 11, 12, 15, 21, 18, 23, 24, 25, 29] and an excellent survey [7] to name just a few references. Of course, this list, as well as the bibliography of the present paper, is by far incomplete.

This text can be regarded as some mathematical variations on the topic of Kozlov papers [23, 24, 25] and of Borisov, Mamaev results [5, 7]. All the necessary preliminary physical information can be found in their papers.

2 Equations of motion

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and $M = (M_1, M_2, M_3)$ be the two vectors of coordinates and momenta, respectively. We postulate that they satisfy to the following differential equations

$$\dot{M} = M \times \omega\,, \qquad \dot{\gamma} = \gamma \times \omega\,. \tag{2.1}$$

For any vector function ω on the dynamical variables $x = \gamma, M$ these equations in $\mathcal{M} = \mathbb{R}^3 \times \mathbb{R}^3$ have the following integrals of motion

$$H_1 = (\gamma, \gamma), \qquad H_2 = (\gamma, M), \qquad H_3 = (M, M)\,. \tag{2.2}$$

Six differential equations can be solved in quadratures if we know its four integrals [17]. So, we want to add some additional integral to the known integrals H_1, H_2 and H_3 (2.2).

If we assume the existence of the following additional integral of motion

$$H_4 = (M, \omega) \tag{2.3}$$

one gets

$$\frac{dH_4}{dt} = (M \times \omega, w) + (M, \dot{\omega}) = (M, \dot{\omega}) = 0\,,$$

it means that the derivative $\dot{\omega}$ has to be perpendicular to M. Below we stint ourselves by second order integrals in momenta M, such as

$$H_4 = (M, \mathbf{A}M) \qquad \text{and} \qquad \omega = \mathbf{A}M\,. \tag{2.4}$$

In generic $\mathbf{A}$ can be a matrix depending on γ, which has to satisfy to the equation

$$(M, \dot{w}) = (\mathbf{A}^\top \mathbf{A}^{-1}\omega, M \times \omega) + (M, \dot{\mathbf{A}}M) = 0\,.$$

This equation can be replaced by the particular system of very simple equations

$$\mathbf{A}^\top \mathbf{A}^{-1} = \mathbf{Id}, \qquad \text{and} \qquad (M, \dot{\mathbf{A}}M) = 0\,, \tag{2.5}$$

which has two trivial solutions

$$\mathbf{A}_1 = \mathbf{A}\,, \qquad \text{and} \qquad \mathbf{A}_2 = f(\gamma)\,\gamma \otimes \gamma\,, \qquad \dot{\mathbf{A}}_{1,2} = 0, \tag{2.6}$$

and one their nontrivial composition

$$\mathbf{A}_3 \quad = \quad \mathbf{A} + \mathrm{g}(\gamma)\,\mathbf{A}\,\gamma \otimes \gamma\,\mathbf{A}\,, \tag{2.7}$$

$$\dot{\mathbf{A}}_3 \quad = \quad \mathrm{g}(\gamma)^2 \mathbf{A}\,(\gamma \otimes \beta + \beta \otimes \gamma)\,\mathbf{A}, \qquad \beta = \big(\eta\gamma - (\gamma, \gamma)\mathbf{A}\gamma\big) \times \mathbf{A}M\,. \tag{2.8}$$

Here $\mathbf{A} = \mathbf{A}^\top$ is a numeric symmetric matrix, function $\mathrm{g}(\gamma)$ is equal to

$$\mathrm{g}(\gamma) = \frac{1}{\eta - (\gamma, \mathbf{A}\gamma)}\,, \tag{2.9}$$

whereas function $f(\gamma)$ and parameter η are arbitrary ones.

Remark 1 There are many other solutions of the system (2.5). As an example, linear in variables γ matrices

$$\mathbf{A}_4 = \mathbf{A} + \mathbf{B}(\gamma \otimes c + c \otimes \gamma)\mathbf{B}^\top$$

satisfy (2.5), if we impose various restrictions on the numerical entries of matrices $\mathbf{A}, \mathbf{B}$ and vector c.

Thus, we obtain the fourth integral of motion H_4 (2.4) for the six equations (2.1) at $\omega = \mathbf{A}_k M$, $k = 1, 2, 3$. We proceed by showing that these dynamical systems are solvable in quadratures in framework of the Euler-Jacobi last multiplier theory [17].

By definition, the Jacobi multiplier $\mu(x)$ of (2.1) is a function on dynamical variables $x = \gamma, M$, which has to satisfy to the equation

$$\sum_{i=1}^{6} \frac{\partial}{\partial x_i}\,\mu(x)\,\dot{x}_i = 0, \qquad \Rightarrow \qquad \dot{\mu} + \mu \sum_{j=1}^{3} \left(\frac{\partial}{\partial \gamma_j}\,(\gamma \times \omega)_j + \frac{\partial}{\partial M_j}\,(M \times \omega)_j \right) = 0\,.$$

For the solutions $\mathbf{A}_{1,2}$ (2.6) this equation is trivial

$$\dot{\mu} = 0 \qquad \text{and} \qquad \mu = c, \qquad c \in \mathbb{R}, \tag{2.10}$$

but for its combination $\mathbf{A}_3$ one gets

$$2\mathrm{g}(\gamma)\dot{\mu} - \mu\dot{\mathrm{g}}(\gamma) = 0\,, \qquad \text{and} \qquad \mu = c\sqrt{\mathrm{g}(\gamma)}\,. \tag{2.11}$$

According to [17] the Jacobi's multiplier is some nontrivial function in the case of constrained systems only. The integrability conditions of the nonholonomic systems formulated by Kozlov [24, 25] include the preservation of measure related with the Jacobi multiplier.

Remark 2 Equations (2.1) can be identified with the Euler-Poisson equations describing the rotation of a rigid body around a fixed point or with the Kirchhoff equations describing the motion of a solid body in an ideal incompressible fluid.

Vector M is the vector of the kinetic momentum of the body, expressed in the so-called body frame. This frame is firmly attached to the body, its origin is in the body's fixed point and $\mathbf{A}_1^{-1}$ is the tensor of inertia with regard to the fixed point and this body frame.

In the Euler-Poisson case, vector γ is the unit Poisson vector, $(\gamma,\gamma) = 1$, along the gravity field, with respect to the body frame. In the Kirchhoff case, M and γ are the vectors of the impulsive momentum and the impulsive force, so (γ,γ) is an arbitrary constant. In both cases vector ω is the angular velocity of a body, and our first solution $\mathbf{A}_1$ in (2.6) is associated with the integrable Euler case [2, 6].

The second solution $\mathbf{A}_2$ is trivial because $H_4 = f(\gamma)H_2^2$ and $\dot{\gamma} = 0$.We will not consider this case below.

The third solution $\mathbf{A}_3$ (2.7) can be related with the nonholonomic Chaplygin ball if we put

$$\eta = \frac{1}{ma^2},$$

where m and a are the mass and the radius of the ball respectively. In this case vector M is the ball's angular momentum with respect to the point of contact, and $\mathbf{A}^{-1}$ is the corresponding tensor of inertia.

Dependence of $\mathbf{A}_3$ on γ is a sequence of nonholonomic constraints imposed on the system [10], see also [7] and references within.

To sum up, we can easily get a lot of additional integrals H_4 (2.4) of the equations (2.1) and, therefore, we can solve these differential equations in quadratures without any notion of the Hamilton structure, integrability by Liouville, the Poisson structure, the Lax matrices, classical r-matrices etc.

However, this additional and in some sense redundant information can be useful in various applications, such as the perturbation theory, the quantization theory and so on. Below we reconstruct this information starting with only integrals.

3 Solvability versus integrability.

In this section our aim is to calculate the Poisson brackets for the given models without any assumptions on underlying Hamiltonian or conformally Hamiltonian structures of the equations of motion (2.1) [5, 7]. We will calculate the desired Poisson brackets only assuming that the foliation $H_i = \alpha_i$ is the direct sum of symplectic and lagrangian foliations.

The Jacobi last multiplier theorem [17] ensures that the six equations (2.1) with the four functionally independent integrals $H_1, \ldots, H_4$ and the multiplier μ are *solvable* in quadratures.

On the other hand, according to the Liouville theorem, equations of motion are *integrable* in quadratures if we have the necessary number of integrals of motion in the involution with respect to some Poisson brackets.

In fact [25], the combination of the Jacobi and Liouville theorems gives rise to the following simple proposition .

Proposition 1 *Solvable equations of motion (2.1) with integrals $H_1,\dots,H_4$ and multiplier μ are integrable by Liouville, if there is the Poisson bivector P such as*

$$\begin{aligned} &\bullet\ [\![P,P]\!]=0, && \textit{the Jacobi identity,} \\ &\bullet\ PdH_i=PdH_k=0, && \textit{reduction of dimension of the phase space} \\ &\bullet\ \{H_l,H_m\}=\langle PdH_l,dH_m\rangle=0, && \textit{the involution of the integrals}. \end{aligned} \tag{3.1}$$

Here $[\![.,.]\!]$ is the Schouten bracket, $H_1,\dots,H_4$ are the four integrals (2.2-2.3), and (i,j,l,m) is the arbitrary permutation of $(1,2,3,4)$.

The first equation in (3.1) guaranties that P is a Poisson bivector. In the second equation we define two Casimir elements H_i and H_k of P. It is a necessary condition because by fixing its values one gets the four dimensional symplectic phase space of our dynamical system. The third equations provides that two remaining integrals H_l and H_m are in the involution with respect to the Poisson bracket associated with P.

The system (3.1) has infinitely many solutions and, therefore, we have to narrow the search space and try to get some particular solutions only.

3.1 The linear in momenta Poisson bivectors.

It is natural to suppose that second order polynomials $H_{3,4}$ in momenta M are the integrals of motion, whereas $H_{1,2}$ are the Casimir functions. Moreover, we assume that the entries of P are the linear functions in momenta M.

Proposition 2 *In the hypotheses mentioned above the system of equations (3.1) has the following linear in momenta solutions:*

$$\begin{aligned} P_1&=\begin{pmatrix}0&\mathbf{\Gamma}\\ \mathbf{\Gamma}&\mathbf{M}\end{pmatrix}, && \mathbf{A}_1=\mathbf{A}, \\ P_2&=(M,\mathbf{A}\gamma)\begin{pmatrix}0&0\\0&\mathbf{\Gamma}\end{pmatrix}, && \mathbf{A}_2=f(\gamma)\,\gamma\otimes\gamma\,, \\ P_3&=\frac{1}{\sqrt{\mathrm{g}(\gamma)}}\begin{pmatrix}0&\mathbf{\Gamma}\\ \mathbf{\Gamma}&\mathbf{M}\end{pmatrix}-\sqrt{\mathrm{g}(\gamma)}\,(M,\mathbf{A}\gamma)\begin{pmatrix}0&0\\0&\mathbf{\Gamma}\end{pmatrix}, && \mathbf{A}_3=\mathbf{A}+\mathrm{g}(\gamma)\,\mathbf{A}\,\gamma\otimes\gamma\,\mathbf{A}\,. \end{aligned} \tag{3.2}$$

Here $f(\gamma)$ is an arbitrary function, $\mathrm{g}(\gamma)$ is given by (2.9) and

$$\mathbf{\Gamma}=\begin{pmatrix}0&\gamma_3&-\gamma_2\\-\gamma_3&0&\gamma_1\\ \gamma_2&-\gamma_1&0\end{pmatrix},\qquad \mathbf{M}=\begin{pmatrix}0&M_3&-M_2\\-M_3&0&M_1\\M_2&-M_1&0\end{pmatrix},$$

In generic case $H_{1,2}\neq 0$ at $k=1,3$ there is only one linear in momenta M solution of (3.1), whereas at $k=2$ we show a particular solution.

At $k=1$ and $k=3$ the Poisson brackets between variables $x=\gamma, M$ look like

$$\{M_i,M_j\}_1=\varepsilon_{ijk}M_k,\qquad \{M_i,\gamma_j\}_1=\varepsilon_{ijk}\gamma_k\qquad \{\gamma_i,\gamma_j\}_1=0, \tag{3.3}$$

and

$$\{M_i,M_j\}_3=\varepsilon_{ijk}\left(\frac{M_k}{\sqrt{\mathrm{g}(\gamma)}}-\sqrt{\mathrm{g}(\gamma)}(M,\mathbf{A}\gamma)\gamma_k\right),\qquad \{M_i,\gamma_j\}_3=\frac{\varepsilon_{ijk}\gamma_k}{\sqrt{\mathrm{g}(\gamma)}}\qquad \{\gamma_i,\gamma_j\}_3=0, \tag{3.4}$$

Here ε_{ijk} is a totally skew-symmetric tensor.

The first bracket $\{.,.\}_1$ is the well studied Lie-Poisson bracket on the Lie algebra $e^*(3)$. Indeed, the Euler top can be expressed as a Hamiltonian system on coadjoint orbits of Lie algebra $e(3)$ of Lie group $E(3)$. The nonholonomic Chaplygin ball is the so-called conformally Hamiltonian system on special nontrivial deformations of the same orbits.

Remark 3 Of course, we can get similar Poisson bivectors for other solutions of (2.5) as well. For instance, if $w = \mathbf{A}_4 M$, where

$$\mathbf{A}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a + b\gamma_3 \end{pmatrix},$$

the integrals H_3 and H_4 are in the involution with respect to the Poisson brackets associated with the Poisson bivector

$$P_4 = \frac{\eta}{\sqrt{ax_3 + b}} \left[\begin{pmatrix} 0 & \mathbf{\Gamma} \\ \mathbf{\Gamma} & \mathbf{M} \end{pmatrix} - \frac{aM_3}{2(ax_3 + b)} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{\Gamma} \end{pmatrix} \right]. \tag{3.5}$$

It is easy to prove that the equations (2.1) are non Hamiltonian equations with respect to this Poisson bracket.

According to the second equation in (3.1), the symplectic leaves of the Poisson bivectors P_1 and P_3 are topologically equivalent to each other. Below we study the difference between their symplectic structures.

3.2 Properties of the linear Poisson bivectors.

Let us remind some necessary facts from the Poisson geometry. The Poisson manifold $\mathcal{M}$ is a smooth (or complex manifold) endowed with the Poisson bivector P fulfilling the Jacobi condition

$$[\![P, P]\!] = 0$$

with respect to the Schouten bracket on the algebra of the multivector fields on $\mathcal{M}$. Other Poisson bivector P' is compatible with P if any of its linear combination $P + \lambda P'$ is the Poisson bivector, i.e. if

$$[\![P, P']\!] = 0\,.$$

The compatible bivectors P' are the 2-cocycles in the Poisson-Lichnerowicz cohomology defined by Poisson bivector P on the Poisson manifold $\mathcal{M}$.

The Poisson-Lichnerowicz cohomology of the Poisson manifold was defined in [27], and it provides a good framework to express the deformation and the quantization obstructions, see [39]. For example, the Lie derivative of P along any vector field X

$$P^{(X)} = \mathcal{L}_X\big(P\big) \qquad \Rightarrow \qquad [\![P, P^{(X)}]\!] = 0 \tag{3.6}$$

is 2-coboundary, i.e. it is a 2-cocycle associated with the Liouville vector field X. If X is such vector field that the Jacobi condition

$$[\![P^{(X)}, P^{(X)}]\!] = 0$$

is satisfied, then $P^{(X)}$ (3.6) is called the *trivial* deformation of the Poisson bivector P.

Remark 4 In the theory of Frobenius manifolds we have another definition of the trivial deformation. Namely, deformation is called trivial if there exists a formal diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ admitting the Taylor expansion, which pulls back P' to P.

The first bivector P_1 (3.2) is the well studied Lie-Poisson bivector on the Lie algebra $e^*(3)$ of Lie group $E(3)$ of Euclidean motions of $\mathbb{R}^3$. It is easy to prove that

$$[\![P_1, P_2]\!] \neq 0 \qquad \text{and} \qquad [\![P_1, P_3]\!] = 0\,.$$

It means that P_3 is a 2-cocycle in the Poisson-Lichnerowicz cohomology of the Lie algebra $e^*(3)$ and, moreover, bivectors P_1 and P_3 have the same Casimir elements, according to their construction (3.1).

Proposition 3 *In the generic case the Poisson bivector*

$$P_3 = \frac{1}{\sqrt{\mathrm{g}(\gamma)}} \begin{pmatrix} 0 & \mathbf{\Gamma} \\ \mathbf{\Gamma} & \mathbf{M} \end{pmatrix} - \sqrt{\mathrm{g}(\gamma)}\,(M, \mathbf{A}\gamma) \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{\Gamma} \end{pmatrix} \tag{3.7}$$

is a nontrivial deformation of the standard Lie-Poisson bivector P_1, i.e. $P_3 \neq \mathcal{L}_X(P_1)$ for any X. In fact

$$P_3 = \mathcal{L}_Y\big(P_1\big) + \frac{\sqrt{\mathrm{g}(\gamma)}\,(\gamma, M)}{2}\,\mathcal{L}_Z\big(P_1\big) \tag{3.8}$$

where entries of the vector fields $Y = \sum Y^j \partial_j$ and $Z = \sum Z^j \partial_j$ are given by

$$Y^i = Z^i = 0, \qquad Y^{i+3} = -\frac{M_j}{\sqrt{\mathrm{g}(\gamma)}}, \quad Z^{i+3} = \Big(\big(\mathrm{tr}\mathbf{A}\cdot\mathbf{Id} - \mathbf{A}\big)\,\gamma\Big)_i, \qquad i = 1, 2, 3\,.$$

It is easy to see, that if $H_2 = (\gamma, M) = 0$ then $P_3 = \mathcal{L}_Y\big(P_1\big)$ is a *trivial* deformation with all the pleasant mathematical and physical consequences, see [27, 39] and [7, 10] respectively.

Remark 5 In the finite-dimensional case local Poisson geometry begins with the splitting theorem, which says that in the neighborhood of any point in the Poisson manifold $\mathcal{M}$, there are coordinates $(q_1, \ldots, q_k,\, p_1, \ldots, p_k,\, C_1, \ldots, C_\ell)$ such as

$$P = \sum_{i=1}^{k} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j=1}^{\ell} \varphi_{ij}(C) \frac{\partial}{\partial C_i} \wedge \frac{\partial}{\partial C_j} \qquad \text{and} \qquad \varphi_{ij}(0) = 0\ .$$

So, if the compatible bivectors P and P' have a common set of Casimirs $C_1, \ldots, C_\ell$, we can identify the Darboux coordinates (q, p) of P with the Darboux coordinates (q', p') of P' and obtain the local map $\phi : \mathcal{M} \to \mathcal{M}$, which pulls back P' to P.

Below we prove that in our case this local map can be extended to the global one.

Remark 6 If we come back to the general theory, the second Poisson-Lichnerowicz cohomology group $\mathcal{H}^2_P$ on $\mathcal{M}$ is precisely the set of bivectors P' solving $[\![P, P']\!] = 0$ modulo the solutions of the form $P^{(X)} = \mathcal{L}_X(P)$.

We can interpret $\mathcal{H}^2_P$ as the space of infinitesimal deformations of the Poisson structure modulo trivial deformations. We should keep in mind that cohomology reflects the topology of the leaf space and the variation in the symplectic structure as one passes from one leaf to another [39].

Summing up, we suppose that the linear in momenta Poisson bivector P_3 (3.7-3.8) is an element of the second Poisson-Lichnerowicz cohomology group $\mathcal{H}^2_P$ of $e^*(3)$. The first time this bivector has been obtained was in [5] by investigating the nonholonomic Chaplygin ball.

3.3 The r-matrices

For any ω equations (2.1) can be rewritten in the Lax form

$$\frac{d\mathbf{L}}{dt} = [\mathbf{L}, \mathbf{\Omega}], \qquad \mathbf{L} = \mathbf{M} + \frac{\mathbf{\Gamma}}{\lambda}, \qquad\qquad \lambda \in \mathbb{R}\,, \tag{3.9}$$

if we identify $(\mathbb{R}^3,\times)$ and $(so(3),[.,.])$ by using a well known isomorphism

$$z=(z_1,z_2,z_3)\to\mathbf{Z}=\begin{pmatrix}0 & z_3 & -z_2\\ -z_3 & 0 & z_1\\ z_2 & -z_1 & 0\end{pmatrix}, \tag{3.10}$$

where $\times$ is a cross product in $\mathbb{R}^3$ and $[.,.]$ is a matrix commutator in $so(3)$. As usual [30], the first Lax matrix $\mathbf{L}$ gives rise to three integrals only

$$-\frac{1}{2}\,\mathrm{trace}\,\mathbf{L}^2=H_3+\frac{2H_2}{\lambda}+\frac{H_1}{\lambda^2}\,.$$

Using this Lax matrix the Poisson brackets $\{.\,,.\}_k$ can be rewritten in the r-matrix form

$$\{\overset{1}{\mathbf{L}}(\lambda),\overset{2}{\mathbf{L}}(\mu)\}_k=[r_{12}(\lambda,\mu)\,,\overset{1}{\mathbf{L}}]-[r_{21}(\lambda,\mu)\,,\overset{2}{\mathbf{L}}(\mu)\,]\,. \tag{3.11}$$

Here $\overset{1}{\mathbf{L}}(\lambda)=\mathbf{L}(\lambda)\otimes\mathbf{Id}\,,\ \overset{2}{\mathbf{L}}(\mu)=\mathbf{Id}\otimes\mathbf{L}(\mu)$ and $r_{12}(\lambda,\mu)$ is a classical r-matrix and

$$r_{21}(\lambda,\mu)=\mathbf{P}r_{12}(\mu,\lambda)\,,$$

where $\mathbf{P}$ is a permutation operator [30].

For the first bracket $\{.\,,.\}_1$ (3.3) associated with the bivector P_1 (3.2) this r matrix

$$r_{12}(\lambda,\mu)=\frac{\mu}{\mu-\lambda}\sum_{i=1}^{3}\mathbf{S}_i\otimes\mathbf{S}_i\,, \tag{3.12}$$

is a well known constant matrix [30]. Here $\mathbf{S}_i$ form a basis in the space of 3×3 antisymmetric matrices

$$\mathbf{S}_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix},\qquad \mathbf{S}_2=\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},\qquad \mathbf{S}_3=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

For the nonholonomic bracket $\{.,.\}_3$ (3.4) associated with the bivector P_3 (3.2) the r-matrix will be a more complicated dynamical r-matrix.

Proposition 4 *At $k=3$ the Lax matrix for nonholonomic Chaplygin ball (3.9) satisfies the linear r-matrix algebra (3.11) with the following r-matrix*

$$r_{12}(\lambda,\mu)=\frac{\mu}{\mu-\lambda}\left(\frac{1}{\sqrt{\mathrm{g}(\gamma)}}-\lambda\sqrt{\mathrm{g}(\gamma)}\,(M,\mathbf{A}\gamma)\right)\sum_{i=1}^{3}\mathbf{S}_i\otimes\mathbf{S}_i\,. \tag{3.13}$$

Thus, we obtain a classical r-matrix for the nonholonomic Chaplygin ball. However, it is dynamical r-matrix meant for a concrete model only and, therefore, we could not make a substantial profit out of the majority of opportunities of the classical r-matrix method [30].

3.4 Change of the time variable

The notion of the Poisson brackets allows us to rewrite the equations (2.1) in the form of

$$\frac{dx}{d\tau}=\{H,x\}_k,\qquad H=\frac{1}{2}\,H_4\,. \tag{3.14}$$

after changing the time variable

$$d\tau=\mu_k\,dt\,,\qquad\qquad \mu_{1,2}=1,\quad \mu_3=\sqrt{\mathrm{g}(\gamma)}\,, \tag{3.15}$$

where μ_k are the Jacobi multipliers (2.10-2.11). Namely this transformation has been used in the Chaplygin work [10] in order to get the solutions as the functions of the time variable. The properties of this change of time are discussed in [7, 25].

It is easy to see that for the equations (3.14) the Jacobi multiplier is equal to μ_k^2, because

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\mu_k \cdot \frac{dx_j}{dt} \right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\mu_k^2 \cdot \frac{dx_j}{d\tau} \right) = 0\,.$$

So, this systems of equations looks like the Hamiltonian system with respect to the Poisson brackets $\{.,.\}_k$, but at $k = 3$ its Jacobi multiplier does not equal to the constant, as it does for the standard Hamiltonian systems [17].

On the other hand, if we make an inverse transformation

$$d\tau' = \mu_k^{-1}\, dt \qquad \Rightarrow \qquad \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\mu_k \cdot \frac{dx_j}{dt} \right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(1 \cdot \frac{dx_j}{d\tau'} \right) = 0$$

we get a system with a constant Jacobi multiplier, but the form of the equations will be non-Hamiltonian

$$\frac{dx}{d\tau'} = \mu^2 \{H, x\}_k\,.$$

Remark 7 The systems with the vector field

$$X = PdH$$

and the Jacobi multiplier $\mu(x)$ depending on x are called conformally Hamiltonian in [7].

So, at $k = 3$ there are two transformations of the time variable, which reduce either the form of the equations or the value of the Jacobi multiplier to a habitual form. However, we can not directly identify these transformations with *canonical transformations* of the extended phase space, which change time and the Hamilton function simultaneously [28, 31, 32, 34].

4 Separation of variables at $(\gamma, M) = 0$.

Now we address the problem of separation of variables within the theoretical scheme of bi-hamiltonian geometry [38]. According to [10] we can start with the case $(\gamma, M) = 0$ and then reduce the generic case to this particular one. This reduction is related to the twisted Poisson maps and its detailed discussion goes beyond the scope of this paper.

Let us remind, that the bi-Hamiltonian manifold $\mathcal{M}$ is a smooth (or complex) manifold endowed with two compatible Poisson bivectors P and P'. Dynamical systems on $\mathcal{M}$ with the integrals of motion in involution with respect to the both brackets

$$\{H_i, H_j\} = \{H_i, H_j\}' = 0, \qquad i, j = 1, \ldots, n, \tag{4.1}$$

are called bi-integrable systems [38].

In fact, the family of bi-integrable systems coincides with the family of separable systems because the bi-involutivity of the integrals of motion (4.1) is equivalent to the existence of the control matrix F defined by

$$P' dH_i = P \sum_{j=1}^{n} F_{ij}\, dH_j, \qquad i = 1, \ldots, n. \tag{4.2}$$

The eigenvalues of F are the coordinates of separation, whereas the suitable normalized left eigenvectors of F form the generalized Stäckel matrix S

$$F = S^{-1}\, \mathrm{diag}\,(q_1, \ldots, q_n)\, S$$

which defines the separation relations

$$\Phi_i(q_i,p_i,H_1,\dots,H_n)=\sum_{j=1}^{n} S_{ij}(q_i,p_i)H_j+U_i(q_i,p_i)=0\,,\qquad i=1,\dots,n\,. \tag{4.3}$$

Here the entries of Stäckel matrix S_{ij} and the Stäckel potentials U_i depend only on one pair (q_i,p_i) of the canonical variables of separation and, in generis case, on the integrals of motion [38].

In our case the Stäckel matrix and the Stäckel potentials depend only on variables of separation, and it allows us to calculate the canonical transformation from the initial variables γ, M to the variables of separation explicitly.

4.1 Coordinates of separation.

In order to get variables of separation according to the general usage of bi-hamiltonian geometry firstly we have to calculate the bi-hamiltonian structure for the given system with integrals of motion $H_1,\dots,H_n$ on the Poisson manifold M with the kinematic Poisson bivector P [38].

Proposition 5 *Let us introduce vector fields $X_k=\sum X_k^j\partial_j$, $k=1,3$, with the following entries:*

$$X_k^i=0,\quad X_k^{i+3}=\Bigl[\gamma\times\mathbf{A}_k(\gamma\times M)\Bigr]_i,\quad i=1,2,3\,. \tag{4.4}$$

If $(\gamma,M)=0$ then the Poisson bivectors

$$P_1'=\mathcal{L}_{X_1}\bigl(P_1\bigr)\qquad \text{and}\qquad P_3'=\mathcal{L}_{X_3}\bigl(P_3\bigr)$$

are compatible with the bivectors P_1 and P_3 (3.2) respectively. These bivectors P_k' have common symplectic leaves

$$P_k'dH_1=0,\qquad P_k'dH_2=0,\qquad k=1,3, \tag{4.5}$$

with the initial kinematic bivectors P_k and the integrals of motion $H_3=(M,M)$ and $H_4=(M,\mathbf{A}_kM)$ are in the bi-involution

$$\{H_3,H_4\}_k=\{H_3,H_4\}_k'=0\,, \tag{4.6}$$

with respect to the corresponding Poisson brackets.

Thus, we proved that the Euler top and the nonholonomic Chaplygin ball are bi-integrable systems at $(\gamma,M)=0$.

If $P_{1,3}$ are given by (3.2) and $P_k'=\mathcal{L}_{X_k}\bigl(P_k\bigr)$ (4.4), then control matrices $F^{(1)}$ and $F^{(3)}$ in

$$P_k'dH=P\Bigl(F^{(k)}dH\Bigr),\qquad dH=\begin{pmatrix} dH_3\\ dH_4\end{pmatrix}$$

are equal to

$$F^{(k)}=\begin{pmatrix} 0 & (\mathbf{A}_k^{\vee}\gamma,\gamma)\\ -(\gamma,\gamma) & \Bigl((\mathrm{tr}\mathbf{A}_k\cdot\mathbf{Id}-\mathbf{A}_k)\gamma,\gamma\Bigr)\end{pmatrix},\qquad k=1,3\,. \tag{4.7}$$

Here $\mathbf{A}_k^{\vee}=(\det\mathbf{A}_k)\,\mathbf{A}_k{}^{-1}$ are adjoint or cofactor matrices.

Proposition 6 *The Darboux-Nijenhuis variables associated with the bivectors P_k' and control matrices $F^{(k)}$ are the roots of their characteristic polynomials*

$$\tau_k(\lambda)=\lambda^2-\Bigl((\mathrm{tr}\mathbf{A}_k\cdot\mathbf{Id}-\mathbf{A}_k)\gamma,\gamma\Bigr)+(\gamma,\gamma)(\mathbf{A}_k{}^{\vee}\gamma,\gamma)=0\,. \tag{4.8}$$

These Darboux-Nijenhuis variables are the variables of separation for the bi-lagrangian foliation defined by integrals $H_1,\dots,H_4$.

At $(\gamma, M)=0$ the passage from the Euler top to the nonholonomic Chaplygin ball consists of the replacement of the numerical matrix $\mathbf{A}_1$ (2.6) on dynamical one $\mathbf{A}_3$ (2.7) in the equations of motion (2.1), in the Hamiltonian $H_4=(M,\mathbf{A}_k M)$ and equations (4.4,4.7,4.8) only.

On the face of it we could get more complicated passage for the conjugated momenta, such as the kinematic Poisson bivectors P_k (3.2) are completely different. Nevertheless, in the next section we prove that the separated momenta can be obtained in a common way as well.

Remark 8 The simplicity of this passage is a sequence of the equation (3.8)

$$P_3=\mathcal{L}_Y(P_1),$$

properties of the Lie derivative $\mathcal{L}$ and of the vector fields Y and X_k.

4.2 Elliptic coordinates

If axes of the body frame attached to the body coincide with the principal inertia axes of the body then the tensor of inertia $\mathbf{A}^{-1}$ is diagonal and

$$\mathbf{A}=\begin{pmatrix} a_1 & 0 & 0\\ 0 & a_2 & 0\\ 0 & 0 & a_3\end{pmatrix}.$$

Moreover, if we put $(\gamma,\gamma)=1$, then by dividing $\tau(\lambda)$ on $\det(\mathbf{A}-\lambda\,\mathbf{Id})$ we get the standard definitions of variables of separation.

Proposition 7 *In the above-mentioned hypotheses the variables of separation* u, v *and* u,v *are zeroes of the functions*

$$e_1(\lambda)=\frac{\gamma_1^2}{\lambda-a_1}+\frac{\gamma_2^2}{\lambda-a_2}+\frac{\gamma_3^2}{\lambda-a_3}=\frac{(\lambda-u)(\lambda-v)}{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)}, \tag{4.9}$$

and

$$e_3(\lambda)=\mathrm{g}(\gamma)\left(\frac{\gamma_1^2(\eta-a_1)}{\lambda-a_1}+\frac{\gamma_2^2(\eta-a_2)}{\lambda-a_2}+\frac{\gamma_3^2(\eta-a_3)}{\lambda-a_3}\right)=\frac{(\lambda-\mathrm{u})(\lambda-\mathrm{v})}{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)}, \tag{4.10}$$

respectively.

Equation (4.9) is a standard definition of the elliptic coordinates u, v on the unit sphere, whereas equation (4.10) determines the nonholonomic elliptic coordinates u,v, introduced in [10], see also [7, 8, 16, 25].

We have to point out that our aim is the *calculation* of the well-known variables of separation without any additional assumptions. Thus, we have to calculate the conjugated momenta p_u, p_v and $\mathrm{p_u},\mathrm{p_v}$ in framework of bi-hamiltonian geometry. It is easy to prove that in our case we have identical Stäckel matrices

$$S^{(1)}=\begin{pmatrix} 1 & 1\\ -u & -v\end{pmatrix}\qquad\text{and}\qquad S^{(3)}=\begin{pmatrix} 1 & 1\\ -\mathrm{u} & -\mathrm{v}\end{pmatrix},$$

and identical Stäckel potentials

$$\begin{array}{ll} U_1^{(1)}=u\,H_3-H_4\,, & U_2^{(1)}=v\,H_3-H_4\,,\\ U_1^{(3)}=\mathrm{u}\,H_3-H_4\,, & U_2^{(3)}=\mathrm{v}\,H_3-H_4\,, \end{array}\qquad H_3=(M,M),\quad H_4=(M,\omega). \tag{4.11}$$

According to [38], the notion of the Stäckel potentials allows us to find unknown conjugated momenta using the Poisson brackets only.

For instance, the following recurrence chain of the Poisson brackets

$$\phi_1 = \{u, U_1^{(1)}\}_1, \qquad \phi_2 = \{u, \phi_1\}_1, \ldots, \quad \phi_i = \{u, \phi_{i-1}\}_1 \tag{4.12}$$

breaks down on the third step $\phi_3 = 0$. It means that $U_1^{(1)}(u, p_u)$ is the second order polynomial in momenta p_u and, therefore, we can define this unknown momenta in the following way

$$p_u = \frac{\phi_1}{\phi_2} = \sum_{ijm} \varepsilon_{ijm} \frac{\gamma_j \gamma_m (a_j - a_m)(a_i - u)}{2(u - v)(a_j - u)(a_m - u)} J_i \tag{4.13}$$

up to the canonical transformations $p_u \to p_u + f(u)$. As above, ε_{ijk} is a totally skew-symmetric tensor.

Similar calculation with $U_2^{(1)}(v, p_v)$ yields to the definition of the second momenta p_u. In nonholonomic case we can perform completely identical calculations too. The results obtained so far can be summarized in the following statement.

Proposition 8 *The initial coordinates $x = \gamma, M$ are expressed via elliptic coordinates u, v and p_u, p_v*

$$\begin{aligned} \gamma_i &= \sqrt{\frac{(u - a_i)(v - a_i)}{(a_j - a_i)(a_m - a_i)}}, \qquad i \neq j \neq m\,, \\ M_i &= \frac{2\varepsilon_{ijm}\gamma_j\gamma_m(a_j - a_m)}{u - v}\Big((a_i - u)p_u - (a_i - v)p_v\Big)\,, \end{aligned} \tag{4.14}$$

In terms of the nonholonomic elliptic coordinates u, v *and* $\mathrm{p}_u, \mathrm{p}_v$ *the same variables look like*

$$\gamma_i = \sqrt{\frac{(a_j - \eta)(a_m - \eta)}{(\mathrm{u} - \eta)(\mathrm{v} - \eta)}} \cdot \sqrt{\frac{(\mathrm{u} - a_i)(\mathrm{v} - a_i)}{(a_j - a_i)(a_m - a_i)}}\,, \qquad i \neq j \neq m\,, \tag{4.15}$$

$$M_i = \frac{2\varepsilon_{ijm}\gamma_j\gamma_m(a_j - a_m)}{\mathrm{u} - \mathrm{v}} \sqrt{\frac{(\eta - \mathrm{u})(\eta - \mathrm{v})}{(\eta - a_1)(\eta - a_2)(\eta - a_3)}} \Big((a_i - \mathrm{u})(\eta - \mathrm{u})\mathrm{p}_u - (a_i - \mathrm{v})(\eta - \mathrm{v})\mathrm{p}_v\Big)\,.$$

In nonholonomic case these variables have been introduced by Chapligin [10]. Our aim was to calculate the variables of separation in framework of bi-hamiltonian geometry without any additional speculations and assumptions.

4.3 The reduction of the Poisson brackets

The variables of separation u, v, p_u, p_v and $\mathrm{u}, \mathrm{v}, \mathrm{p}_u, \mathrm{p}_v$ are the Darboux variables with respect to the brackets $\{.,.\}_1$ and $\{.,.\}_3$, respectively. Thus, according to Remark 5, we can identify these variables and get the diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$, which pulls back the nonholonomic bracket $\{.,.\}_3$ to the Lie-Poisson bracket $\{.,.\}_1$ on the Lie algebra $e^*(3)$.

Proposition 9 *At $(\gamma, M) = 0$ the Poisson bracket $\{.,.\}_3$ (3.4) between the variables γ, M are reduced to the Lie-Poisson bracket $\{.,.\}_1$ (3.3) between variables*

$$\begin{aligned} \hat{\gamma}_j &= \sqrt{\mathrm{g}(\gamma)\Big(\eta - (\gamma,\gamma)\,a_j\Big)}\,\gamma_j\,, \qquad j = 1, 2, 3, \\ \hat{M}_j &= \sqrt{\frac{1}{\prod_{i\neq j}\Big(\eta - (\gamma,\gamma)\,a_i\Big)}}\left(\frac{M_j}{\sqrt{\mathrm{g}(\gamma)}} + \sqrt{\mathrm{g}(\gamma)}\,(M, \mathbf{A}\gamma)\,\gamma_j\right)\,. \end{aligned} \tag{4.16}$$

This mapping identifies the Chaplygin variables u, v *with the usual elliptic coordinates u, v on the sphere.*

So, at $(\gamma, M) = 0$ we can map the nonholonomic Poisson bracket to the standard Poisson bracket on cotangent bundle to the sphere. It means that any integrable system on the sphere has an integrable counterpart with respect to the nonholonomic bracket and vise versa. The list of the known integrable systems on the sphere can be found in [4, 6, 41].

On the other hand, we can say that mapping (4.16) relates the usual metric on the sphere with some nonholonomic metric on the same sphere. It can be interesting to study this metric and the corresponding variational problem according to [20].

In the next section we prove that we can not identify the Euler top and the nonholonomic Chaplygin ball using this mapping because they have different separated relations even at $(\gamma, M) = 0$.

4.4 Separation relations

Substituting variables γ, M (4.14) and (4.15) into the Stäckel potentials (4.11), we obtain a pair of separation relations (4.3) for the Euler top and the Chaplygin ball. These separated equations define some algebraic curves and we can say that the equations of motion (2.1) are linearized on the symmetrized product of these curves.

Proposition 10 *In holonomic case at $\omega = \mathbf{A}_1 M$ the variables of separation lie on two copies of the hyperelliptic curve of genus one*

$$\mathcal{C}^{(1)}: \qquad 4(a_1-\mathrm{x})(a_2-\mathrm{x})(a_3-\mathrm{x})\,\mathrm{y}^2 - (\mathrm{x}H_3 - H_4) = 0, \qquad \mathrm{x} = u, v, \quad \mathrm{y} = p_u, p_v\,. \tag{4.17}$$

In nonholonomic case at $\omega = \mathbf{A}_3 M$ the variables of separation lie on two copies of the following hyperelliptic curve of genus two

$$\mathcal{C}^{(3)}: \qquad 4(\eta-\mathrm{x})(a_1-\mathrm{x})(a_2-\mathrm{x})(a_3-\mathrm{x})\,\mathrm{y}^2-(\mathrm{x}H_3-H_4) = 0, \qquad \mathrm{x} = \mathrm{u}, \mathrm{v}, \quad \mathrm{y} = \mathrm{p}_u, \mathrm{p}_v\,. \tag{4.18}$$

In fact, we obtain the variables of separation and the separated equations geometrically, i.e. without the equations of motion, the time variable and the underlying Hamiltonian or conformally Hamiltonian structures. We only suppose that the foliation defined by the integrals H_3 and H_4 on symplectic leaves of the corresponding Poisson brackets is bi-lagrangian foliation.

However, in order to get the solutions of the separated equations $\mathrm{x}(t)$ and $\mathrm{y}(t)$ we have to explicitly introduce a time variable t. Solving separated equations with respect to H_3 and H_4 one gets the Hamilton functions for the Euler top

$$H_4 = \frac{4v(a_1-u)(a_2-u)(a_3-u)}{u-v}\,p_u^2 + \frac{4u(a_1-v)(a_2-v)(a_3-v)}{v-u}\,p_v^2, \tag{4.19}$$

and for the Chaplygin ball

$$H_4 = \frac{4\mathrm{v}(\eta-\mathrm{u})(a_1-\mathrm{u})(a_2-\mathrm{u})(a_2-\mathrm{u})}{\mathrm{u}-\mathrm{v}}\,\mathrm{p}_u^2 + \frac{4\mathrm{u}(\eta-\mathrm{v})(a_1-\mathrm{v})(a_2-\mathrm{v})(a_3-\mathrm{v})}{\mathrm{v}-\mathrm{u}}\,\mathrm{p}_v^2\,. \tag{4.20}$$

By definition the variables of separation are canonical variables and, therefore, we have

$$\{H_4, \mathrm{x}_1\}_k = \frac{4\mathrm{x}_2\sqrt{\mathrm{P}_k(\mathrm{x}_1)}}{\mathrm{x}_1-\mathrm{x}_2} \qquad \text{and} \qquad \{H_4, \mathrm{x}_2\}_k = \frac{4\mathrm{x}_1\sqrt{\mathrm{P}_k(\mathrm{x}_2)}}{\mathrm{x}_2-\mathrm{x}_1},. \tag{4.21}$$

Here variables $\mathrm{x}_{1,2}$ are coordinates of separation u, v or u, v at $k = 1, 3$, respectively. Polynomials $\mathrm{P}_k(\mathrm{x})$ are the polynomials of degree 4 and 5 in x variable

$$\begin{aligned} \mathrm{P}_1(\mathrm{x}) &= (a_1-\mathrm{x})(a_2-\mathrm{x})(a_3-\mathrm{x})(\mathrm{x}H_3-H_4), \\ \mathrm{P}_3(\mathrm{x}) &= (\eta-\mathrm{x})(a_1-\mathrm{x})(a_2-\mathrm{x})(a_3-\mathrm{x})(\mathrm{x}H_3-H_4). \end{aligned}$$

On the other hand, according to (3.14), the brackets (4.21) are equal to

$$\{H_4, \mathrm{x}_1\}_k = \frac{2}{\mu_k}\frac{d\mathrm{x}_1}{dt} \qquad \text{and} \qquad \{H_4, \mathrm{x}_2\}_k = \frac{2}{\mu_k}\frac{d\mathrm{x}_2}{dt}$$

where

$$\mu_1 = 1, \qquad \text{and} \qquad \mu_3 = \sqrt{\mathrm{g}(\gamma)} = \sqrt{\frac{(\eta - \mathrm{u})(\eta - \mathrm{v})}{(\eta - a_1)(\eta - a_2)(\eta - a_3)}}\,.$$

So, in order to get the solutions $\mathrm{x}_{1,2}(t)$ of the equations of motion we have to consider the Jacobi inversion problem for the equations

$$\begin{aligned} \beta_1 - 2\int \mu_k\, dt &= \int \frac{d\mathrm{x}_1}{\sqrt{\mathrm{P}_k(\mathrm{x}_1)}} + \int \frac{d\mathrm{x}_2}{\sqrt{\mathrm{P}_k(\mathrm{x}_2)}}, \\ \beta_2 &= \int \frac{\mathrm{x}_1 d\mathrm{x}_1}{\sqrt{\mathrm{P}_k(\mathrm{x}_1)}} + \int \frac{\mathrm{x}_2 d\mathrm{x}_2}{\sqrt{\mathrm{P}_k(\mathrm{x}_2)}}, \end{aligned} \tag{4.22}$$

where $\beta_{1,2}$ are the constants of integration. The change of time variable (3.15) reduces these equations to the standard Abel-Jacobi equations [10, 25]. The solution of this Jacobi inversion problem by Wurzelfunktionen is discussed in [8].

Remark 9 It is easy to prove that the right hand side in β_2 (4.22) coincides with an additional Euler-Jacobi quadrature emerged in the Jacobi last multiplier theory. Of course, for the Chaplygin ball this quadrature can be obtained without any change of time variable.

5 The 2×2 Lax matrices.

In variables of separation we deal with the uniform Stäckel systems (4.19-4.20) and, therefore, we can get 2×2 Lax matrices associated with the Abel-Jacobi equations (4.22) in a standard way, see [13, 26, 32, 33, 36] as well as the relevant references therein.

According to [32, 33, 36], let us introduce the following functions on the canonical variables of separation and spectral parameter λ

$$\begin{aligned} h_1(\lambda) &= -\frac{1}{8}\Big\{ H_3, e_1(\lambda) \Big\}_1 \\ h_3(\lambda) &= -\frac{1}{8(\eta - \lambda)}\Big\{ H_3, e_3(\lambda) \Big\}_3, \end{aligned}$$

and

$$\begin{aligned} f_1(\lambda) &= \frac{1}{4}\left(\Big\{ H_3, h_1(\lambda) \Big\}_1 - e_1(\lambda) H_3 \right), \\ f_3(\lambda) &= \frac{1}{4(\eta - \lambda)}\left(\Big\{ H_3, h_3(\lambda) \Big\}_3 - \left(1 + \frac{\mathrm{tr}\mathbf{A} - 2(\mathrm{u} + \mathrm{v})}{\eta - \lambda}\right) e_3(\lambda) H_3 - \frac{e_3(\lambda) H_4}{\eta - \lambda} \right). \end{aligned}$$

Here $e_{1,3}(\lambda)$ are given by (4.9-4.10), H_3 is a leading term in the polynomial $(\mathrm{x}H_3 - H_4)$ from the separation relations (4.17-4.18), whereas the Hamilton function is equal to $H_4/2$ (4.19-4.20).

In order to obtain these functions in initial variables we have to use the definitions of $e_{1,3}(\lambda)$, the integrals $H_{3,4}$ and the brackets $\{.,.\}_{1,3}$ in (γ, M) variables and, in addition, it is necessary to substitute

$$\mathrm{u} + \mathrm{v} = Res|_{\lambda=\infty}\, e_3(\lambda)(\lambda - \mathrm{tr}\mathbf{A})$$

in the last formulae.

Proposition 11 *At $(\gamma, M) = 0$ the Lax matrices*

$$\mathscr{L}_k = \begin{pmatrix} h_k & e_k \\ f_k & -h_k \end{pmatrix}, \qquad \mathscr{A}_k = \frac{1}{\mu_k\, e_k}\begin{pmatrix} -e_k' & 0 \\ 2h_k' & e_k' \end{pmatrix}, \qquad k = 1, 3, \tag{5.1}$$

satisfy to the Lax equation

$$\frac{d}{dt}\mathscr{L}_k(\lambda)=\frac{\mu_k}{2}\Big\{H_4,\mathscr{L}_k\Big\}_k=\Big[\mathscr{L}_k(\lambda),\mathscr{A}_k(\lambda)\Big]\,.$$

Here $z'=\{z,H_4\}_{1,3}$ is a time derivative up to Jacobi multiplier $\mu_{1,3}$ (3.15).

As usual, substituting $\lambda=\mathrm{x}$ into the determinants of the Lax matrces

$$\det\mathscr{L}_k(\lambda)=-h_k^2(\lambda)-e_k(\lambda)f_k(\lambda)\,,$$

which are equal to

$$\begin{aligned}\det\mathscr{L}_1(\lambda)&=-\frac{\lambda H_3-H_4}{4(a_1-\lambda)(a_2-\lambda)(a_1-\lambda)}\,,\\ \det\mathscr{L}_3(\lambda)&=-\frac{\lambda H_3-H_4}{4(\eta-\lambda)(a_1-\lambda)(a_2-\lambda)(a_1-\lambda)}\,,\end{aligned}$$

one gets separated relations (4.17) and (4.18) because $e_k(\mathrm{x})=0$ and $h_k(\mathrm{x})=\mathrm{y}$.

Remark 10 In [10] Chaplygin reduces the generic case at $(\gamma,M)\neq 0$ to the particular case at $(\gamma,M)=0$. By applying the inverse map to the Lax matrices (3.9) one gets the Lax matrices for the generic case of the nonholonomic Chaplygin ball. These matrices and the corresponding r-matrix algebra will be studied in a forthcoming separate publication.

Matrices $\mathscr{L}_k(\lambda)$ are associated with the uniform Stäckel systems and, therefore, they satisfy to the linear r-matrix algebra (3.11) with the well-studied dynamical r-matrices [13, 26, 32, 33, 36]. In contrast with the previous 3×3 Lax matrices (3.9) it allows us to obtain some well studied generalizations of these 2×2 matrices in the next paragraphs.

5.1 Chaplygin ball and separable potentials.

We are going to demonstrate that the Chaplygin ball at $(\gamma,M)=0$ is still integrable in the force fields associated with a huge family of the so-called separable potentials [4, 13, 41].

It is well known of how to get various generalizations of the separable systems using the deformations of their separated equations [17]. For instance, let us consider following deformations of the separation relations (4.17) and (4.18)

$$4(a_1-\mathrm{x})(a_2-\mathrm{x})(a_3-\mathrm{x})\,\mathrm{y}^2-(\mathrm{x}H_3-H_4)+V(\mathrm{x})=0$$

or

$$4(\eta-\mathrm{x})(a_1-\mathrm{x})(a_2-\mathrm{x})(a_3-\mathrm{x})\,\mathrm{y}^2-(\mathrm{x}H_3-H_4)+V(\mathrm{x})=0\,,$$

where potential V is some function on x. Usually, potential V is a linear combination of the trivial separable potentials $V_m=\alpha_m\mathrm{x}^m$, where m is a positive or negative integer [4, 41].

In order to get the same deformations in the initial variables γ,M we can use the generating function [41]

$$\Phi(\lambda)=\frac{\phi(\lambda)}{e_k(\lambda)}\,,$$

or the determinant of the corresponding deformations the Lax matrix $\mathscr{L}_k(\lambda)$ (5.1)

$$f_k\to f_k+\left[\frac{\phi(\lambda)}{e_k(\lambda)}\right]_{MN}\,.$$

Here $\phi(\lambda)$ is a parametric function on spectral parameter and $[\xi(\lambda)]_{MN}$ is a linear combination of the Laurent projections of $\xi(\lambda)$ by λ [13, 32, 33].

For example, if $V=\alpha\mathrm{x}^2$ one gets the integrable system

$$H_3=(M,M)+\alpha(\gamma,\mathbf{A}\gamma)\,,\qquad H_4=(M,\mathbf{A}_1M)-\frac{\alpha}{a_1a_2a_3}(\gamma,\mathbf{A}^{-1}\gamma)\,, \tag{5.2}$$

which can be identified with the Neumann system on the sphere, and its nonholonomic counterpart

$$\begin{aligned} H_3 &= (M,M)+\alpha\mathrm{g}(\gamma)\Big(\eta(\gamma,\mathbf{A}\gamma)-(\mathbf{A}\gamma,\mathbf{A}\gamma)\Big)\,, \\ H_4 &= (M,\mathbf{A}_3M)-\alpha\,\mathrm{g}(\gamma)\,\left(\frac{\eta(\gamma,\mathbf{A}^{-1}\gamma)}{a_1a_2a_3}-a_1a_2a_3\right)\,. \end{aligned} \tag{5.3}$$

Remark 11 Another nonholonomic analog of the Neumann system with the polynomial in γ potential has been proposed by Kozlov [24] at

$$V=-\alpha\mathrm{x}^2+(a_1+a_2+a_3)\mathrm{x}+\frac{\alpha(a_1-\mathrm{x})(a_2-\mathrm{x})(a_3-\mathrm{x})}{\eta-\mathrm{x}}\,, \tag{5.4}$$

see integrals of motion in (6.10). At $(\gamma,M)=0$ this system is separable in the Chaplygin coordinates [16].

If $V=\beta\mathrm{x}^3$ we obtain a forth order polynomial potential on the sphere

$$H_4=(M,\mathbf{A}_1M)+\beta\frac{(\gamma,\mathbf{A}^{-1}\gamma)}{a_1a_2a_3}\Big((\gamma,\mathbf{A}\gamma)-\mathrm{tr}\mathbf{A}\Big)\,, \tag{5.5}$$

and its nonholonomic analog

$$H_4=(M,\mathbf{A}_3M)+\beta\,\mathrm{g}(\gamma)\left(\frac{\eta(\gamma,\mathbf{A}^{-1}\gamma)}{a_1a_2a_3}-a_1a_2a_3\right)\Big[\mathrm{g}(\gamma)\Big(\eta(\gamma,\mathbf{A}\gamma)-(\mathbf{A}\gamma,\mathbf{A}\gamma)\Big)-\mathrm{tr}\mathbf{A}\Big]\,. \tag{5.6}$$

Similarly we can get other well-known integrable systems on the sphere [4, 41], such as Braden and Rosochatius systems, and their nonholonomic counterparts separable in the nonholonomic elliptic coordinates.

The second type of perturbations of the Chaplygin system is related with other possible deformations of the separated equations (4.18). For instance, we can insert second arbitrary parameter ζ into the separation relations

$$4(\eta-\mathrm{x})(\zeta-\mathrm{x})(a_1-\mathrm{x})(a_2-\mathrm{x})(a_3-\mathrm{x})\,\mathrm{y}^2-(\mathrm{x}H_3-H_4)=0,\qquad \mathrm{x}=\mathrm{u},\mathrm{v}. \tag{5.7}$$

As for the initial Chaplygin ball, we have to change the definition of the physical variables γ in terms of the variables of separation (4.10). By analogy it can look like

$$e(\lambda)=\widetilde{\mathrm{g}}(\gamma)\sum_{i=1}^{3}\frac{\gamma_j^2(\eta-a_1)(\zeta-a_1)}{\lambda-a_i}=\frac{(\lambda-\mathrm{u})(\lambda-\mathrm{v})}{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)}\,, \tag{5.8}$$

where function $\widetilde{\mathrm{g}}(\gamma)$ is determined from an obvious condition

$$\mathrm{Res}\,e(\lambda)|_{\lambda=\infty}=-1\,,$$

and an assumption $(\gamma,\gamma)=1$. For the definition of the momenta M and of the Poisson bracket between γ and M we can use a transformation similar to (4.16). In this case physical meaning of the equations (2.1) is unknown.

5.2 Analogs of the Chaplygin system on Riemannian spaces of constant curvature.

At $(\gamma, M) = 0$ the Euler top is a dynamical system describing free motion on the two-dimensional sphere. It is well-known how to describe similar N-dimensional free motion on any Riemannian space of constant curvature using a well-studied theory of the orthogonal coordinate systems and the corresponding Killing tensors [3]. For example, according to [26] we can get 2×2 Lax matrices (5.1) for such free motion and then add separable potentials [32, 33, 35], which form depends on the chosen orthogonal coordinate system.

We can try to determine the N-dimensional Chaplygin systems on the Riemannian spaces of constant curvature using known generalizations of the Chaplygin ball associated to the semi-simple Lie groups [15, 18, 21].

Let us start with elliptic coordinates (q,p) on the sphere $\mathbb{S}_N$ in $N+1$-dimensional Euclidean space $\mathbb{R}_N$. In this case the transformation of the free motion to its Chaplygin analog consists of three steps:

- we have to change the separation relations from

$$\prod_{i=1}^{N+1}(a_i-q_j)\,p_j^2-\sum_{i=1}^{N} q_j^m H_m=0\,,\qquad j=1,\dots,N, \tag{5.9}$$

  to

$$(\eta-q_j)\prod_{i=1}^{N+1}(a_i-q_j)\,p_j^2-\sum_{i=1}^{N} q_j^m H_m=0\,, \tag{5.10}$$

  compare with (5.9-5.9);

- we have to change the definition of q_j in term of cartesian coordinates in $\mathbb{R}_N$ from

$$e_1(\lambda)=\sum_{k=1}^{N+1}\frac{\hat\gamma_k^2}{\lambda-a_k}=\frac{\prod_{j=1}^{N}(\lambda-q_j)}{\prod_{i=1}^{N+1}(\lambda-a_i)}\,, \tag{5.11}$$

  that implies $\sum_{i=1}^{N+1}\hat\gamma_i^2=1$, to

$$e_3(\lambda)=\mathrm{g}(\gamma)\sum_{k=1}^{N+1}\frac{\gamma_k^2\,(\eta-a_k)}{\lambda-a_k}=\frac{\prod_{j=1}^{N}(\lambda-q_j)}{\prod_{i=1}^{N+1}(\lambda-a_i)}\,, \tag{5.12}$$

  where $\mathrm{g}(\gamma)$ is defined by residue of $e_3(\lambda)$ at infinity;

- we have to change the time variable in order to attach some nonholonomic physical meaning to the proposed pure mathematical integrals $H_1,\dots,H_N$ and the corresponding equations of motion, see [15, 18, 21].

Remark 12 Similar to (4.16) the definitions (5.11) and (5.12) determine the canonical point transformation between cartesian coordinates $\hat\gamma$ and γ

$$\hat\gamma_j=\sqrt{\mathrm{g}(\gamma)\Bigl(\eta-(\gamma,\gamma)\,a_j\Bigr)}\,\gamma_j\,,\qquad j=1,\dots,N+1,$$

which allows us to determine the "nonholonomic" momenta p_{γ_j} in terms of canonical momenta $\hat p_{\gamma_j}$ with brackets $\{\hat\gamma_i,\hat p_{\gamma_j}\}=\delta_{ij}$ and to get a "nonholonomic" Poisson bracket in $\mathbb{R}_N$, similar to the bracket $\{.,.\}_3$ (3.4).

Remark 13 Instead of (5.11) we can start with any other coordinate system and the corresponding separated equations on the Riemannian spaces of constant curvature [3]. For instance, we can take elliptic coordinates

$$e(\lambda)=1+\sum_{k=1}^{N}\frac{\gamma_k^2}{\lambda-e_k}=\frac{\prod_{j=1}^{N}(\lambda-q_j)}{\prod_{i=1}^{N}(\lambda-e_i)}\,. \tag{5.13}$$

or parabolic coordinates

$$e(\lambda)=\lambda-2\gamma_N-\sum_{k=1}^{N-1}\frac{\gamma_k^2}{\lambda-e_k}=\frac{\prod_{j=1}^{N}(\lambda-q_j)}{\prod_{i=1}^{N-1}(\lambda-e_i)} \tag{5.14}$$

in N-dimensional Euclidean space.

So, there are not mathematical problems in the construction of such "nonholonomic" dynamical systems associated with any orthogonal coordinate system and their potential generalizations. The main problem is the introduction of a suitable time variable.

6 Generalizations of the nonholonomic Chaplygin system at $(\gamma, M)\neq 0$.

In the previous section we consider various deformations of our dynamical systems at $(\gamma, M)=0$ using the variables of separation. We proceed by discussing some possible deformations of the Chaplygin ball in generic case.

Let us consider the deformation of the equations (2.1)

$$\dot M=M\times\omega+\gamma\times b\,,\qquad \dot\gamma=\gamma\times\omega\,, \tag{6.1}$$

where $\omega=\mathbf{A}_{1,3}M$ and vector b is an arbitrary function on $x=\gamma, M$.

It is clear that the functions $H_{1,2}$ (2.2) remain the integrals of equations (6.1). It allows us to look for two additional integrals H_3 and H_4 in the involution with respect to the same Poisson brackets $\{.,.\}_{1,3}$

$$\{H_3,H_4\}_3=0\,.$$

For the first Poisson bracket all integrable deformations are well known [2, 6, 30]. At $\eta\to\infty$ we have $\mathbf{A}_3\to\mathbf{A}_1$ (2.7), and the corresponding equations of motion (6.1) coincide to each other. It means that we can try to get nonholonomic counterparts of the Lagrange and Kowalevski tops, or of the Kirchhoff, Clebsh and Steklov-Lyapunov systems.

Remark 14 If the Hamilton function reads as

$$2H=H_4=(M,\omega)+2V(\gamma)\,,\qquad \omega=\mathbf{A}_{1,3}M,$$

then in holonomic case at $k=1$ equations (6.1) are identified with the Euler-Poisson equations [2, 6]

$$\dot M=M\times\frac{\partial H}{\partial M}+\gamma\times\frac{\partial H}{\partial\gamma}\,,\qquad \dot\gamma=\gamma\times\frac{\partial H}{\partial M}\,,\qquad H=\frac12\,H_4, \tag{6.2}$$

whereas in nonholonomic case at $k=3$ first equation has to be replaced to

$$\dot M=M\times\frac{\partial H}{\partial M}+\gamma\times\frac{\partial V}{\partial\gamma}\,, \tag{6.3}$$

according to the procedure of elimination of the undetermined Lagrange multipliers [7, 10].

Remark 15 There are other interesting deformations of equations of motion, which require a changing these Poisson brackets, see [7, 29].

6.1 Linear integrals of motion.

Let us briefly consider the Lagrange top [2, 6, 30] and its nonholonomic twin [9, 19].

Proposition 12 *If* $\omega = \mathbf{A}_{1,3}M$,

$$a_1 = a_2, \qquad \textit{and} \qquad b = (0,0,b_3)\,,$$

then the integrals of the equations (6.2)-(6.3)

$$H_3 = (M,M) + 2a_1^{-1}(b,\gamma) \qquad \textit{and} \qquad H_4 = (M,\omega) + 2(b,\gamma), \tag{6.4}$$

are in the involution with respect to the Poisson brackets $\{.,.\}_1$ *and* $\{.,.\}_3$*, respectively.*

In holonomic case the linear in momenta integral

$$K = (b, M) = M_3, \qquad \{\, H_m, K\,\}_1 = 0,$$

can be obtained from the quadratic integrals (6.4) in a standard way

$$\sqrt{H_4 - a_1 H_3} = \sqrt{a_3 - a_1}M_3 = \sqrt{a_3 - a_1}K. \tag{6.5}$$

In nonholonomic case the linear integral looks like

$$K = \sqrt{\mathrm{g}(\gamma)}\,\left(M_3 + \frac{a_1 x_3(\gamma, M)}{\eta - a_1(\gamma,\gamma)}\right)\,, \qquad \{\, H_m, K\,\}_3 = 0\,. \tag{6.6}$$

It can be represented via quadratic integrals (6.4) and the Casimir functions according to the relation

$$H_4 - a_1 H_3 + (a_1 - a_3)\big(\eta - a_1(\gamma,\gamma)\big)\,K^2 - \frac{a_1^2}{\eta - a_1(\gamma,\gamma)}\,(\gamma, M)^2 = 0\,.$$

In both cases the equations of motion (6.1) are equivalent to the Lax equation

$$\frac{d\mathbf{L}}{dt} = [\mathbf{L}, \mathbf{\Omega} + \lambda\mathbf{B}], \qquad \mathbf{L} = \mathbf{M} + \frac{\mathbf{\Gamma}}{\lambda}, \qquad\qquad \lambda \in \mathbb{R}\,.$$

It is obvious, that the Lax matrix $\mathbf{L}$ satisfies the linear r-matrix brackets (3.11) with the same r-matrices (3.12) and (3.13).

In holonomic case by $\omega = \mathbf{A}_1 M$ we have $b \times (M - \omega) = 0$ at $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$. It allows us to get another well-known Lax representation for the Lagrange top [2, 30]:

$$\frac{d\mathbf{L}}{dt} = [\mathbf{L}, \mathbf{\Omega} + \lambda\mathbf{B}], \qquad \mathbf{L} = \lambda\mathbf{B} + \mathbf{M} + \frac{\mathbf{\Gamma}}{\lambda}\,. \tag{6.7}$$

In nonholonomic case $b \times (M - \omega) \neq 0$ and we have no such Lax matrix at all. Of course, it is a superficial argument because the main point is that the nonholonomic system is related with the genus three algebraic curve instead of the elliptic curve for the Lagrange top.

Namely, using the Euler angles and their conjugated momenta, for the Lagrange top we can easily prove that the pair of canonical variables

$$u = \gamma_3 = cos(\theta), \qquad p_u = \frac{\gamma_2 M_1 - \gamma_1 M_2}{\gamma_1^2 + \gamma_2^2} = -\frac{p_\theta}{\sin(\theta)}, \qquad \{u, p_u\}_1 = 1,$$

lies on the elliptic curve defined by equation

$$a_1 p_u^2 + \frac{2ub + \beta - a_3\,\alpha^2}{u^2 - 1} + \frac{a_1(\alpha u + \ell)^2}{(u^2-1)^2} = 0\,,$$

where we fix the values of the integrals of motion

$$H_1 = (\gamma,\gamma) = 1, \qquad H_2 = (\gamma, M) = \ell, \qquad K = \alpha, \qquad H_4 = \beta\,. \tag{6.8}$$

For the nonholonomic system canonical variables

$$u = u, \qquad \hat{p}_u = \frac{p_u}{\sqrt{\mathrm{g}}}, \qquad \{u, \hat{p}_u\}_3 = 1,$$

satisfy to the following separated equation

$$a_1\hat{p}_u^2 + \frac{2ub+\beta-a_3\rho\,\alpha^2}{u^2-1}\,\mathrm{g} + \frac{a_1\eta\,(\alpha\,u+\varrho\,\ell)^2}{\big(\eta+a_1(u^2-1)\big)(u^2-1)^2} = 0\,. \tag{6.9}$$

Here we fix the values of the integrals of motion as in (6.8) and

$$\mathrm{g} = \frac{1}{\eta-a_1+(a_1-a_3)u^2}\,,\qquad \rho = \frac{(\eta-a_1)^2}{\eta+a_1(u^2-1)}\,,\qquad \varrho = -\frac{\eta+a_1(u^2-1)}{\eta-a_1}\sqrt{\mathrm{g}}\,.$$

If $\ell=(\gamma,M)=0$, one gets the elliptic curve, but in generic case rewriting the equation (6.9) in polynomial form we obtain the algebraic curve of genus three.

Only at $(\gamma,M)=0$ we can get the solutions in the terms of elliptic functions and, therefore, only in this particular case we can try to reconstruct the Lax matrix associated with the elliptic curve.

Remark 16 Three different bi-Hamiltonian structures for the Lagrange top have been obtained in [37]. These structures are related with different variables of separation and, therefore, different quadratures. If we get similar dynamical Poisson bivectors for its nonholonomic counterpart, one gets various quadratures, which could be associated with the distinct Lax matrices and underlying r-matrix algebras.

6.2 Second order integrals of motion

For the Kirchhoff problem, the integrable cases by Kirchhoff, Clebsch, and Steklov-Lyapunov are known. In this section we begin with the Clebsch case

Proposition 13 *If $\omega=\mathbf{A}_{1,3}M$ then the integrals of the equations (6.2)-(6.3)*

$$H_3=(M,M)-(\mathbf{A}\gamma,\gamma)\qquad and \qquad H_4=(M,\omega)+(\mathbf{A}^\vee\gamma,\gamma) \tag{6.10}$$

are in the involution with respect to the Poisson brackets $\{.,.\}_1$ and $\{.,.\}_3$, respectively.

In holonomic case we have the well-studied Clebsch problem. In nonholonomic case this deformation of the Chaplygin system has been proposed by Kozlov [24].

There are some different Lax matrices for the Clebsch model [6, 30]. For example,

$$\mathbf{L}=\lambda\mathbf{A}_1+\mathbf{M}+\frac{\gamma\times\gamma}{\lambda}\,.$$

We can not directly generalize this matrix to the nonholonomic case, because $\dot{\mathbf{A}}_3\neq 0$ (2.8) in contrast with $\dot{\mathbf{A}}_1=0$ above. We suppose that the nonholonomic Kozlov system is related with the algebraic curve of higher genus and, therefore, the corresponding Lax matrices will be more complicated deformations of the known Lax matrices for the Clebsch problem.

At $(\gamma,M)=0$ the Clebsch system becomes the so-called Neumann system on the sphere, which is separable in the elliptic coordinates u,v (4.9) [6]. Its nonholonomic counterpart is the separable system in Chaplygin coordinates u,v (4.10) [16] and, therefore, we can get 2×2 Lax matrices $\mathscr{L}(\lambda)$ (5.1) for this nonholonomic system as well.

Remark 17 The Clebsch case is equivalent to the Brun case of integrability in the Euler-Poisson equations [6] and, moreover, it is trajectory isomorphic to the Kowalevski gyrostat [22]. We can hope to get a nonholonomic analog of the Kowalevski top by using similar isomorphism.

Now let us consider the integrable Steklov-Lyapunov case in the Kirchhoff equation and the corresponding integrals of motion

$$\begin{aligned} H_3&=(M,M)-2(M,\mathbf{A}\gamma)+(\gamma,\mathbf{C}^2\gamma)\,,\\ H_4&=(M,\omega)+2(M,\mathbf{A}^\vee\gamma)+(\mathbf{A}\gamma,\mathbf{C}^2\gamma)\,, \end{aligned} \tag{6.11}$$

where $\mathbf{C} = \mathrm{diag}(a_2 - a_3, a_3 - a_1, a_1 - a_2)$. These integrals are in the involution with respect to the first bracket $\{H_3, H_4\}_1 = 0$.

If we replace $\omega = \mathbf{A}_1 M$ on $\omega = \mathbf{A}_3 M$ in H_4 then $\{H_3, H_4\}_3 = 0$ only if two parameters a_i coincide with each other. So, for the nonholonomic bracket $\{.,.\}_3$ we have to propose some more complicated deformations of the integrals (6.11).

Remark 18 It is known that the Steklov-Lyapunov system is equivalent to the integrable system on the sphere with forth order potential (5.5) [35]. We suppose that a similar transformation of the system (5.6) separable in nonholonomic elliptic coordinates allows us to get a nonholonomic counterpart of the Steklov-Lyapunov system.

7 Conclusion

We consider two very similar dynamical systems, which evolve on coadjoint orbits of Lie algebra $e(3)$ and their non-trivial symplectic deformations.

Close ties between the integrable Euler top and the nonholonomic Chaplygin ball allow us to get Lax matrices, r-matrices and bi-hamiltonian structure for this nonholonomic system. Moreover, in framework of the Jacobi method of separation of variables we describe a huge family of separable potentials, which can be added to nonholonomic Hamiltonian and briefly discuss how to get the N-dimensional nonholonomic systems on the Riemannian spaces of constant curvature.

In [10] Chaplygin transforms the generic case of the rolling ball to the particular case of horizontal angular momentum $(\gamma, M) = 0$. It allows us to solve the equations of motion using the same variables of separation u, v (4.10), which will be the *non-canonical* variables with respect to initial Poisson bracket $\{.,.\}_3$ (3.4) after this map. We will discuss this Chaplygin map in framework of the Poisson geometry in separate publication, as well as the corresponding 2×2 Lax matrices and the underlying r-matrix algebra.

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