Supersymmetric Reciprocal Transformation

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The supersymmetric analog of the reciprocal transformation is introduced. This is used to establish a transformation between one of the supersymmetric Harry Dym equations and supersymmetric modified Korteweg-de Vries equation. A supersymmetric generalization of the Kawamoto equation is constructed and associated to the supersymmetric Sawada-Kotera equation.

I. INTRODUCTION

The Miura transformation linking the Korteweg-de Vries (KdV) and the Modified KdV (MKdV) equations played a key role in the development of the Inverse Scattering Method as a technique to solve integrable nonlinear partial differential equations. On the other side, the gauge type of transformations among the associated linear problems may be used to generate transformations between the corresponding nonlinear problems. Typically the gauge transformations of the scattering operator may either change the explicit form of the linear problem - corresponding to different nonlinear equations - or they will keep the linear problem invariant. The first case encodes the Miura transformation while the latter case represents auto - Bäcklund transformation. Moreover the reciprocal transformation, in conjunction with the gauge transformation also plays a key role in links between different scattering problems. It was demonstrated in[23] that AKNS and WKI scattering schemes are linked, when the transformations of the independent variables are taken into account. Furthermore in the context of soliton theory, the Harry Dym (HD) hierarchy[3 12], known to be invariant under a reciprocal transformation, is also connected with the KdV hierarchy. The Camassa-Holm equation is connected via the reciprocal link with the first negative flow of the KdV hierarchy[10], while the Sawada-Kotera (SK) and Kaup-Kupershmidt (KK) equations are linked by reciprocal transformation with the Kawamoto equation[13].

The aim of the present paper is to describe how we should modify the scenario of the reciprocal transformation to the supersymmetric equations. We consider in details the transformations among the supersymmetrci HD, MKdV, Kawamoto and SK equations.

Our approach is motivated by recent interest to the supersymmetric nonlinear partial differential equations. These equations have long history and appeared almost in parallel to the usage of the supersymmetry in the quantum field theory. The first results, concerned the construction of classical field theories with fermionic and bosonic fields depending on time and one space variable, can be found in[4,7,11,14,15,18]. In many cases, the addition of fermion fields does not guarantee that the final theory becomes supersymmetric invariant. Therefore this method was named as the fermionic extension in order to distinguish it from the fully supersymmetric method which was developed later[5,16,17,19,21].

There are many recipes how classical models could be embedded in fully supersymmetric superspace. The main idea is simple: in order to get such generalization we should construct the supermultiplet which contains the classical functions. It means that we have to add to a system of k bosonic equations kN fermions and $k(N-1)$ bosons ($k = 1, 2, ..., N = 1, 2, ...$) in such a way that they create superfields. Now working with this supermultiplet we can step by step apply integrable Hamiltonians methods to our considerations depending what we would like to construct. In that way the basic solitonic equations have been supersymmetrized as for example the KdV equation, Boussinesq equation, Two-Boson equation, HD equation and recently the SK equation. All these supersymmetric equations are integrable in the sense that they possess the recursion operator or the bi-Hamiltonian structure and consequently they have infinite number of conserved densities. Interestingly *not all solitonic equations have been successfully embedded into the superspace*, as for example we do not know up to now the supersymmetric version of the Kaup-Kupershmidt (KK) equation.

Due to the large number of the supersymmetric equations it is reasonable to find the reciprocal link between these supersymmetric equations. Our paper concerns to this problem and is divided into five sections. In the first section we recapitulate basic notations and ideas used in the classical reciprocal link between the HD equation and the MKdV equation. The second section, after introducing the supersymmetric notation, deals with the supersymmetric generalization of the reciprocal link for the supersymmetric HD and MKdV equations. The third section introduces the supersymmetric Kawamoto equation, its Lax representation and construction of the supersymmetric reciprocal link to the recently discovered the SUSY SK equation. The last section contains concluding remarks.

II. REVIEW ON THE CLASSICAL CASE

The integrability of the Harry Dym equation

$$u_t = -u^3 u_{3x} \tag{1}$$

follows from its nonstandard Lax representation

$$\left[\frac{\partial}{\partial t} + B, L\right] = 0 \ , \tag{2}$$

where

$$L = u^2 \partial_x^2 \ , \quad B = 4(L^{2/3})_{\geq 2} = 4u^3 \partial_x^3 + 6u^2 u_x \partial_x^2 \ .$$

To associate the HD equation (1) to the MKdV equation, we recall the Liouville transformation. Namely,

$$\frac{\partial}{\partial x} = u^{-1}\frac{\partial}{\partial y} \ , \tag{3}$$

then the Lax operator L is transformed to

$$\hat{L} = \partial_y^2 + v\partial_y \ ,$$

where v is given by the Cole-Hopf transformation

$$v = -\frac{\partial}{\partial y}\log u \ . \tag{4}$$

It is straightforward to check that the Lax equation

$$\left[\frac{\partial}{\partial \tau} + \hat{B}, \hat{L}\right] = 0 \tag{5}$$

implies the MKdV equation

$$v_\tau = -v_{3y} + \frac{3}{2}v^2 v_y \ , \tag{6}$$

where

$$\hat{B} = 4(L^{3/2})_{\geq 1} = 4\partial_y^3 + 6v\partial_y^2 + \left(3v_y + \frac{3}{2}v^2\right)\partial_y \ .$$

Now, the relation between ∂_t and ∂_τ is inferred from formulas (2) and (5), which is given by

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau} + \hat{B}\ |_{v=-u_y/u} - B\ |_{\partial_x = u^{-1}\partial_y} \\
&= \frac{\partial}{\partial \tau} + \left(u^{-1}u_{2y} - \frac{3}{2}u^{-2}u_y^2\right)\frac{\partial}{\partial y} \\
&= \frac{\partial}{\partial \tau} + \left(uu_{2x} - \frac{1}{2}u_x^2\right)\frac{\partial}{\partial y} \ .
\end{aligned}$$

Thus, we recover the reciprocal link between the HD equation (1) and the MKdV equation (6), i.e.

$$\begin{aligned}
\frac{\partial}{\partial x} &= u^{-1}\frac{\partial}{\partial y} \ , \\
\frac{\partial}{\partial t} &= \left(uu_{2x} - \frac{1}{2}u_x^2\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial \tau} \ ,
\end{aligned}$$

and the relation between two fields (4). The explicit transformations for the independent variables are given by

$$y = \int^x u^{-1}\mathtt{d}x \ , \qquad \tau = t \ .$$

With respect to the new independent variables (y, τ), the equation for u is

$$\frac{\partial}{\partial \tau} u = -\frac{\partial}{\partial y}\left(u_{2y} - \frac{3}{2}\frac{u_y^2}{u}\right) \ . \tag{7}$$

Introduce a potential $u = w_y$, we have the Schwarzian KdV equation

$$w_\tau = -\left(w_{3y} - \frac{3}{2}\frac{w_{2y}^2}{w_y^2}\right) \equiv -\{w, y\} \ , \tag{8}$$

where $\{w, y\}$ denotes the Schwarzian derivative of w with respect to y.

To obtain the MKdV equation (6) from the Schwarzian KdV equation (8), one only has to make the Cole-Hopf transformation

$$v = -\frac{\partial}{\partial y}\log w_y \ .$$

We remark that above construction of the reciprocal transformation heavily relies on the Lax representation. An alternative is based on the conservation laws. Indeed, the HD equation (1) can be reformulated as follows

$$\frac{\partial}{\partial t}(u^{-1}) = \frac{\partial}{\partial x}\left(u u_{2x} - \frac{1}{2}u_x^2\right) \ .$$

Thus, it is natural to introduce[13]

$$\mathtt{d}y = u^{-1}\mathtt{d}x + \left(u u_{2x} - \frac{1}{2}u_x^2\right)\mathtt{d}t, \qquad \mathtt{d}\tau = \mathtt{d}t$$

which is nothing but the reciprocal transformation discussed. The advantage of this latter approach is that everything follows from the equation *only*.

III. SUSY RECIPROCAL TRANSFORMATION: SUSY HARRY DYM CASE

In this section, taking one of the supersymmetric HD equations proposed in[2] as an example, we exhibit the reciprocal link between the supersymmetric HD equation and the supersymmetric MKdV equation.

The supersymmetric HD equation we will consider takes the form[2]

$$\begin{aligned} W_t = \frac{1}{16}\Big[& 8\mathcal{D}^5((\mathcal{D}W)^{-1/2}) - 3\mathcal{D}(W_{xx}W_x(\mathcal{D}W)^{-5/2}) \\ & + \frac{3}{4}(\mathcal{D}W_x)^2 W_x (\mathcal{D}W)^{-7/2} - \frac{3}{4}\mathcal{D}^{-1}\left((\mathcal{D}W_x)^3(\mathcal{D}W)^{-7/2}\right)\Big] \ . \end{aligned}$$

where $W = W(x, \theta, t)$ is a fermionic super field and $\mathcal{D} = \partial_\theta + \theta\partial_x$ is the super derivative.

By means of

$$U = (\mathcal{D}W)^{-1/2} ,$$

this equation can be conveniently written as

$$U_t = \frac{1}{4}U_{3x}U^3 - \frac{3}{8}(\mathcal{D}U_{2x})(\mathcal{D}U)U^2 , \tag{9}$$

which admits the Lax representation

$$\left[\frac{\partial}{\partial t} - (L_h^{3/2})_{\geq 3}, L_h\right] = 0 , \tag{10}$$

where the Lax operator is given by

$$L_h = U\mathcal{D}U\partial_x\mathcal{D} .$$

A. The super analogy of Liouville transformation

Our aim now is to convert the Lax opertaor L_h of the SUSY HD equation into that of the SUSY MKdV equation. To this end, we propose the following super analogy of Liouville transformation

$$\mathcal{D} = U^{-1/2}\mathbb{D} , \tag{11}$$

where $\mathbb{D}$ denotes the transformed superderivative given by

$$\mathbb{D} = \partial_\varrho + \varrho\partial_y .$$

Through the transformation (11), we obtain

$$L_m = \partial_y^2 + (\mathbb{D}\Psi)\partial_y + \left(\frac{1}{2}\Psi_x + \frac{1}{4}\Psi(\mathbb{D}\Psi)\right)\mathbb{D} ,$$

where Ψ is a fermionic super field related with U by the super Cole-Hopf transformation

$$\Psi = -\mathbb{D}\log U . \tag{12}$$

We claim that the operator L_m is the Lax operator of the supersymmetric MKdV equation. In fact, an easy calculation shows that the Lax equation

$$\left[\frac{\partial}{\partial \tau} - (L_m^{3/2})_{\geq 1},\ L_m\right] = 0 \tag{13}$$

implies the SUSY MKdV equation

$$\Psi_\tau = \frac{1}{4}\Psi_{3y} - \frac{3}{16}\Psi_y(\mathbb{D}\Psi)^2 - \frac{3}{16}\Psi(\mathbb{D}\Psi_y)(\mathbb{D}\Psi)\ . \tag{14}$$

The transformation between ∂_t and ∂_τ is found by

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau} - (L_m^{3/2})_{\geq 1}\,|_{\Psi=-(\mathbb{D}U)/U} + (L_h^{3/2})_{\geq 3}\,|_{\mathcal{D}=U^{-1/2}\mathbb{D}} \\ &= \frac{\partial}{\partial \tau} + \frac{1}{8}\left(-2\frac{U_{2y}}{U} + 3\frac{U_y^2}{U^2} + \frac{(\mathbb{D}U_y)(\mathbb{D}U)}{U^2}\right)\frac{\partial}{\partial y} \\ &\quad + \frac{1}{16}\left(-2\frac{(\mathbb{D}U_{2y})}{U} + 5\frac{(\mathbb{D}U_y)U_y}{U^2} + 3\frac{(\mathbb{D}U)U_{2y}}{U^2} - 6\frac{(\mathbb{D}U)U_y^2}{U^3}\right)\mathbb{D} \\ &= \frac{\partial}{\partial \tau} + \left(-\frac{1}{4}U_{2x}U + \frac{1}{8}U_x^2 + \frac{1}{4}(\mathcal{D}U_x)(\mathcal{D}U)\right)\frac{\partial}{\partial y} \\ &\quad + \left(-\frac{1}{8}(\mathcal{D}U_{2x})U^{3/2}\right)\mathbb{D}\ .\end{aligned} \tag{15}$$

Therefore, we succeeded in constructing between the supersymmetric HD eqution (9) and the supersymmetric MKdV equation (14) a reciprocal transformation

$$(x, \theta, t, U) \to (y, \varrho, \tau, \Psi)\ ,$$

which is given by formulas (11), (12) and (15).

In terms of the new super space-time (y, ϱ, τ), the bosonic super field U satisfies

$$U_\tau = \frac{1}{16}\left(4U_{3y} - 12\frac{U_{2y}U_y}{U} + 6\frac{U_y^3}{U^2} - 3\frac{(\mathbb{D}U_y)(\mathbb{D}U)U_y}{U^2}\right), \tag{16}$$

from which we obtain, through the transformation $U = (\mathbb{D}\Lambda)^{-2}$, the supersymmetric Schwarzian KdV equation[20]

$$\Lambda_\tau = \frac{1}{4}\left(\Lambda_{3y} - 3\frac{\Lambda_{2y}(\mathbb{D}\Lambda_y)}{(\mathbb{D}\Lambda)}\right). \tag{17}$$

The link between the supersymmetric Schwarzian KdV equation (17) and the supersymmetric MKdV equation (14) is supplied by the super Cole-Hopf transformation

$$\Psi = 2\mathbb{D}\log(\mathbb{D}\Lambda)\ .$$

B. The superconformal transformation

Relying on the linear spectral problem, we constructed the reciprocal link between the SUSY HD equation and the SUSY MKdV equation in last subsection III A. It would be nice if the reciprocal transformation can be established without any knowledge of linear problems. Next we will show that this is indeed the case: the reciprocal transformation is the result of the equation itself. To this end, we need the superconformal transformation, which is the super diffeomorphism such that the superderivative transforms covariantly[1,9,20]. Namely, $\mathcal{D} = G\mathbb{D}$, where G is a bosonic super field.

Proposition 1 *Let*

$$\frac{\partial G}{\partial t} = \mathcal{D}\Xi \tag{18}$$

be a conservation law, and suppose that a potential W can be introduced by

$$\mathcal{D}W = 2\Xi G \ , \tag{19}$$

then a superconformal transformation may be defined consistently.

Proof: First, we consider the following change of variables

$$(x, \theta, t) \to (y, \varrho, \tau) = (y(x, \theta, t), \varrho(x, \theta, t), t) \tag{20}$$

where (y, ϱ) is the new super spatial variable and τ is the new temporal variable. Next we use notation $\mathbb{D} = \frac{\partial}{\partial \varrho} + \varrho \frac{\partial}{\partial y}$ as our new superderivative.

To ensure that (20) is a superconformal transformation, we impose

$$\mathcal{D} = G\mathbb{D} \ , \tag{21}$$

and by a direct calculation we have

$$\begin{aligned}
\frac{\partial}{\partial x} &= \left[\frac{\partial y}{\partial x} - \left(\frac{\partial \varrho}{\partial x}\right)\varrho\right]\frac{\partial}{\partial y} + \frac{\partial \varrho}{\partial x}\mathbb{D} \equiv S\frac{\partial}{\partial y} + \Gamma\mathbb{D} \ , \\
\frac{\partial}{\partial \theta} &= \left[\frac{\partial y}{\partial \theta} - \left(\frac{\partial \varrho}{\partial \theta}\right)\varrho\right]\frac{\partial}{\partial y} + \frac{\partial \varrho}{\partial \theta}\mathbb{D} \equiv \Lambda\frac{\partial}{\partial y} + T\mathbb{D} \ , \\
\frac{\partial}{\partial t} &= \left[\frac{\partial y}{\partial t} - \left(\frac{\partial \varrho}{\partial t}\right)\varrho\right]\frac{\partial}{\partial y} + \frac{\partial \varrho}{\partial t}\mathbb{D} + \frac{\partial}{\partial \tau} \equiv \hat{W}\frac{\partial}{\partial y} + \hat{\Xi}\mathbb{D} + \frac{\partial}{\partial \tau} \ .
\end{aligned}$$

Naturally, all the coefficients S, T, $\hat{W}$, Γ, Λ and $\hat{\Xi}$ have to be selected consistently such that the compatibility conditions

$$\begin{aligned}
&\frac{\partial}{\partial \theta}\Gamma = \frac{\partial}{\partial x}T \ , \qquad \tfrac{\partial}{\partial \theta}\hat{\Xi} = \tfrac{\partial}{\partial t}T \ , \qquad \frac{\partial}{\partial x}\hat{\Xi} = \frac{\partial}{\partial t}\Gamma \ , \\
&\frac{\partial}{\partial \theta}S = \frac{\partial}{\partial x}\Lambda + 2\Gamma T \ , \quad \tfrac{\partial}{\partial t}\Lambda = \tfrac{\partial}{\partial \theta}\hat{W} - 2\hat{\Xi}T \ , \quad \frac{\partial}{\partial t}S = \frac{\partial}{\partial x}\hat{W} + 2\hat{\Xi}\Gamma \ ,
\end{aligned}$$

hold.

We notice that (21) implies

$$\frac{\partial}{\partial x} = \mathcal{D}^2 = G\mathbb{D}G\mathbb{D} = G^2\frac{\partial}{\partial y} + (\mathcal{D}G)\mathbb{D} \ ,$$

and also we have

$$\begin{aligned}\frac{\partial}{\partial \theta} &= \mathcal{D} - \theta\frac{\partial}{\partial x} = G\mathbb{D} - \theta\left(G^2\frac{\partial}{\partial y} + (\mathcal{D}G)\mathbb{D}\right) \\ &= -\theta G^2\frac{\partial}{\partial y} + \Big(G - \theta(\mathcal{D}G)\Big)\mathbb{D} \ .\end{aligned}$$

These last two equations suggest the following identifications

$$S = G^2 \ , \qquad \Gamma = (\mathcal{D}G) \ , \qquad \Lambda = -\theta G^2 \ , \qquad T = G - \theta(\mathcal{D}G) \ ,$$

furthermore, we ask for

$$\hat{W} = W \ , \qquad \hat{\Xi} = \Xi \ .$$

Then, a straightforward calculation shows that all the compatibility conditions are satisfied and thus, the proposition is proved. Q.E.D.

Let us now consider the supersymmetric HD equation (9). In order to take advantage of the proposition, we notice that the SUSY HD equation (9) has the following conservation law

$$\frac{\partial}{\partial t}\Big(U^{-1/2}\Big) = \mathcal{D}\left(-\frac{1}{8}(\mathcal{D}U_{2x})U^{3/2}\right) , \tag{22}$$

which leads to

$$G = U^{-1/2} \ , \qquad \Xi = -\frac{1}{8}(\mathcal{D}U_{2x})U^{3/2} \ ,$$

also,

$$2\Xi G = -\frac{1}{4}(\mathcal{D}U_{2x})U = \mathcal{D}\left(-\frac{1}{4}U_{2x}U + \frac{1}{8}U_x^2 + \frac{1}{4}(\mathcal{D}U_x)(\mathcal{D}U)\right) \ ,$$

hence we choose

$$\mathcal{D} = U^{-1/2}\mathbb{D} \ ,$$

which coincides with the super analogy of Liouville transformation (11) and

$$W = -\frac{1}{4}U_{2x}U + \frac{1}{8}U_x^2 + \frac{1}{4}(\mathcal{D}U_x)(\mathcal{D}U) \ .$$

According to our proposition, the relation between ∂_t and ∂_τ is given by

$$\frac{\partial}{\partial t} = \left(-\frac{1}{4}U_{2x}U + \frac{1}{8}U_x^2 + \frac{1}{4}(\mathcal{D}U_x)(\mathcal{D}U)\right)\frac{\partial}{\partial y} + \left(-\frac{1}{8}(\mathcal{D}U_{2x})U^{3/2}\right)\mathbb{D} + \frac{\partial}{\partial \tau} \ ,$$

which of course recovers the relation (15), as is expected.

IV. SUSY RECIPROCAL TRANSFORMATION: SUSY KAWAMOTO CASE

In this section, we consider fifth order equations. The analog of the HD equation now is the Kawamoto equation

$$v_t = v^5 v_{5x} + 5v^4 v_x v_{4x} + \frac{5}{2} v^4 v_{2x} v_{3x} + \frac{15}{4} v^3 v_x^2 v_{3x} \ , \tag{23}$$

which is integrable. Indeed, its Lax operator reads as[6]

$$L = v^3 \partial_x^3 + 3 v_x v^2 \partial_x^2 \ .$$

As Kawamoto showed, the equation (23) is reciprocally associated to the following equation

$$w_t = w_{5x} - 5w_x w_{3x} - 5w_{2x}^2 - 5w_x^3 - 20 w w_x w_{2x} - 5w^2 w_{3x} + 5w^4 w_x \ . \tag{24}$$

It is a well known fact that[8], through the Miura transformations

$$u = w_x - w^2,$$

and

$$u = -w_x - \frac{1}{2} w^2$$

respectively, the equation (24) is the common modification, sometimes known as Fordy-Gibbons equation, of the SK equation

$$u_t = u_{5x} + 5u_{3x} u + 5u_{2x} u_x + 5u_x u^2 \ ,$$

and of the KK equation

$$u_t = u_{5x} + 10u_{3x} u + 25u_{2x} u_x + 20u_x u^2 \ . \tag{25}$$

Now let us turn to the supersymmetric case. The equation (23) is embedded in its supersymmetric analogy as

$$\begin{aligned} V_t = {} & V^5 V_{5x} + 5V^4 V_{4x} V_x + \frac{5}{2} V^4 V_{3x} V_{2x} + \frac{15}{4} V^3 V_{3x} V_x^2 \\ & -\frac{5}{2} V^4 (\mathcal{D}V_{4x})(\mathcal{D}V) - 5V^4 (\mathcal{D}V_{3x})(\mathcal{D}V_x) \\ & -\frac{15}{2} V^3 (\mathcal{D}V_{3x})(\mathcal{D}V) V_x - \frac{15}{2} V^3 (\mathcal{D}V_{2x})(\mathcal{D}V_x) V_x \\ & -\frac{15}{4} V^3 (\mathcal{D}V_{2x})(\mathcal{D}V) V_{2x} - \frac{15}{8} V^2 (\mathcal{D}V_{2x})(\mathcal{D}V) V_x^2 \ , \end{aligned} \tag{26}$$

whose integrability follows from the Lax representation

$$\left[\frac{\partial}{\partial t} + 9(L_k^{5/3})_{\geq 3}, L_k\right] = 0$$

where the Lax operator is given by

$$L_k = V^{3/2}\mathcal{D}^3 V^{3/2}\mathcal{D}^3 \tag{27}$$

and V is a bosonic super field.

The equation (26) is our supersymmetric Kawamoto equation and the rest of this section will be devoted to it. Let us introduce the super Liouville transformation

$$\mathcal{D} = V^{-1/2}\mathbb{D} \ ,$$

then, the operator L_k takes the form

$$\begin{aligned} L_{msk} = \partial_y^3 - 3(\mathbb{D}\Psi)\partial_y^2 + [2(\mathbb{D}\Psi)^2 - 2(\mathbb{D}\Psi_y) - \Psi_y\Psi]\partial_y \\ +[\Psi_y(\mathbb{D}\Psi) - \Psi_{2y} + \Psi(\mathbb{D}\Psi_y)]\mathbb{D} \ , \end{aligned} \tag{28}$$

where the fermionic super field Ψ is related with the bosonic V by the Cole-Hopf transformation

$$\Psi = -\mathbb{D}\log(V^{-1/2}) \ . \tag{29}$$

Using the transformed operator L_{msk} we consider the following Lax equation

$$\left[\frac{\partial}{\partial \tau} + 9(L_{msk}^{5/3})_{\geq 1}, L_{msk}\right] = 0 \ ,$$

which provides the SUSY nonlinear evolution equation

$$\begin{aligned} \Psi_t = \mathbb{D}[(\mathbb{D}\Psi_{4y}) - 5(\mathbb{D}\Psi_{2y})(\mathbb{D}\Psi_y) - 5(\mathbb{D}\Psi_{2y})(\mathbb{D}\Psi)^2 \\ -5(\mathbb{D}\Psi_y)^2(\mathbb{D}\Psi) + (\mathbb{D}\Psi)^5 - 5\Psi_{3y}\Psi_y - 5\Psi_{2y}\Psi_y(\mathbb{D}\Psi) \\ -10\Psi_y\Psi(\mathbb{D}\Psi_{2y}) - 10\Psi_y\Psi(\mathbb{D}\Psi_y)(\mathbb{D}\Psi)] \ . \end{aligned} \tag{30}$$

The above equation is a modification of the SK equation proposed in[25]. To see it, we can further bring L_{msk}, through gauge transformation, to a new operator, i.e.

$$L_{sk} = e^{-(\mathbb{D}^{-1}\Psi)} L_{msk}\, e^{(\mathbb{D}^{-1}\Psi)} = \Big(\mathbb{D}^3 + \Psi_y - \Psi(\mathbb{D}\Psi)\Big)^2 = (\mathcal{D}^3 + \Phi)^2 \ , \tag{31}$$

which is nothing but the Lax operator for the SK equation, which reads as

$$\Phi_t = \Phi_{5y} + 5\Phi_{3y}(\mathbb{D}\Phi) + 5\Phi_{2y}(\mathbb{D}\Phi_y) + 5\Phi_y(\mathbb{D}\Phi)^2 = 0 \ . \tag{32}$$

The Lax operator and infinite number of supersymmetric conserved densities for this equation have been found in[25] while the odd Bi-Hamiltonian structure in[22]. Due to the existence of the Miura transformation, which follow from the equation (31)

$$\Phi = \Psi_y - \Psi(\mathcal{D}\Psi) \tag{33}$$

it is possible to transforms the odd Bi-Hamiltonian structure of the supersymmetric Sawada - Kotera equation onto the odd Bi-Hamiltonian structure of the supersymmetric Kawamoto equation (30).

To complete the construction of the reciprocal transformation, one has to find the relation between ∂_t and ∂_τ. As in the SUSY HD case, this can be done in two ways, either using linear problem or making use of conservation law. We now take the latter approach and will show that the result follows from Proposition 1. To do it, we notice that the equation (26) admits the conservation law

$$\begin{aligned}\frac{\partial}{\partial t}\left(V^{-1/2}\right) = \frac{1}{8}\mathcal{D}\Big(&- 4(\mathcal{D}V_{4x})V^{7/2} - 16(\mathcal{D}V_{3x})V_xV^{5/2}\\ &-14(\mathcal{D}V_{2x})V_{2x}V^{5/2} - 5(\mathcal{D}V_{2x})V_x^2V^{3/2}\\ &+20(\mathcal{D}V_{2x})(\mathcal{D}V_x)(\mathcal{D}V)V^{3/2} - 4(\mathcal{D}V)V_{4x}V^{5/2}\\ &+4(\mathcal{D}V_x)V_{3x}V^{5/2} - 10(\mathcal{D}V)V_{3x}V_xV^{3/2}\Big) ,\end{aligned}$$

which implies that we can identify

$$G = V^{-1/2} ,$$

and

$$\begin{aligned}\Xi = \frac{1}{8}\Big(&- 4(\mathcal{D}V_{4x})V^{7/2} - 16(\mathcal{D}V_{3x})V_xV^{5/2}\\ &-14(\mathcal{D}V_{2x})V_{2x}V^{5/2} - 5(\mathcal{D}V_{2x})V_x^2V^{3/2}\\ &+20(\mathcal{D}V_{2x})(\mathcal{D}V_x)(\mathcal{D}V)V^{3/2} - 4(\mathcal{D}V)V_{4x}V^{5/2}\\ &+4(\mathcal{D}V_x)V_{3x}V^{5/2} - 10(\mathcal{D}V)V_{3x}V_xV^{3/2}\Big) .\end{aligned}$$

Taking into account of the formula (19) we find

$$\begin{aligned}W = \frac{1}{4}\Big(&- 4V_{4x}V^3 - 8V_{3x}V_xV^2 + 8(\mathcal{D}V_{3x})(\mathcal{D}V)V^2 + V_{2x}V_x^2V\\ &-V_{2x}^2V^2 - \frac{1}{4}V_x^4 + 12(\mathcal{D}V_{2x})(\mathcal{D}V_x)V^2 + 6(\mathcal{D}V_{2x})(\mathcal{D}V)V_xV\\ &+2(\mathcal{D}V_x)(\mathcal{D}V)V_{2x}V - (\mathcal{D}V_x)(\mathcal{D}V)V_x^2\Big).\end{aligned}$$

Therefore, our reciprocal transformation in this case reads as

$$\mathcal{D} = V^{-1/2}\mathbb{D} \ , \tag{34}$$

$$\frac{\partial}{\partial t} = W\frac{\partial}{\partial y} + \Xi\mathbb{D} + \frac{\partial}{\partial \tau} \ . \tag{35}$$

Under the transformation (34) and (35), the equation (26) is changed to

$$\begin{aligned} V_\tau = & V_{5y} - 5V_{4y}V_yV^{-1} - \frac{25}{2}V_{3y}V_{2y}V^{-1} + \frac{85}{4}V_{3y}V_y^2V^{-2} + \frac{145}{4}V_{2y}^2V_yV^{-2} \\ & -\frac{265}{4}V_{2y}V_y^3V_{-3} + \frac{405}{16}V_y^5V^{-4} - \frac{5}{2}(\mathbb{D}V_{3y})(\mathbb{D}V_y)V^{-1} \\ & +\frac{5}{2}(\mathbb{D}V_{3y})(\mathbb{D}V)V_yV^{-2} + \frac{25}{4}(\mathbb{D}V_{2y})(\mathbb{D}V_y)V_yV^{-2} \\ & -\frac{25}{4}(\mathbb{D}V_{2y})(\mathbb{D}V)V_y^2V^{-3} - 5(\mathbb{D}V_y)(\mathbb{D}V)V_{3y}V_{-2} \\ & +\frac{25}{2}(\mathbb{D}V_y)(\mathbb{D}V)V_{2x}V_yV^{-3} - \frac{15}{4}(\mathbb{D}V_y)(\mathbb{D}V)V_y^3V^{-4} \ , \end{aligned}$$

which is related to the equation (30) through the transformation (29).

V. CONCLUSION

The reciprocal link, sometimes also named as hodograph transformation, is a useful instrument which allows us to transforms one equation to the other equation which in some cases are very well known. One would like to say the same in the supersymmetric case but then the situation is more complicated. In this paper we constructed the supersymmetric analogon of the reciprocal link between supersymmetric HD equation and MKdV equation. We also proposed a supersymmetric Kawamoto equation together with its Lax representation and established the supersymmetric link to the supersymmetric Sawada-Kotera equation.

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Appendix A

It is well known that a nonlinear differential equation could have many conservation laws and different conservation law may lead to different reciprocal transformation. In this

Appendix, we will show that it also is the case for the supersymmetric systems.

In addition to the conservation law (22), the supersymmetric HD equation (9) has another one given by

$$\frac{\partial}{\partial t}\Big(U^{-1}\Big) = \mathcal{D}\left(\frac{1}{8}(\mathcal{D}U)U_{2x} - \frac{1}{8}(\mathcal{D}U_x)U_x - \frac{1}{4}(\mathcal{D}U_{2x})U\right) \ . \tag{A1}$$

Therefore, a potential can be introduced for the quantity

$$2\left(\frac{1}{8}(\mathcal{D}U)U_{2x} - \frac{1}{8}(\mathcal{D}U_x)U_x - \frac{1}{4}(\mathcal{D}U_{2x})U\right)U^{-1} \ ,$$

i.e.

$$\begin{aligned}&\mathcal{D}\left(\tfrac{1}{4}\tfrac{(\mathcal{D}U_x)(\mathcal{D}U)}{U} - \tfrac{1}{2}U_{2x}\right)\\ =&2\left(\tfrac{1}{8}(\mathcal{D}U)U_{2x} - \tfrac{1}{8}(\mathcal{D}U_x)U_x - \tfrac{1}{4}(\mathcal{D}U_{2x})U\right)U^{-1}.\end{aligned} \tag{A2}$$

According the Proposition, one can apply the reciprocal transformation

$$\mathcal{D} = U^{-1}\mathbb{D} \ , \tag{A3}$$

$$\begin{aligned}\frac{\partial}{\partial t} =& \left(\frac{1}{4}\frac{(\mathcal{D}U_x)(\mathcal{D}U)}{U} - \frac{1}{2}U_{2x}\right)\frac{\partial}{\partial y}\\ &+\left(\frac{1}{8}(\mathcal{D}U)U_{2x} - \frac{1}{8}(\mathcal{D}U_x)U_x - \frac{1}{4}(\mathcal{D}U_{2x})U\right)\mathbb{D} + \frac{\partial}{\partial \tau} \ .\end{aligned} \tag{A4}$$

to the supersymmetric HD equation (9).

A direct calculation gives us the following result

$$\begin{aligned}U_\tau =& \frac{1}{4}U_{3y}U^{-3} - \frac{3}{2}U_{2y}U_yU^{-4} + \frac{3}{8}(\mathbb{D}U_{2y})(\mathbb{D}U)U^{-4}\\ &+\frac{3}{2}U_y^3U^{-5} - \frac{3}{2}(\mathbb{D}U_y)(\mathbb{D}U)U_yU^{-5} \ ,\end{aligned} \tag{A5}$$

which, by $U = \hat{U}^{-1}$, is transformed to the supersymmetric HD equation

$$\hat{U}_\tau = \frac{1}{4}\hat{U}_{3y}\hat{U}^3 - \frac{3}{8}(\mathbb{D}\hat{U}_{2y})(\mathbb{D}\hat{U})U^2.$$

The invariance of the supersymmetric HD equation (9) under the supersymmetric reciprocal transformation (A3) and (A4) can be viewed as a supersymmetric generalization of the invariance of the HD equation (1) under the transformation

$$\mathtt{d}y = u^{-2}\mathtt{d}x + 2u_{2x}\mathtt{d}t \ , \qquad \mathtt{d}\tau = \mathtt{d}t$$

which follows from the conservation law

$$\frac{\partial}{\partial t}\Big(u^{-2}\Big) = \frac{\partial}{\partial x}\Big(2u_{2x}\Big) \ ,$$

and was first reported in[24].

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