Two extensions of 1D Toda hierarchy

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Abstract

The extended Toda hierarchy of Carlet, Dubrovin and Zhang is reconsidered in the light of a $2+1$D extension of the 1D Toda hierarchy constructed by Ogawa. These two extensions of the 1D Toda hierarchy turn out to have a very similar structure, and the former may be thought of as a kind of dimensional reduction of the latter. In particular, this explains an origin of the mysterious structure of the bilinear formalism proposed by Milanov.

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1 Introduction

Geometry of 2D topological field theories has a profound relationship with integrable hierarchies [1]. Of particular interest is the topological sigma model (geometrically, the Gromov-Witten invariants) of the Riemann sphere $\mathbf{CP}^1$, which is related to the 1D Toda hierarchy and its dispersionless limit. To describe the correlation functions of "descendants" of primary observables, however, one has to extend the usual 1D Toda hierarchy by an extra set of commuting flows [2, 3, 4]. In the following, we refer to this extension as "logarithmic", because the Lax equations of these extra commuting flows are formulated by a kind of logarithm of the Lax operator. Carlet, Dubrovin and Zhang presented a rigorous formulation of the logarithm of the Lax operator, and thereby formulated a Lax formalism of the extended Toda hierarchy [5].

Recently, Milanov presented a bilinear (or Hirota) formalism of this extended Toda hierarchy [6]. According to Milanov's results, the tau function of the usual 1D Toda hierarchy can be extended to this hierarchy and satisfies a bilinear equation. This equation is certainly an extension of the familiar bilinear equation (of the contour integral type [7]) of the 1D Toda hierarchy, reducing to the latter as some of arbitrary constants in the equation are set to 0. For nonzero values of those arbitrary constants, however, Milanov's bilinear equation takes a quite mysterious form, the meaning of which has remained to be elucidated.

In this paper, we propose to understand the logarithmic extension in the light of the so called "$2+1$D extension". Here "1" means the (lowest) temporal dimension, and "2" an extension of the spatial dimension (in the case of the Toda hierarchy, a 1D lattice) by an extra spatial dimension. For example, a $2+1$D extension of the KdV equation was introduced by Calogero [8], Bogoyavlensky [9] and Schiff [10] in different contexts. A Lie algebraic interpretation of the same equation and the associated hierarchy of commuting flows was discovered later [11, 12, 13] and generalized to the nonlinear Schrödinger hierarchy [14]. An important outcome of the Lie-algebraic studies is a systematic derivation of a bilinear formalism of those $2+1$D extensions. As regards the 1D Toda hierarchy, two different $2+1$D extensions (based on two different reductions of the 2D Toda hierarchy [15] to the 1D Toda hierarchy) were constructed by Ogawa [16]. We shall show that the logarithmic extension of the 1D Toda hierarchy can be rewritten to a form that resembles one of Ogawa's $2+1$D extension. This enables us to consider the logarithmic extension as a kind of "dimensional reduction" of the $2+1$D extension. We can thus derive a bilinear formalism of the logarithmic extension by the same method as used for the $2+1$D extensions [13, 14, 16]. Milanov's results can be thus recovered from the perspectives of $2+1$D extensions.

This paper is organized as follows. Section 2 is a review of the Lax formalism and the bilinear formalism of the 1D Toda hierarchy. Since the 1D Toda hierarchy can be derived from the 2D Toda hierarchy, we omit the proof of the existence of the dressing operators and the tau function (which is parallel to the case of the 2D Toda hierarchy) and explain the derivation of bilinear equations in detail. Section 3 is a review of one of Ogawa's $2+1$D extensions that is relevant to the subject of this paper. Since Ogawa's paper [16] is rather sketchy on this case, we give a rather detailed account of its Lax and bilinear formalisms. Armed with the knowledge on the $2+1$D extension, we turn to Carlet, Dubrovin and Zhang's logarithmic extension in Section 4. We conclude this paper with a few remarks in Section 5.

2 1D Toda hierarchy

2.1 Lax equations

Let s denote the spatial coordinate of the 1D Toda hierarchy. Unlike the usual formulation on a 1D lattice, s is now understood to be a continuous variable. The Lax operator of the 1D Toda hierarchy is a difference operator of the form

$$\mathcal{L} = e^{\partial_s} + b(s) + c(s)e^{-\partial_s},$$

where $e^{n\partial_s}$'s ($\partial_s = \partial/\partial_s$) denote the shift operators that act on a function of s as $e^{n\partial_s}f(s) = f(s+n)$, and $b(s)$ and $c(s)$ are dynamical variables. Time evolutions $\mathcal{L} = \mathcal{L}(\boldsymbol{t})$, $\boldsymbol{t} = (t_1, t_2, \ldots)$, of the Lax operator are defined by the Lax equations

$$\frac{\partial \mathcal{L}}{\partial t_n} = [A_n, \mathcal{L}], \quad n = 1, 2, \ldots. \tag{2.1}$$

The generators A_n of time evolutions are constructed from $\mathcal{L}$ as

$$A_n = \frac{1}{2} (\mathcal{L}^n)_{\geq 0} - \frac{1}{2} (\mathcal{L}^n)_{<0},$$

where $(\quad)_{\geq 0}$ and $(\quad)_{<0}$ denote the nonnegative and negative power parts of difference operators:

$$\left(\sum_{n=-\infty}^{\infty} a_n(s) e^{n\partial_s}\right)_{\geq 0} = \sum_{n\geq 0} a_n(s) e^{n\partial_s},$$
$$\left(\sum_{n=-\infty}^{\infty} a_n(s) e^{n\partial_s}\right)_{<0} = \sum_{n<0} a_n(s) e^{n\partial_s}.$$

The lowest ($n = 1$) Lax equation consists of the equations

$$\frac{\partial b(s)}{\partial t_1} = c(s+1) - c(s), \quad \frac{\partial c(s)}{\partial t_1} = c(s)(b(s) - b(s-1)),$$

which can be converted to the usual 1D Toda equation

$$\frac{\partial^2 \phi(s)}{\partial t_1^2} = e^{\phi(s)-\phi(s+1)} - e^{\phi(s-1)-\phi(s)}$$

by the change of variables

$$b(s) = \frac{\partial \phi(s)}{\partial t_1}, \quad c(s) = e^{\phi(s-1)-\phi(s)}.$$

2.2 Wave functions and auxiliary linear equations

Let W and $\bar{W}$ be dressing operators of the form

$$W = 1 + \sum_{n=1}^{\infty} w_n(s) e^{-n\partial_s}, \quad \bar{W} = \sum_{n=0}^{\infty} \bar{w}_n(s) e^{n\partial_s}$$

by which the Lax operator is expressed as

$$\mathcal{L} = W e^{\partial_s} W^{-1} = \bar{W} e^{-\partial_s} \bar{W}^{-1}. \tag{2.2}$$

One can tune these dressing operators to satisfy the evolution equations

$$\frac{\partial W}{\partial t_n} = A_n W - \frac{1}{2} W e^{n\partial_s}, \quad \frac{\partial \bar{W}}{\partial t_n} = A_n \bar{W} + \frac{1}{2} \bar{W} e^{-n\partial_s} \tag{2.3}$$

as well. These equations can be converted to the auxiliary linear equations

$$\frac{\partial \Psi(s,z)}{\partial t_n} = A_n \Psi(s,z), \quad \frac{\partial \bar{\Psi}(s,z)}{\partial t_n} = A_n \bar{\Psi}(s,z) \tag{2.4}$$

for the wave functions

$$\Psi(s,z) = W z^s e^{\xi(\boldsymbol{t},z)/2} = \left(1 + \sum_{n=1}^{\infty} w_n(s) z^{-n}\right) z^s e^{\xi(\boldsymbol{t},z)/2},$$
$$\bar{\Psi}(s,z) = \bar{W} z^s e^{\xi(\boldsymbol{t},z^{-1})/2} = \left(\sum_{n=0}^{\infty} \bar{w}_n(s) z^n\right) z^s e^{\xi(\boldsymbol{t},z^{-1})/2},$$

where

$$\xi(\boldsymbol{t},z) = \sum_{n=1}^{\infty} t_n z^n.$$

The dressing relations (2.2), too, become auxiliary linear equations of the form

$$\mathcal{L}\Psi(s,z) = z\Psi(s,z), \quad \mathcal{L}\bar{\Psi}(s,z) = z^{-1}\bar{\Psi}(s,z). \tag{2.5}$$

2.3 Bilinear equations for wave functions

Let us introduce the difference operators

$$V = e^{-\partial_s}(W^*)^{-1} e^{\partial_s}, \quad \bar{V} = e^{-\partial_s}(V^*)^{-1} e^{\partial_s},$$

where A^* denotes the formal adjoint of a difference operator A, namely,

$$\left(\sum_{n=-\infty}^{\infty} a_n(s) e^{n\partial_s}\right)^* = \sum_{n=-\infty}^{\infty} e^{-n\partial_s} a_n(s),$$

and define the "dual wave functions" as

$$\Psi^*(s,z) = V z^{-s} e^{-\xi(\boldsymbol{t},z)/2}, \quad \bar{\Psi}^*(s,z) = \bar{V} z^{-s} e^{-\xi(\boldsymbol{t},z^{-1})/2}.$$

As we show below, the wave functions $\Psi(s,z), \bar{\Psi}(s,z)$ and their duals satisfy the bilinear equation

$$\oint \frac{dz}{2\pi i} z^k \Psi(s',\boldsymbol{t}',z)\Psi^*(s,\boldsymbol{t},z) = \oint \frac{dz}{2\pi i} z^{-k} \bar{\Psi}(s',\boldsymbol{t}',z)\bar{\Psi}^*(s,\boldsymbol{t},z) \tag{2.6}$$

for $k = 0, 1, 2, \ldots$ and arbitrary values of $s', s, \boldsymbol{t}', \boldsymbol{t}$ except for the condition [1]

$$s' - s \in \mathbf{Z}. \tag{2.7}$$

[1] If the spatial variable s is integer-valued, this condition is obviously satisfied. Since s is now a continuous variable, this condition is necessary to ensure single-valuedness of the integrands in (2.6).

In the present setting, both hand sides of the bilinear equation may be thought of as the residue of formal Laurent series, namely,

$$\oint \frac{dz}{2\pi i} \sum_{n=-\infty}^{\infty} a_n z^n = a_{-1},$$

though, in a complex analytic setting, they are understood to be the contour integrals along simple closed curves C_∞, C_0 encircling the points $z = \infty, 0$.

A technical clue to the derivation of (2.6) is the the identity (see, e.g., Ogawa's paper [16])

$$\oint \frac{dz}{2\pi i} \psi(s,z)\phi^*(s,z) = (Ae^{\partial_s}B^*)_{s's} = (Be^{-\partial_s}A^*)_{ss'} \tag{2.8}$$

that holds, under condition (2.7), for any difference operators

$$A = \sum_{n=-\infty}^{\infty} a_n(s)e^{n\partial_s}, \quad B = \sum_{n=-\infty}^{\infty} b_n(s)e^{n\partial_s}$$

and the associated "wave functions"

$$\psi(s,z) = Az^s = \sum_{n=-\infty}^{\infty} a_n(s)z^{n+s}, \quad \phi^*(s,z) = Bz^{-s} = \sum_{n=-\infty}^{\infty} b_n(s)z^{-n-s}.$$

$(\quad)_{s's}$ denotes the "(s', s)-matrix element" [2] of difference operators:

$$\left(\sum_{n=-\infty}^{\infty} a_n(s)e^{n\partial_s}\right)_{s's} = a_{s-s'}(s').$$

We apply this formula to the operator relation

$$We^{\partial_s}V^* = e^{\partial_s} = \bar{W}e^{\partial_s}\bar{W}^*$$

and obtain the bilinear equation

$$\oint \frac{dz}{2\pi i} \Psi(s', \boldsymbol{t}, z)\Psi^*(s, \boldsymbol{t}, z) = \oint \frac{dz}{2\pi i} \bar{\Psi}(s', \boldsymbol{t}, z)\bar{\Psi}^*(s, \boldsymbol{t}, z), \tag{2.9}$$

which is a special case of (2.6) where $k = 0$ and $\boldsymbol{t}' = \boldsymbol{t}$. We can deform this equation to (2.6) by two steps as follows.

The first step is to insert $z^{\pm k}$, $k = 0, 1, 2, \ldots$ into the contour integrals. To this end, we apply $\mathcal{L}^k$ to both hand sides of (2.9) with respect to the variable s' as

$$\oint \frac{dz}{2\pi i} \mathcal{L}^k\Psi(s', \boldsymbol{t}, z) \cdot \Psi^*(s, \boldsymbol{t}, z) = \oint \frac{dz}{2\pi i} \mathcal{L}^k\bar{\Psi}(s', \boldsymbol{t}, z) \cdot \bar{\Psi}^*(s, \boldsymbol{t}, z).$$

[2] If the spatial variable s is integer-valued, this is indeed the matrix element of a $\mathbf{Z} \times \mathbf{Z}$ matrix that represents the difference operator.

By (2.5), this equation turns into the equation

$$\oint \frac{dz}{2\pi i} z^k \Psi(s', \boldsymbol{t}, z)\Psi^*(s, \boldsymbol{t}, z) = \oint \frac{dz}{2\pi i} z^{-k} \bar{\Psi}(s', \boldsymbol{t}, z)\bar{\Psi}^*(s, \boldsymbol{t}, z). \tag{2.10}$$

The second step is to shift the value of $\boldsymbol{t}$ in $\Psi(s', \boldsymbol{t}, z)$ and $\bar{\Psi}(s', \boldsymbol{t}, z)$. To this end, let us note that the auxiliary linear equations (2.4) can be extended to higher orders as

$$\prod_{i=1}^{\infty} \left(\frac{\partial}{\partial t_i}\right)^{l_i} \Psi(s, z) = A_{l_1, l_2, \ldots} \Psi(s, z),$$

$$\prod_{i=1}^{\infty} \left(\frac{\partial}{\partial t_i}\right)^{l_i} \bar{\Psi}(s, z) = A_{l_1, l_2, \ldots} \bar{\Psi}(s, z),$$

where $A_{l_1, l_2, \ldots}$'s are difference operators of finite order in s that are recursively determined by A_n's. For example,

$$\frac{\partial^2 \Psi(s, z)}{\partial t_m \partial t_n} = \frac{\partial}{\partial t_m}\left(\frac{\partial \Psi(s, z)}{\partial t_n}\right) = \left(\frac{\partial A_n}{\partial t_m} + A_n A_m\right)\Psi(s, z),$$

hence

$$A_{m,n} = \frac{\partial A_n}{\partial t_m} + A_n A_m.$$

The same equation holds for $\bar{\Psi}(s, z)$ as well. Applying $A_{l_1, l_2, \ldots}$ to both hand sides of (2.10) with respect to s', we have the equations

$$\oint \frac{dz}{2\pi i} z^k \prod_{i=1}^{\infty} \left(\frac{\partial}{\partial t_i}\right)^{l_i} \Psi(s', \boldsymbol{t}, z) \cdot \Psi^*(s, \boldsymbol{t}, z)$$
$$= \oint \frac{dz}{2\pi i} z^{-k} \prod_{i=1}^{\infty} \left(\frac{\partial}{\partial t_i}\right)^{l_i} \bar{\Psi}(s', \boldsymbol{t}, z) \cdot \bar{\Psi}^*(s, \boldsymbol{t}, z)$$

for all values of $l_1, l_2, \ldots$. Since the derivatives of $\Psi(s', \boldsymbol{t}, z)$ and $\bar{\Psi}(s', \boldsymbol{t}, z)$ can be collected to the generating functions

$$\sum_{l_1, l_2, \ldots = 0}^{\infty} \prod_{i=1}^{\infty} \frac{a_i^{l_i}}{l_i!} \left(\frac{\partial}{\partial t_i}\right)^{l_i} \Psi(s', \boldsymbol{t}, z) = \Psi(s', \boldsymbol{t} + \boldsymbol{a}, z),$$

$$\sum_{l_1, l_2, \ldots = 0}^{\infty} \prod_{i=1}^{\infty} \frac{a_i^{l_i}}{l_i!} \left(\frac{\partial}{\partial t_i}\right)^{l_i} \bar{\Psi}(s', \boldsymbol{t}, z) = \bar{\Psi}(s', \boldsymbol{t} + \boldsymbol{a}, z)$$

of new variables $\boldsymbol{a} = (a_1, a_2, \ldots)$, the last bilinear equations can be converted to the generating functional form

$$\oint \frac{dz}{2\pi i} z^k \Psi(s', \boldsymbol{t} + \boldsymbol{a}, z) \cdot \Psi^*(s, \boldsymbol{t}, z)$$
$$= \oint \frac{dz}{2\pi i} z^{-k} \bar{\Psi}(s', \boldsymbol{t} + \boldsymbol{a}, z) \cdot \bar{\Psi}^*(s, \boldsymbol{t}, z). \tag{2.11}$$

Replacing $\boldsymbol{t}+\boldsymbol{a}\to\boldsymbol{t}'$, we obtain the bilinear equation (2.6).

Though we omit details, one can conversely derive the auxiliary linear equations (2.4) and (2.5) from (2.6).

2.4 Tau function and bilinear equations

The wave functions and their duals can be expressed in terms of the tau function $\tau(s,\boldsymbol{t})$ as

$$\begin{aligned}
\Psi(s,z) &= \frac{\tau(s,\boldsymbol{t}-[z^{-1}])}{\tau(s,\boldsymbol{t})} z^s e^{\xi(\boldsymbol{t},z)/2},\\
\Psi^*(s,z) &= \frac{\tau(s,\boldsymbol{t}+[z^{-1}])}{\tau(s,\boldsymbol{t})} z^{-s} e^{-\xi(\boldsymbol{t},z)/2},\\
\bar{\Psi}(s,z) &= \frac{\tau(s+1,\boldsymbol{t}+[z])}{\tau(s,\boldsymbol{t})} z^s e^{\xi(\boldsymbol{t},z^{-1})/2},\\
\bar{\Psi}^*(s,z) &= \frac{\tau(s-1,\boldsymbol{t}-[z])}{\tau(s,\boldsymbol{t})} z^{-s} e^{-\xi(\boldsymbol{t},z^{-1})/2},
\end{aligned} \tag{2.12}$$

where

$$[z] = \left(z, \frac{z^2}{2}, \ldots, \frac{z^n}{n}, \ldots\right).$$

The bilinear equation (2.6) for the wave functions thereby turns into the bilinear equation

$$\begin{aligned}
&\oint \frac{dz}{2\pi i} z^{k+s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2} \tau(s',\boldsymbol{t}'-[z^{-1}])\tau(s,\boldsymbol{t}+[z^{-1}])\\
&\quad = \oint \frac{dz}{2\pi i} z^{-k+s'-s} e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2} \tau(s'+1,\boldsymbol{t}'+[z])\tau(s-1,\boldsymbol{t}-[z])
\end{aligned} \tag{2.13}$$

for the tau function, which holds for $k=0,1,\ldots$ and arbitrary values of $s,s',\boldsymbol{t},\boldsymbol{t}'$ under the condition (2.7).

Let us mention a few consequences of (2.13).

1. We can replace $z^{\pm k}$ by an arbitrary formal power series $f(z^{\pm 1}) = \sum_{k=0}^\infty f_k z^{\pm k}$ as

$$\begin{aligned}
&\oint \frac{dz}{2\pi i} f(z) z^{s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2} \tau(s',\boldsymbol{t}'-[z^{-1}])\tau(s,\boldsymbol{t}+[z^{-1}])\\
&\quad = \oint \frac{dz}{2\pi i} f(z^{-1}) z^{s'-s} e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2} \tau(s'+1,\boldsymbol{t}'+[z])\tau(s-1,\boldsymbol{t}-[z]).
\end{aligned}$$

   In particular, if we choose $f(z)$ as $f(z) = z^k e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2}$, we have the bilinear equation

$$\begin{aligned}
&\oint \frac{dz}{2\pi i} z^{s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)} \tau(s',\boldsymbol{t}'-[z^{-1}])\tau(s,\boldsymbol{t}+[z^{-1}])\\
&\qquad = \oint \frac{dz}{2\pi i} z^{s'-s} \tau(s'+1,\boldsymbol{t}'+[z])\tau(s-1,\boldsymbol{t}-[z]),
\end{aligned} \tag{2.14}$$

which is equivalent to (2.13), hence may be thought of as yet another bilinear representation of the 1D Toda hierarchy.

2. When $k = 0$ and $s' \geq s$, (2.14) reduces to the bilinear equation

$$\oint \frac{dz}{2\pi i} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)} \tau(s, \boldsymbol{t}' - [z^{-1}])\tau(s, \boldsymbol{t} + [z^{-1}]) = 0$$

of the KP ($s' = s$) or modified KP ($s' > s$) hierarchy.

3. If we choose $s' = s$ and $\boldsymbol{t}' = \boldsymbol{t}$, (2.13) reduces to

$$\begin{aligned}&\oint \frac{dz}{2\pi i} z^k \tau(s, \boldsymbol{t} - [z^{-1}])\tau(s, \boldsymbol{t} + [z^{-1}]) \\ &= \oint \frac{dz}{2\pi i} z^{k-2} \tau(s+1, \boldsymbol{t} + [z^{-1}])\tau(s-1, \boldsymbol{t} - [z^{-1}]), \quad k = 0, 1, 2, \ldots.\end{aligned}$$

These equations imply that $\tau(s, \boldsymbol{t} - [z^{-1}])\tau(s, \boldsymbol{t} + [z^{-1}]) - z^{-2}\tau(s+1, \boldsymbol{t} + [z^{-1}])\tau(s-1, \boldsymbol{t} + [z^{-1}])$ is independent of z, hence a function of s and $\boldsymbol{t}$ only. Letting $z \to \infty$ shows that this function is equal to $\tau(s, \boldsymbol{t})^2$. Thus we obtain the bilinear functional equation

$$\begin{aligned}&\tau(s, \boldsymbol{t} - [z^{-1}])\tau(s, \boldsymbol{t} + [z^{-1}]) \\ &\qquad = z^{-2}\tau(s+1, \boldsymbol{t} + [z^{-1}])\tau(s-1, \boldsymbol{t} - [z^{-1}]) + \tau(s, \boldsymbol{t})^2\end{aligned}$$

with a parameter z. Expanded in powers of z^{-1}, the z^{-2} part of this equation gives the Hirota equation

$$\frac{\partial^2 \tau(s, \boldsymbol{t})}{\partial t_1^2} \tau(s, \boldsymbol{t}) - \left(\frac{\partial \tau(s, \boldsymbol{t})}{\partial t_1}\right)^2 = \tau(s+1, \boldsymbol{t})\tau(s-1, \boldsymbol{t})$$

of the 1D Toda equation.

2.5 Reduction from 2D Toda hierarchy

The 2D Toda hierarchy has two series of time variables $\boldsymbol{t} = (t_1, t_2, \ldots)$ and $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$. The Lax equations are formulated in terms of two Lax operators

$$\begin{aligned}L &= e^{\partial_s} + u_1(s) + u_2(s)e^{-\partial_s} + \cdots, \\ \bar{L} &= \bar{u}_0(s)e^{\partial_s} + \bar{u}_1(s)e^{2\partial_s} + \cdots\end{aligned}$$

and the generators of time evolutions

$$B_n = (L^n)_{\geq 0}, \quad \bar{B}_n = (\bar{L}^{-n})_{<0}$$

as

$$\begin{aligned}\frac{\partial L}{\partial t_n} &= [B_n, L], \quad \frac{\partial L}{\partial \bar{t}_n} = [\bar{B}_n, L], \\ \frac{\partial \bar{L}}{\partial t_n} &= [B_n, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{L}], \quad n = 1, 2, \ldots.\end{aligned}$$

This hierarchy reduces to the 1D Toda hierarchy by adding the constraint [3]

$$(\mathcal{L} :=)\ L = \bar{L}^{-1}.$$

Defining $\mathcal{L}$ thus by both hand sides of this constraint and comparing the $(\quad)_{\geq 0}$ and $(\quad)_{<0}$ parts, we can readily see that $\mathcal{L}$ can be written as

$$\mathcal{L} = B_1 + C_1,$$

hence a difference operator of the form $e^{\partial_s} + b(s) + c(s)e^{-\partial_s}$. Moreover, under this constraint, we have the identities

$$B_n + \bar{B}_n = \mathcal{L}^n,$$

which imply that the time evolutions in the diagonal direction of the $(t_n, \bar{t}_n)$ plane are trivial:

$$\frac{\partial \mathcal{L}}{\partial t_n} + \frac{\partial \mathcal{L}}{\partial \bar{t}_n} = [\mathcal{L}^n, \mathcal{L}] = 0, \quad n = 1, 2, \ldots.$$

The residual time evolutions of $\mathcal{L}$ generated by

$$A_n = \frac{1}{2}B_n - \frac{1}{2}\bar{B}_n$$

can be identified with the 1D Toda hierarchy.

As regards the tau function, this reduction procedure amounts to adding the constraints

$$\frac{\partial \tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}})}{\partial t_n} + \frac{\partial \tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}})}{\partial \bar{t}_n} = 0, \quad n = 1, 2, \ldots$$

to the tau function $\tau(\boldsymbol{t}, \bar{\boldsymbol{t}})$ of the 2D Toda hierarchy, which thereby becomes a function $\tau(s, \boldsymbol{t} - \bar{\boldsymbol{t}})$ of s and $\boldsymbol{t} - \bar{\boldsymbol{t}}$. The reduced function $\tau(s, \boldsymbol{t})$ is exactly the tau function of the 1D Toda hierarchy.

3 $2+1$D extension

3.1 Lax equations and auxiliary linear equations

Following Ogawa [16], we introduce a new spatial variable y and an infinite number of time variables $\boldsymbol{x} = (x_1, x_2, \ldots)$. The dynamical variables $b(s)$ and $c(s)$ now depend on $y, \boldsymbol{x}$ and $\boldsymbol{t}$. The $2+1$D extension consists of the Toda flows with respect to $\boldsymbol{t}$ and the commuting flows with respect to $\boldsymbol{x}$ defined by Lax equations of the form

$$\frac{\partial \mathcal{L}}{\partial x_n} = \mathcal{L}^n \frac{\partial \mathcal{L}}{\partial y} + [P_n, \mathcal{L}] = [\mathcal{L}^n \partial_y + P_n, \mathcal{L}], \quad n = 1, 2, \ldots, \tag{3.1}$$

[3]Another reduction to the 1D Toda hierarchy is achieved by the constraint $L + L^{-1} = \bar{L} + \bar{L}^{-1}$ [15]. This reduction is suited for the soliton solutions of the 1D Toda lattice.

where ∂_y denotes $\partial/\partial y$, and P_n's are difference operators of finite order specified below. The associated auxiliary linear equations for $\Psi(s,z)$ and $\bar{\Psi}(s,z)$ read

$$\frac{\partial\Psi(s,z)}{\partial x_n} = (\mathcal{L}^n\partial_y + P_n)\Psi(s,z), \quad \frac{\partial\bar{\Psi}(s,z)}{\partial x_n} = (\mathcal{L}^n\partial_y + P_n)\bar{\Psi}(s,z). \tag{3.2}$$

The dressing operators W and $\bar{W}$ thereby satisfy the evolution equations

$$\frac{\partial W}{\partial x_n} = \mathcal{L}^n\frac{\partial W}{\partial y} + P_n W, \quad \frac{\partial \bar{W}}{\partial x_n} = \mathcal{L}^n\frac{\partial \bar{W}}{\partial y} + P_n \bar{W}. \tag{3.3}$$

P_n's are determined by (3.3) themselves as follows. Let us rewrite (3.3) as

$$P_n = \frac{\partial W}{\partial x_n}W^{-1} - \mathcal{L}^n\frac{\partial W}{\partial y}W^{-1} = \frac{\partial \bar{W}}{\partial x_n}\bar{W}^{-1} - \mathcal{L}^n\frac{\partial \bar{W}}{\partial y}\bar{W}^{-1}.$$

The $(\quad)_{\geq 0}$ and $(\quad)_{<0}$ parts of these equations give

$$(P_n)_{\geq 0} = -\left(\mathcal{L}^n\frac{\partial W}{\partial y}W^{-1}\right)_{\geq 0}, \quad (P_n)_{<0} = -\left(\mathcal{L}^n\frac{\partial \bar{W}}{\partial y}\bar{W}^{-1}\right)_{<0}.$$

Thus P_n's are determined as

$$P_n = -\left(\mathcal{L}^n\frac{\partial W}{\partial y}W^{-1}\right)_{\geq 0} - \left(\mathcal{L}^n\frac{\partial \bar{W}}{\partial y}\bar{W}^{-1}\right)_{<0}. \tag{3.4}$$

The auxiliary linear equations have another expression of the form

$$\frac{\partial\Psi(s,z)}{\partial x_n} = (z^n\partial_y + Q_n)\Psi(s,z), \quad \frac{\partial\bar{\Psi}(s,z)}{\partial x_n} = (z^{-n}\partial_y + Q_n)\bar{\Psi}(s,z), \tag{3.5}$$

where

$$Q_n = P_n - \frac{\partial\mathcal{L}^n}{\partial y} = -\left(\frac{\partial W}{\partial y}e^{n\partial_s}W^{-1}\right)_{\geq 0} - \left(\frac{\partial \bar{W}}{\partial y}e^{-n\partial_s}\bar{W}^{-1}\right)_{<0}. \tag{3.6}$$

3.2 Bilinear equation for wave functions

Let us start from the bilinear equation

$$\oint\frac{dz}{2\pi i}z^k\Psi(s',\boldsymbol{x},\boldsymbol{t}',z)\Psi^*(s,\boldsymbol{x},\boldsymbol{t},z) = \oint\frac{dz}{2\pi i}z^{-k}\bar{\Psi}(s',\boldsymbol{x},\boldsymbol{t}',z)\bar{\Psi}^*(s,\boldsymbol{x},\boldsymbol{t},z)$$

of the 1D Toda hierarchy, and deform it incorporate to the auxiliary linear equations (3.5). To this end, we extend (3.5) to higher orders as

$$\prod_{i=1}^{\infty}\left(\frac{\partial}{\partial x_i} - z^i\frac{\partial}{\partial y}\right)^{l_i}\Psi(s,z) = Q_{l_1,l_2,\ldots}\Psi(s,z),$$
$$\prod_{i=1}^{\infty}\left(\frac{\partial}{\partial x_i} - z^{-i}\frac{\partial}{\partial y}\right)^{l_i}\bar{\Psi}(s,z) = Q_{l_1,l_2,\ldots}\bar{\Psi}(s,z),$$

where $Q_{l_1,l_2,\ldots}$ are difference operators of finite order in s. Applying $Q_{l_1,l_2,\ldots}$ to both hand sides of the bilinear equation with respect to s', we have the equations

$$\oint \frac{dz}{2\pi i} z^k \prod_{i=1}^{\infty} \left(\frac{\partial}{\partial x_i} - z^i \frac{\partial}{\partial y} \right)^{l_i} \Psi(s', \boldsymbol{x}, \boldsymbol{t}', z) \cdot \Psi^*(s, \boldsymbol{x}, \boldsymbol{t}, z)$$
$$= \oint \frac{dz}{2\pi i} z^{-k} \prod_{i=1}^{\infty} \left(\frac{\partial}{\partial x_i} - z^{-i} \frac{\partial}{\partial y} \right)^{l_i} \bar{\Psi}(s', \boldsymbol{x}, \boldsymbol{t}', z) \cdot \bar{\Psi}^*(s, \boldsymbol{x}, \boldsymbol{t}, z)$$

for all values of $l_1, l_2, \ldots$. These bilinear equations can be packed into the generating functional form

$$\oint \frac{dz}{2\pi i} z^k \Psi(s', y - \xi(\boldsymbol{a}, z), \boldsymbol{x} + \boldsymbol{a}, \boldsymbol{t}', z) \Psi^*(s, y, \boldsymbol{x}, \boldsymbol{t}, z)$$
$$= \oint \frac{dz}{2\pi i} z^{-k} \bar{\Psi}(s', y - \xi(\boldsymbol{a}, z^{-1}), \boldsymbol{x} + \boldsymbol{a}, \boldsymbol{t}', z) \bar{\Psi}^*(s, y, \boldsymbol{x}, \boldsymbol{t}, z) \quad (3.7)$$

with new variables $\boldsymbol{a} = (a_1, a_2, \ldots)$. Note that this equation, like (2.6), holds for $k = 0, 1, 2, \ldots$ and arbitrary values of $s', s, \boldsymbol{x}, \boldsymbol{t}', \boldsymbol{t}$ under condition (2.7).

Moreover, we can extend (3.7) to the slightly more general (but actually equivalent) form

$$\oint \frac{dz}{2\pi i} z^k \Psi(s', y - \xi(\boldsymbol{a}, z), \boldsymbol{x} + \boldsymbol{a}, \boldsymbol{t}', z) \Psi^*(s, y - \xi(\boldsymbol{b}, z), \boldsymbol{x} + \boldsymbol{b}, \boldsymbol{t}, z)$$
$$= \oint \frac{dz}{2\pi i} z^{-k} \bar{\Psi}(s', y - \xi(\boldsymbol{a}, z^{-1}), \boldsymbol{x} + \boldsymbol{a}, \boldsymbol{t}', z) \bar{\Psi}^*(s, y - \xi(\boldsymbol{b}, z^{-1}), \boldsymbol{x} + \boldsymbol{b}, \boldsymbol{t}, z), \quad (3.8)$$

where $\boldsymbol{b} = (b_1, b_2, \ldots)$ is yet another set of variables. This equation, too, holds for $k = 0, 1, 2, \ldots$ and arbitrary values of $s', s, \boldsymbol{x}, \boldsymbol{t}', \boldsymbol{t}'$ except for the condition (2.7). To derive this equation, we apply the operator $(-c\partial/\partial y)^l/l!$ (where c is a constant and $l = 0, 1, 2, \ldots$) to both hand sides of (3.7), shift k to $k + ln$ ($n = 1, 2, \ldots$), and take the summation over $l = 0, 1, 2, \ldots$. The outcome is the equation

$$\oint \frac{dz}{2\pi i} z^k \Psi(s', y - \xi(\boldsymbol{a}, z) - cz^n, \boldsymbol{x} + \boldsymbol{a}, \boldsymbol{t}', z) \Psi^*(s, y - cz^n, \boldsymbol{x}, \boldsymbol{t}, z)$$
$$= \oint \frac{dz}{2\pi i} z^{-k} \bar{\Psi}(s', y - \xi(\boldsymbol{a}, z) - cz^{-n}, \boldsymbol{x} + \boldsymbol{a}, \boldsymbol{t}', z) \bar{\Psi}^*(s, y - cz^{-n}, \boldsymbol{x}, \boldsymbol{t}, z).$$

Repeating this procedure for $n = 1, 2, \ldots$ with independent constants $c = b_n$, we can derive the equation

$$\oint \frac{dz}{2\pi i} z^k \Psi(s', y - \xi(\boldsymbol{a} + \boldsymbol{b}, z), \boldsymbol{x} + \boldsymbol{a}, \boldsymbol{t}', z) \Psi^*(s, y - \xi(\boldsymbol{b}, z), \boldsymbol{x}, \boldsymbol{t}, z)$$
$$= \oint \frac{dz}{2\pi i} z^{-k} \bar{\Psi}(s', y - \xi(\boldsymbol{a} + \boldsymbol{b}, z), \boldsymbol{x} + \boldsymbol{a}, \boldsymbol{t}', z) \bar{\Psi}^*(s, y - \xi(\boldsymbol{b}, z^{-1}), \boldsymbol{x}, \boldsymbol{t}, z).$$

Replacing $\boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{b}$ and $\boldsymbol{a} \to \boldsymbol{a} - \boldsymbol{b}$ in this equation, we obtain (3.8).

3.3 Bilinear equation for tau function

Let $\tau(s,\boldsymbol{x},\boldsymbol{t})$ be a tau function in the sense of the 1D Toda hierarchy, namely, a function with which the wave functions are expressed as (2.12). Note that such a tau function is unique up to a multiplier that depends on only $\boldsymbol{x}$.

The bilinear equation (3.7) for the wave functions turns into an equation for the tau function of the form

$$\begin{aligned}&\oint\frac{dz}{2\pi i}z^{k+s'-s}e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2}\\&\quad\times\frac{\tau(s',y-\xi(\boldsymbol{a},z),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}'-[z^{-1}])\tau(s,y,\boldsymbol{x},\boldsymbol{t}+[z^{-1}])}{\tau(s',y-\xi(\boldsymbol{a},z),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}')\tau(s,y,\boldsymbol{x},\boldsymbol{t})}\\&\qquad=\oint\frac{dz}{2\pi i}z^{-k+s'-s}e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2}\\&\qquad\quad\times\frac{\tau(s'+1,y-\xi(\boldsymbol{a},z^{-1}),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}'+[z])\tau(s-1,y,\boldsymbol{x},\boldsymbol{t}-[z])}{\tau(s',y-\xi(\boldsymbol{a},z^{-1}),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}')\tau(s,y,\boldsymbol{x},\boldsymbol{t})}.\end{aligned}$$

We can now use the same trick as used in Section 2.4. Namely, we can replace $z^{\pm k}$ by an arbitrary power series $f(z^{\pm 1})=\sum_{k=0}^{\infty}f_kz^{\pm k}$ of z as

$$\begin{aligned}&\oint\frac{dz}{2\pi i}f(z)z^{s'-s}e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2}\\&\quad\times\frac{\tau(s',y-\xi(\boldsymbol{a},z),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}'-[z^{-1}])\tau(s,y,\boldsymbol{x},\boldsymbol{t}+[z^{-1}])}{\tau(s',y-\xi(\boldsymbol{a},z),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}')\tau(s,y,\boldsymbol{x},\boldsymbol{t})}\\&\qquad=\oint\frac{dz}{2\pi i}f(z^{-1})z^{s'-s}e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2}\\&\qquad\quad\times\frac{\tau(s'+1,y-\xi(\boldsymbol{a},z^{-1}),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}'+[z])\tau(s-1,y,\boldsymbol{x},\boldsymbol{t}-[z])}{\tau(s',y-\xi(\boldsymbol{a},z^{-1}),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}')\tau(s,y,\boldsymbol{x},\boldsymbol{t})}.\end{aligned}$$

In particular, if we choose $f(z)$ as

$$f(z)=z^k\tau(s',y-\xi(\boldsymbol{a},z),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}')\tau(s,y,\boldsymbol{x},\boldsymbol{t}),$$

the denominators disappear and we obtain the bilinear equation

$$\begin{aligned}&\oint\frac{dz}{2\pi i}z^{k+s'-s}e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2}\\&\quad\times\tau(s',y-\xi(\boldsymbol{a},z),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}'-[z^{-1}])\tau(s,y,\boldsymbol{x},\boldsymbol{t}+[z^{-1}]),\\&\qquad=\oint\frac{dz}{2\pi i}z^{-k+s'-s}e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2}\\&\qquad\quad\times\tau(s'+1,y-\xi(\boldsymbol{a},z^{-1}),\boldsymbol{x}+\boldsymbol{a},\boldsymbol{t}'+[z])\tau(s-1,y,\boldsymbol{x},\boldsymbol{t}-[z])\end{aligned}\tag{3.9}$$

for the tau function.

In the same way, the bilinear equation (3.8) of a slightly more general form can be converted to

$$\begin{aligned}&\oint \frac{dz}{2\pi i} z^{k+s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2}\tau(s', y-\xi(\boldsymbol{a},z), \boldsymbol{x}+\boldsymbol{a}, \boldsymbol{t}'-[z^{-1}])\\ &\quad\times \tau(s, y-\xi(\boldsymbol{b},z), \boldsymbol{x}+\boldsymbol{b}, \boldsymbol{t}+[z^{-1}]),\\ &= \oint \frac{dz}{2\pi i} z^{-k+s'-s} e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2}\tau(s'+1, y-\xi(\boldsymbol{a},z^{-1}), \boldsymbol{x}+\boldsymbol{a}, \boldsymbol{t}'+[z])\\ &\qquad\times \tau(s-1, y-\xi(\boldsymbol{b},z^{-1}), \boldsymbol{x}+\boldsymbol{b}, \boldsymbol{t}-[z]).\end{aligned} \tag{3.10}$$

4 Logarithmic extension

4.1 Lax equations

Following Carlet, Dubrovin and Zhang [5], we define the logarithm $\log\mathcal{L}$ of the Lax operator $\mathcal{L}$ as

$$\log\mathcal{L} = \frac{1}{2}W\partial_s W^{-1} - \frac{1}{2}\bar{W}\partial_s\bar{W}^{-1}.$$

This definition can be rewritten as

$$\log\mathcal{L} = -\frac{1}{2}[\partial_s, W]W^{-1} + \frac{1}{2}[\partial_s, \bar{W}]\bar{W}^{-1} = -\frac{1}{2}\frac{\partial W}{\partial s}W^{-1} + \frac{1}{2}\frac{\partial \bar{W}}{\partial s}\bar{W}^{-1},$$

which shows that $\log\mathcal{L}$ becomes a difference operator (of infinite order).

The logarithmic extension of the Toda hierarchy consists of the Toda flows with respect to $\boldsymbol{t}$ and another set of commuting flows with respect to $\boldsymbol{x} = (x_1, x_2, \ldots)$. The extended flows are defined by the Lax equations [5]

$$\frac{\partial\mathcal{L}}{\partial x_n} = [C_n, \mathcal{L}], \quad n = 1, 2, \ldots, \tag{4.1}$$

where

$$C_n = (\mathcal{L}^n\log\mathcal{L})_{\geq 0} - (\mathcal{L}^n\log\mathcal{L})_{<0}\,.$$

Note that $\mathcal{L}^n\log\mathcal{L}$ can be expressed in terms of the dressing operators as

$$\mathcal{L}^n\log\mathcal{L} = \frac{1}{2}We^{n\partial_s}\partial_s W^{-1} - \frac{1}{2}\bar{W}e^{-n\partial_s}\partial_s\bar{W}^{-1}. \tag{4.2}$$

A few remarks are in order.

1. This definition of C_n's differs from the usual definition

   $$C_n = (\mathcal{L}^n(\log\mathcal{L} - c_n))_{\geq 0} - (\mathcal{L}^n(\log\mathcal{L} - c_n))_{<0}\,,$$

   where c_n's are numerical constants of the form

   $$c_n = 1 + 2 + \cdots + \frac{1}{n}$$

that plays an important role in the application to 2D topological field theories [2, 3, 4]. In the context of integrable structure, however, this difference is superficial.

2. Since C_n can be expressed as

$$C_n = 2\left(\mathcal{L}^n \log \mathcal{L}\right)_{\geq 0} - \mathcal{L}^n \log \mathcal{L} = -2\left(\mathcal{L}^n \log \mathcal{L}\right)_{<0} + \mathcal{L}^n \log \mathcal{L}$$

and $\mathcal{L}^n \log \mathcal{L}$ commutes with $\mathcal{L}$, we can rewrite the Lax equations as

$$\frac{\partial \mathcal{L}}{\partial x_n} = [2\left(\mathcal{L}^n \log \mathcal{L}\right)_{\geq 0}, \mathcal{L}] = [-2\left(\mathcal{L}^n \log \mathcal{L}\right)_{<0}, \mathcal{L}]. \tag{4.3}$$

4.2 Auxiliary linear equations

For comparison with the $2+1$D extension, let us rewrite the Lax equations (4.3). Note that $2\left(\mathcal{L}^n \log \mathcal{L}\right)_{\geq 0}$ can be expressed as

$$\begin{aligned} 2\left(\mathcal{L}^n \log \mathcal{L}\right)_{\geq 0} &= -\left(\mathcal{L}^n \frac{\partial W}{\partial s} W^{-1}\right)_{\geq 0} + \left(\mathcal{L}^n \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}\right)_{\geq 0} \\ &= P_n + \mathcal{L}^n \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}, \end{aligned}$$

where

$$P_n = -\left(\mathcal{L}^n \frac{\partial W}{\partial s} W^{-1}\right)_{\geq 0} - \left(\mathcal{L}^n \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}\right)_{<0}. \tag{4.4}$$

We can further rewrite the right hand side as

$$\begin{aligned} 2\left(\mathcal{L}^n \log \mathcal{L}\right)_{\geq 0} &= P_n + \mathcal{L}^n [\partial_s, \bar{W}] \bar{W}^{-1} \\ &= P_n + \mathcal{L}^n \partial_s - \bar{W} e^{-n\partial_s} \partial_s \bar{W}^{-1}. \end{aligned}$$

Since the last term $\bar{W} e^{-n\partial_s} \partial_s \bar{W}^{-1}$ commutes with $\mathcal{L}$, we can remove it and obtain the equations

$$\frac{\partial \mathcal{L}}{\partial x_n} = [\mathcal{L}^n \partial_s + P_n, \mathcal{L}]. \tag{4.5}$$

Written in this form, the Lax equations of the logarithmic extension exhibit remarkable similarity with the Lax equations (3.1) of the $2+1$D extensions. The only difference is that the role of y is now played by s. Thus the logarithmic extension may be thought of as a kind of dimensional reduction (identifying ∂_y with ∂_s) of the $2+1$D extension. Inspired by this observation, we can readily find the evolution equations

$$\frac{\partial W}{\partial x_n} = \mathcal{L}^n \frac{\partial W}{\partial s} + P_n W, \quad \frac{\partial \bar{W}}{\partial x_n} = \mathcal{L}^n \frac{\partial \bar{W}}{\partial s} + P_n \bar{W} \tag{4.6}$$

for the dressing operators as counterparts of (3.3).

This is, however, a place where a significant difference also shows up. In the present case, we can further rewrite (4.6) to such a form as

$$\begin{aligned}\frac{\partial W}{\partial x_n} &= (\mathcal{L}^n \partial_s + P_n)W - We^{n\partial_s}\partial_s,\\ \frac{\partial \bar{W}}{\partial x_n} &= (\mathcal{L}^n \partial_s + P_n)\bar{W} - \bar{W}e^{-n\partial_s}\partial_s,\end{aligned} \tag{4.7}$$

which rather resembles (2.3). Note here that the roles of $e^{\pm n\partial_s}/2$ in (2.3) are now played by $e^{\pm n\partial_s}\partial_s$, which are connected with $\mathcal{L}^n \log \mathcal{L}$ by the dressing operators as shown in (4.2). These "undressed" generators of time evolutions determine the exponential factors of the wave functions. The exponential factors $e^{\xi(\boldsymbol{t},z^{\pm 1})/2}$ are thus generated from z^s by the first set of generators $e^{\pm n\partial_s}/2$ as

$$\exp\left(\sum_{n=1}^{\infty} t_n e^{\pm n\partial_s}/2\right) z^s = z^s e^{\xi(\boldsymbol{t},z^{\pm 1})/2}.$$

In the same sense, the second set of generators $e^{\pm n\partial_s}\partial_s$ give the power (rather than exponential) functions $z^{\xi(\boldsymbol{x},z^{\pm 1})}$ as

$$\exp\left(\sum_{n=1}^{\infty} x_n e^{\pm n\partial_s}\partial_s\right) z^s = z^{s+\xi(\boldsymbol{x},z^{\pm 1})}.$$

Bearing the last observation in mind, we introduce the wave functions

$$\begin{aligned}\Psi(s,z) &= Wz^{s+\xi(\boldsymbol{x},z)}e^{\xi(\boldsymbol{t},z)/2} = \left(1 + \sum_{n=1}^{\infty} w_n(s)z^{-n}\right) z^{s+\xi(\boldsymbol{x},z)}e^{\xi(\boldsymbol{t},z)/2},\\ \bar{\Psi}(s,z) &= \bar{W}z^{s+\xi(\boldsymbol{x},z^{-1})}e^{\xi(\boldsymbol{t},z^{-1})/2} = \left(\sum_{n=0}^{\infty} \bar{w}_n(s)z^{n}\right) z^{s+\xi(\boldsymbol{x},z^{-1})}e^{\xi(\boldsymbol{t},z^{-1})/2}.\end{aligned}$$

(4.7) can be thereby converted to the auxiliary linear equations

$$\frac{\partial \Psi(s,z)}{\partial x_n} = (\mathcal{L}^n \partial_s + P_n)\Psi(s,z), \quad \frac{\partial \bar{\Psi}(s,z)}{\partial x_n} = (\mathcal{L}^n \partial_s + P_n)\bar{\Psi}(s,z). \tag{4.8}$$

As in the case of the 2+1-dimensional extension, these auxiliary linear equations have another expression of the form

$$\frac{\partial \Psi(s,z)}{\partial x_n} = (z^n \partial_s + Q_n)\Psi(s,z), \quad \frac{\partial \bar{\Psi}(s,z)}{\partial x_n} = (z^{-n} \partial_s + Q_n)\bar{\Psi}(s,z), \tag{4.9}$$

where

$$Q_n = P_n - \frac{\partial \mathcal{L}^n}{\partial s} = -\left(\frac{\partial W}{\partial s}e^{n\partial_s}W^{-1}\right)_{\geq 0} - \left(\frac{\partial \bar{W}}{\partial s}e^{-n\partial_s}\bar{W}^{-1}\right)_{<0}. \tag{4.10}$$

4.3 Bilinear equations

Since the structure of the auxiliary linear equations (4.9) is almost the same as those of the $2+1$D extension, we can convert these auxiliary linear equations into a bilinear form in exactly the same way. Thus, defining the dual wave functions as

$$\Psi^*(s,z) = V^* z^{-s-\xi(\boldsymbol{x},z)} e^{-\xi(\boldsymbol{t},z)/2}, \quad \bar{\Psi}^*(s,z) = \bar{V}^* z^{-s-\xi(\boldsymbol{x},z^{-1})} e^{-\xi(\boldsymbol{t},z^{-1})/2},$$

we obtain the bilinear equation

$$\begin{aligned} &\oint \frac{dz}{2\pi i} z^k \Psi(s' - \xi(\boldsymbol{a},z), \boldsymbol{x}+\boldsymbol{a}, \boldsymbol{t}', z)\Psi^*(s - \xi(\boldsymbol{b},z), \boldsymbol{x}+\boldsymbol{b}, \boldsymbol{t}, z) \\ = &\oint \frac{dz}{2\pi i} z^{-k} \bar{\Psi}(s' - \xi(\boldsymbol{a},z^{-1}), \boldsymbol{x}+\boldsymbol{a}, \boldsymbol{t}', z)\bar{\Psi}^*(s - \xi(\boldsymbol{b},z^{-1}), \boldsymbol{x}+\boldsymbol{b}, \boldsymbol{t}, z), \end{aligned} \tag{4.11}$$

which holds for $k = 0, 1, 2, \ldots$ and arbitrary values of $s', s, \boldsymbol{x}, \boldsymbol{t}', \boldsymbol{t}$ except for the condition (2.7).

It deserves to be stressed here that the integrands in the contour integrals are single-valued. The multi-valuedness of the power functions $z^{s+\xi(\boldsymbol{x},z^{\pm 1})}$ in the wave functions and the dual wave functions cancels each other. This cancellation mechanism is based on the special shift

$$s' \to s' - \xi(\boldsymbol{a}, z^{\pm 1}), \quad \boldsymbol{x} \to \boldsymbol{x}+\boldsymbol{a}, \quad s \to s - \xi(\boldsymbol{b}, z^{\pm 1}), \quad \boldsymbol{x} \to \boldsymbol{x}+\boldsymbol{b}$$

of the s and $\boldsymbol{x}$ variables in the integrand. Actually, this special shift was a main mystery of Milanov's bilinear formalism; we can now explain its origin in the $2+1$D extension.

Lastly, by the same trick as used in the derivation of (3.9) and (3.10), we can derive from (4.11) the bilinear equation

$$\begin{aligned} &\oint \frac{dz}{2\pi i} z^{k+s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)/2} \tau(s' - \xi(\boldsymbol{a},z), \boldsymbol{x}+\boldsymbol{a}, \boldsymbol{t}' - [z^{-1}]) \\ &\quad \times \tau(s - \xi(\boldsymbol{b},z), \boldsymbol{x}+\boldsymbol{b}, \boldsymbol{t} + [z^{-1}]), \\ &\quad = \oint \frac{dz}{2\pi i} z^{-k+s'-s} e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z^{-1})/2} \tau(s' + 1 - \xi(\boldsymbol{a},z^{-1}), \boldsymbol{x}+\boldsymbol{a}, \boldsymbol{t}' + [z]) \\ &\qquad \times \tau(s - 1 - \xi(\boldsymbol{b},z^{-1}), \boldsymbol{x}+\boldsymbol{b}, \boldsymbol{t} - [z]) \end{aligned} \tag{4.12}$$

for the tau function. This equation contains Milanov's bilinear equation as a special case.

5 Conclusion

We have thus shown that the $2+1$D extension and the logarithmic extension have a quite parallel structure. Relevant equations of these two extended Toda hierarchy can be paired as follows:

- Lax equations: (3.1) $\leftrightarrow$ (4.1)
- Auxiliary linear equations: (3.2), (3.4) $\leftrightarrow$ (4.8), (4.4)
- Evolution equations of dressing operators: (3.3) $\leftrightarrow$ (4.6)
- Another form of auxiliary linear equations: (3.5), (3.6) $\leftrightarrow$ (4.9), (4.10)
- Bilinear equations of wave functions: (3.8) $\leftrightarrow$ (4.11)
- Bilinear equations of tau functions: (3.10) $\leftrightarrow$ (4.12)

A new feature of the logarithmic extension is the emergence of the multi-valued factor $z^{\xi(\boldsymbol{x},z^{\pm 1})}$ in the wave functions. The multi-valuedness, however, disappears in the integrand of the bilinear equations. This fact plays a role in the heuristic part of Milanov's derivation of bilinear equations [6]. In our approach, this cancellation mechanics of multi-valuedness is rather a consequence of dimensional reduction of the $2+1$D extension.

Our approach can be readily generalized to the reduction of the 2D Toda hierarchy defined by the constraint

$$(\mathcal{L} :=)\ L^N = \bar{L}^{-\bar{N}},$$

where N and $\bar{N}$ are arbitrary positive integers. The reduced Lax operator $\mathcal{L}$ thus defined takes such a form as

$$\mathcal{L} = B_N + \bar{B}_{\bar{N}} = e^{N\partial_s} + b_1(s)e^{(N-1)\partial_s} + b_N(s) + c_1(s)e^{-\partial_s} + \cdots + c_{\bar{N}}(s)e^{-\bar{N}\partial_s}.$$

The logarithmic extension of this reduced hierarchy coincides with Carlet's "extended bigraded Toda hierarchy" [17]. We can derive bilinear equations for the wave functions and the tau functions, which contains bilinear equations derived by Milanov and Tseng [18] as a special case.

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References

[1] B. Dubrovin, Geometry of 2d topological field theories, in "Integrable systems and quantum groups", Lecture Notes in Math. vol. 1620 (Springer, Berlin, 1996), pp. 120–348.

[2] T. Eguchi and S.-K. Yang, The topological CP^1 model and the large-N matrix integral, Modern Phys. Lett. **A9** (1994), 2893–2902.

[3] E. Getzler, The Toda conjecture, in K. Fukaya et al. (eds.), "Symplectic geometry and mirror symmetry" (World Scientific, Singapore, 2001), pp. 51–79.

[4] Y. Zhang, On the CP^1 topological sigma model and the Toda lattice hierarchy, J. Geom. Phys. **40** (2002), 215–232.

[5] G. Carlet, B. Dubrovin and Y. Zhang, The extended Toda hierarchy, Moscow Math. J. **4** (2004), 313-332, 534.

[6] T.E. Milanov, Hirota quadratic equations for the extended Toda hierarchy, Duke Math. J. **138** (2007), 161–178.

[7] M. Jimbo and T. Miwa, Soliton equations and infinite dimensional Lie algebras, Publ. RIMS, Kyoto University, **19** (1983), 943–1001.

[8] F. Calogero, A method to generate solvable nonlinear evolution equations, Lett. Nuovo Cimento **14** (1975), 443–447.

[9] O.I. Bogoyavlensky, Breaking solitons in $2+1$-dimensional integrable systems, Russian Math. Surveys **45:4** (1990), 1–86.

[10] J. Schiff, Integrability of Chern-Simons-Higgs vortex equations and an reduction of the self-dual Yang-Mills equations to three dimensions, in "Painlevé Transcendents", NATO ASI Series B, vol. 278 (Plenumn Press, 1992), pp. 393–405.

[11] Y. Billig, An extension of the KdV hierarchy arising from a representation of a toroidal Lie algebra, J. Algebra **217** (1999), 40–64.

[12] K. Iohara, Y. Saito and M. Wakimoto, Hirota bilinear forms with 2-toroidal symmetry, Phys. Lett. **A254** (1999), 37–46; Notes on differential equations arising from a representation of 2-toroidal Lie algebras, Prog. Theor. Phys. **135** (1999), 166–181.

[13] T. Ikeda and K. Takasaki, Toroidal Lie algebras and Bogoyavlensky's $(2+1)$-dimensional equation, Intern. Math. Res. Notices **7** (2001), 329–369.

[14] S. Kakei, T. Ikeda and K. Takasaki, Hierarchy of $(2+1)$-dimensional nonlinear Scrödinger equations, self-dual Yang-Mills equation, and toroidal Lie algebras, Ann. Henri Poincaré **3** (2002), 817–845.

[15] K. Ueno and K. Takasaki, The Toda lattice hierarchy, in "Group Representations and Systems of Differential Equations", Adv. Pure Math. vol. 4 (North-Holland and Kinokuniya, 1984), pp. 1–95.

[16] Y. Ogawa, On the $(2+1)$-dimensional extension of 1-dimensional Toda lattice hierarchy, J. Nonlin. Math. Phys. **15** (2008), 48–65.

[17] G. Carlet, The extended bigraded Toda hierarchy, J. Phys. A: Math. Gen. **39** (2006), 9411–9435.

[18] T.E. Milanov and H.H Tseng, The spaces of Laurent polynomials, Gromov-Witten theory of P^1-orbifolds, and integrable hierarchies, J. Reine Angew. Math. **622** (2008), 189–235.