A mapping function approach applied to some classes of nonlinear equations

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Abstract

In this work, we study some models of scalar fields in 1+1 dimensions with non-linear self-interactions. Here, we show how it is possible to extend the solutions recently reported in the literature for some classes of nonlinear equations like the nonlinear Klein-Gordon equation, the generalized Camassa-Holm and the Benjamin-Bona-Mahony equations. It is shown that the solutions obtained by Yomba [1], when using the so-called auxiliary equation method, can be reached by mapping them into some known nonlinear equations. This is achieved through a suitable sequence of translation and power-like transformations. Particularly, the parent-like equations used here are the ones for the $\lambda\phi^4$ model and the Weierstrass equation. This last one, allow us to get oscillating solutions for the models under analysis. We also systematize the approach in order to show how to get a larger class of nonlinear equations which, as far as we know, were not taken into account in the literature up to now.

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In the last few years, a growing number of works have been devoted to obtain novel analytical solutions for some classes of nonlinear differential equations, like the nonlinear Klein-Gordon equation, the generalized Camassa-Holm and Benjamin-Bona-Mahony equations. For this, many approaches were developed. Here we intend to show that through a sequence of transformations, were one alternates translations and power-like transformations, some of the solutions reported recently in the literature can be reproduced and, better, increased. In particular, we treat the cases studied by E. Yomba in a recent work [1], as well as by J. Nickel [2]. In those works, the authors introduced interesting methods in order to deal essentially with an auxiliary equation like

$$\left(\frac{dF(s)}{ds}\right)^2 = \sum_{i=0}^{N} h_i\, F^i. \tag{1}$$

The idea is to show that those solutions can be obtained from a direct mapping with an already known equation like, for instance, the one coming from the so-called $\lambda\phi^4$ model.

Let us begin with the Cases 1 and 2 of Yomba [1]. Starting with the equation

$$\left(\frac{d\phi(s)}{ds}\right)^2 = A\,\phi^4 + B\,\phi^2 + C, \tag{2}$$

and performing the transformation $\phi = \left(F^{-2} + \beta\right)^{-\frac{1}{2}}$, we arrive at

$$\left(\frac{dF(s)}{ds}\right)^2 = C + (B + 3\beta C)\,F^2 + \left(A + 2\beta B + 3\beta^2 C\right)F^4 + \beta\left(A + \beta\,B + \beta^2\,C\right)F^6, \tag{3}$$

which is clearly written in the form appearing in the work [1], namely

$$\left(\frac{dF(s)}{ds}\right)^2 = h_0 + h_2\,F^2 + h_4\,F^4 + h_6\,F^6. \tag{4}$$

At this point we should observe that, in order to make contact with the solution appearing in [1], we must fix the arbitrary parameter β such that $\beta = \frac{3\,h_4}{8\,h_2}$. Once this particular choice is made, the restrictions appearing in the Case 1 of [1] ($h_0 = \frac{8\,h_2^2}{27\,h_4}$, $h_6 = \frac{h_4^2}{4\,h_2}$) are recovered. So, one can conclude that the freedom introduced by means of the arbitrary parameter β, allows us to get less restrictive solutions.

In fact, as we are going to see below, this mapping allows to get, besides the trigonometric and hyperbolic solutions, some other oscillating ones coming from the solutions of the Weierstrass equation [3], [4]. Furthermore, as it can be seen, if one chooses the particular case where $C = 0$ (Case 2) in the equation (2), one obtains the relations,

$$B = h_2,\ A = \epsilon\sqrt{\Delta},\ \beta = \frac{h_4 + \epsilon\sqrt{\Delta}}{2\,h_2}, \tag{5}$$

where $\epsilon = \pm 1$ and $\Delta = h_4^2 - 4\,h_2 h_6$, as defined in the Table 2 of [1]. Note that, due to the restricted choice of the parameter C ($C = 0$), the solutions of equation (2) shall be given by

$$\phi(s) = \frac{2\,h_2\,e^{\pm\sqrt{h_2}s}}{1 - \epsilon\,h_2\,\Delta\,e^{\pm 2\sqrt{h_2}s}}, \tag{6}$$

and the solution for the function $F(s)$ is simply obtained by direct substitution in the equation $F_{(2)} = \pm\left(\phi^2 - \beta\right)^{-1/2}$.

Before going further in our analysis, we should mention that the Case 5 of [1] can be obtained from this last case treated above, by making the additional transformation $F_{(2)} = \pm F_{(5)}^{1/2}$. Then, we obtain the following equation

$$\left(\frac{dF_{(5)}(s)}{ds}\right)^2 = h_{2(5)}\,F_{(5)}^2 + h_{3(5)}\,F_{(5)}^3 + h_{4(5)}\,F_{(5)}^4, \tag{7}$$

where one shall make the identifications: $h_{2(5)} = 4\,h_{2(2)}$, $h_{3(5)} = 4\,h_{4(2)}$ and $h_{4(5)} = 4\,h_{6(2)}$. Thus, we see that the *discriminant* defined for this case comes from the one of the Case 2 and is given by $\Delta_{(5)} = h_{3(5)}^2 - 4\,h_{2(5)}\,h_{4(5)} = 16\ \Delta_{(2)}$. The notation $h_{i(j)}$ stands for the i-th coefficient of the j-th case.

Now, we define "generations" of equations which can be obtained from the important nonlinear Weierstrass differential equation

$$\left(\frac{d\wp}{ds}\right)^2 = 4\,\wp^3 - g_2\,\wp - g_3, \tag{8}$$

which, as it is known, can present solutions in many regimes, according to the specific choice of the *invariants* g_2 and g_3 and of the *discriminant* $\Delta = {g_2}^3 - 27{g_3}^2$. Those solutions include, trigonometric and hyperbolic solutions as well as other oscillating solutions in terms of the Jacobi elliptic functions.

What we call the "zero-th" generation is trivially obtained from the Weierstrass equation by simply performing a translation and a dilation like $F_0(s) = a\,\wp(s) + b$. In this generation, one has a quadratic term in the differential equation. One can check that the Case 4 of [1] is completely recovered, including the same constraints relating the coefficients of the polynomial on the right hand side of the equation.

The next generation is obtained by doing the transformation $\wp(s) = r\,(F_I)^\alpha$ (r and α are arbitrary constants) and, if one is interested only in polynomials with positive and integer degrees in F_I, one must choose $\alpha = -1$ or $\alpha = -2$. In such cases, one obtains:

$$\left(\frac{dF_{Ia}(s)}{ds}\right)^2 = 4\,r\,F_{Ia} - \frac{g_2}{r}\,F_{Ia}^3 - \frac{g_3}{r^2}\,F_{Ia}^4, \;\; \text{for}\,\alpha = -1, \tag{9}$$

and

$$\left(\frac{dF_{Ib}(s)}{ds}\right)^2 = r - \frac{g_2}{4\,r}F_{Ib}^4 - \frac{g_3}{4\,r^2}F_{Ib}^6, \;\; \text{for}\,\alpha = -2. \tag{10}$$

The second generation comes from a translation performed on the functions of the first generation. For instance, by doing the transformation $F_{Ia}(s) = F_{IIa}(s) + k$ one is left with

$$\left(\frac{dF_{IIa}(s)}{ds}\right)^2 = h_0 + h_1\,F_{IIa} + h_2\,F_{IIa}^2 + h_3\,F_{IIa}^3 + h_4\,F_{IIa}^4, \tag{11}$$

where

$$\begin{aligned} h_0 &= -\frac{k}{r^2}\left(g_3 k^3 + g_2\,r\,k^2 - 4\,r^3\right), \;\; h_1 = \frac{1}{k\,r^2}\left(4\,r^3 - 4\,k^3 g_3 - 3\,r\,k^2 g_2\right), \\ h_2 &= -\frac{1}{r^2}\left[3\,k\,(2\,k\,g_3 + r\,g_2)\right], \;\; h_3 = -\frac{1}{r^2}\,(4\,k\,g_3 + r\,g_2) \;\text{ and }\; h_4 = -\frac{g_3}{r^2}. \end{aligned}$$

It is evident that we have no longer the coefficients independent of each other. At this point we can make the identifications defined in [1], $\alpha = h_4$, $\beta = \frac{h_3}{4}$, $\gamma = \frac{h_2}{6}$, $\epsilon = h_0$, and verify that the constraints appearing in the Case 3 of [1] and also in [2], namely

$$g_2 = \alpha\,\epsilon - 4\,\beta\,\delta + 3\,\gamma^2 \;\text{ and }\; g_3 = \alpha\,\gamma\,\epsilon + 2\,\beta\,\gamma\,\delta - \alpha\,\delta^2 - \gamma^3 - \epsilon\,\beta^2$$

are reproduced and, as a consequence, the discriminant of the system will be necessarily the same one for the Weierstrass equation: $\Delta = g_2^3 - 27\,g_3^2$. In this case the solution of the functions F_{IIa} are written in terms of the Weierstrass function as $F_{IIa}(s) = r\wp^{-1}(s) - k$.

Finally, the Case 6 of [1] is precisely the equation (2), which we have commenced the mapping of the first two cases of [1].

Now, we show that, for the cases where the three constants A, B, C are arbitrary, equation (2) can be mapped into the Weierstrass equation. For this we can set $\phi(s) = \sqrt{C}\,\left(\wp(s) - \frac{B}{3}\right)^{-\frac{1}{2}}$ or $\phi(s) = \frac{1}{\sqrt{A}}\left(\wp(s) - \frac{B}{3}\right)^{\frac{1}{2}}$, leading us to the equation (8) with

$$g_2 = 4\,C\left(\frac{B^2}{3\,C} - A\right) \;\text{ and }\; g_3 = 4\,C\,B\left(\frac{A}{3} - \frac{2\,B^2}{27\,C}\right). \tag{12}$$

for the first case or

$$g_2 = 4\,A\left(\frac{B^2}{3\,A} - C\right) \text{ and } g_3 = 4\,A\,B\left(\frac{C}{3} - \frac{2\,B^2}{27\,A}\right). \tag{13}$$

in the second case. In fact, if one is interested in non-singular solutions, one must choose the first solution, since the Weierstrass function always presents singular points which, in that case, will lead to well-behaved solutions. In the Figure 1, some examples of those continuous solutions are presented.

Since we have shown how to obtain all the cases studied in [1] by means of a mapping approach, we can go further with the procedure by getting novel equations whose solution can be reached by using this method. In fact, we can obtain terms of higher degrees polynomial on the right hand side of the auxiliary differential through a suitable combination of power-like and translation transformations on the Weierstrass differential equation. However, in order to achieve this goal, one must choose the translation constant appropriately. For example, let us start with the Weierstrass equation and make the transformation $\wp(s) = (k + F(s)^2)^{-2}$, which leads us to the equation

$$\left(\frac{dF(s)}{ds}\right)^2 = a_0 + a_2\,F^2 + a_4\,F^4 + a_6\,F^6 + a_8\,F^8 + a_{10}\,F^{10}, \tag{14}$$

where we have fixed the invariant g_3 as $g_3 = \frac{4\,-\,g_2\,k^4}{k^6}$ in order to avoid a term with negative power on the right hand side of the differential equation. Under this condition, the coefficients a_i are given by

$$\begin{aligned} a_0 &= (-12 + g_2k^4)/(8\,k),\ \ a_2 = 3(-20 + 3g_2k^4)/16k^2,\ \ a_4 = (-5 + g_2k^4)/k^3, \\ a_6 &= (-30 + 7g_2k^4)/8k^4,\ \ a_8 = 3(-4 + g_2k^4)/8k^5,\ \ a_{10} = (-4 + g_2k^4)/16k^6. \end{aligned} \tag{15}$$

Note that, even if we include an additional parameter through a scaling of the function, we still have only three free parameters, the remaining ones should be fixed in order to grant the exact solvability of the equation. Some solutions for this example are illustrated in the Figure 2.

One can see that the approach developed here generates higher order exactly solvable differential equations. We think that those can be useful to a better understanding of some classes of nonlinear equations like the generalized Camassa-Holm and the Benjamin-Bona-Mahony equations, as done by Yomba [1] and the nonlinear Klein-Gordon equation and their generalizations as the sine-Gordon, the sinh-Gordon and their modified versions, as considered by [5].

In order to illustrate the mapping approach even more, we would like to consider here non-linear models involving the Jacobi elliptic functions, particularly what has been called the Jacobi model [6], whose auxiliary differential equation is

$$\left(\frac{d\phi}{ds}\right)^2 = 4\ sn^2(\frac{\phi}{2}|m) - 4c^2, \tag{16}$$

where c is a constant of the integration and $sn(\varphi|m)$ is the Jacobi elliptic function defined by $sn(\varphi, m) = \sin\theta$, with respect to the integral $\varphi = \int_0^\theta d\alpha/(1-m\sin^2\alpha)^{1/2}$, where θ is called the *amplitude* and m $(0 \leq m \leq 1)$ is the *elliptic parameter*. The equation (16) retrieves the one for the so-called sine-Gordon model for $m = 0$. We have noted, by using the mapping approach, that the above equation has oscillating solutions for $0 < c^2 < 1$ and soliton solutions for $c^2 = 0$. The redefinition of the fields $F(s) = cn(\varphi(s)|m)$, where $cn(\varphi|m)$ is the Jacobi elliptic function defined by $cn(\varphi|m) = \cos\theta$, maps the equation (16) into an equation identical to (1) with $N = 6$ and the coefficients given by

$$h_0 = 1 - m - c^2 + c^2 m,\ h_2 = -2 + c^2 + 3m - 2c^2 m,\ h_4 = 1 - 3m + c^2 m,\ h_6 = m. \tag{17}$$

By using the following mapping

$$F = \pm\sqrt{h_0}(\wp + \kappa)^{-1/2} \tag{18}$$

with $\kappa = 1/3(2 - c^2 - 3m + 2c^2 m)$, we recover the Weierstrass differential equation (8) satisfied by the Weierstrass elliptic function $\wp(s; g_2, g_3)$ with the invariants

$$\begin{aligned} g_2 &= 4/3(1 - c^2(1+m) + c^4(1 - m + m^2)), \\ g_3 &= -4/27(2 - 3c^2(1+m) - 3c^4(1 - 4m + m^2) + c^6(2 - 3m - 3m^2 + 2m^3). \end{aligned} \tag{19}$$

At this point we can use the relation 18.9.11 of the reference [4], which relates the Weierstrass and the Jacobi elliptic functions, since the discriminant $\Delta > 0$. Then we obtain

$$\wp(s) = e_3 + (e_1 - e_3)/sn^2((e_1 - e_3)^{1/2} s|\nu), \tag{20}$$

where

$$e_1 = 1/3(1 + c^2 - 2c^2 m),\ e_2 = 1/3(1 - 2c^2 + c^2 m),\ e_3 = 1/3(-2 + c^2 + c^2 m) \tag{21}$$

and

$$\nu = (e_2 - e_3)/(e_1 - e_3) = (1 - c^2)(1 - c^2 m). \tag{22}$$

Finally the solution for the Jacobi elliptic model can be written as

$$\phi_{Jacobi}(s) = 2\ cn^{-1}(F(s)|m), \tag{23}$$

with

$$F(s) = \pm\frac{\sqrt{(1-c^2)(1-m)}sn((1-c^2m)s|\nu)}{\sqrt{(1-c^2m)-(1-c^2)m\ sn^2((1-c^2m)s|\nu)}}. \tag{24}$$

The above expression is the solution for the equation (1) with $N = 6$ and coefficients h_i given in (17). According to the classification scheme mentioned above, it belongs to the 2nd generation (F_{IIb}). By taking $c = 0$ in equations (22)-(24) we obtain the *(anti-)kink soliton* solutions for the Jacobi elliptic model

$$\phi_{Jacobi}(s) = 2\ cn^{-1}\left(\pm\frac{\sqrt{(1-m)}\tanh s}{\sqrt{1-m\tanh^2 s}}\,\middle|\, m\right) \tag{25}$$

The *kink* solution above can be rewritten as $\phi_{kink-Jacobi}(s) = 2K[m] + 2\ sn^{-1}(\tanh s|m)$, which is the *kink* found in [6], and $K[m]$ is the elliptic quarter period.

The solutions for the sine-Gordon model are retrieved by taking $m = 0$. In this limit the differential equation for $F(s)$ is the equation (1) with $N = 4$; the coefficients can be obtained from (17) by taking $m = 0$. The solution for the sine-Gordon model is

$$\phi_{s-G}(s) = 2\cos^{-1}(\pm\sqrt{1-c^2}sn(s|1-c^2)), \tag{26}$$

which is in agreement with the solution presented in [7] and [8] with an adequate reparametrization of the variable s and a reescaling of the field. For $c = 0$ we obtain the *(anti-)kink* solution for the sine-Gordon model

$$\phi_{s-G}(s) = 2\cos^{-1}(\pm\tanh s) = 4\tan^{-1}(e^{\mp s}). \tag{27}$$

Finally we would like to mention that generalizations of non-polynomial non-linear models as for instance, the generalized versions of the sine-Gordon and the sinh-Gordon models [5], can also be approached by means of this mapping function method described here.

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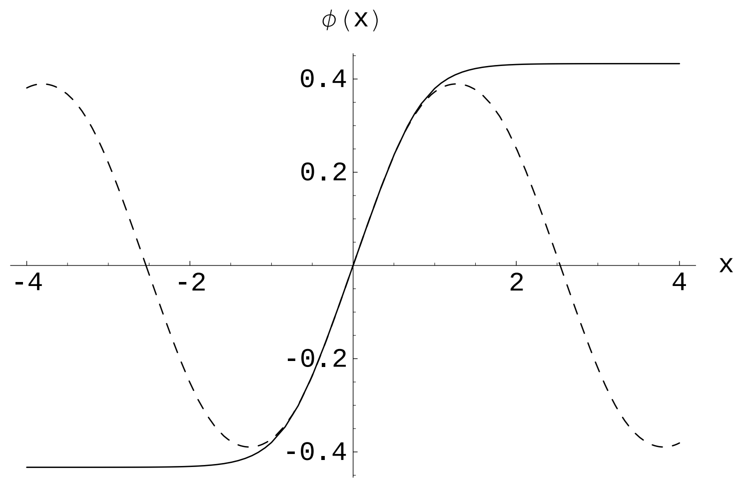

Figure 1: Solutions of the equation (2) with $\Delta = 0$, $g_2 = 12$ and $g_3 = -8$ (solid line) and $\Delta > 0$, $g_2 = 13$ and $g_3 = -8$ (dashed line)

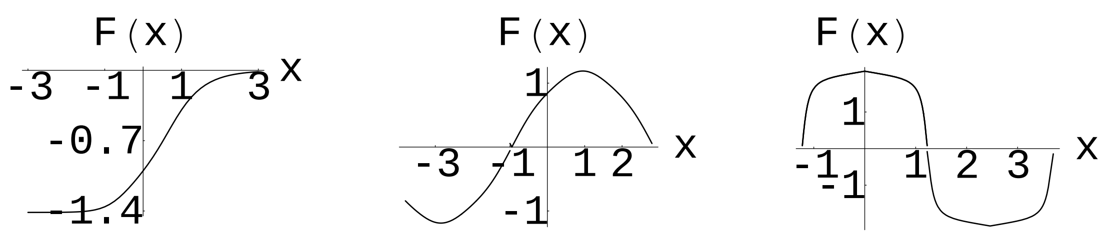

Figure 2: Solutions of the equation (14) for $\Delta = 0$, $g_2 = 12$ and $g_3 = -8$ (left figure); $\Delta > 0$, $g_2 = 20$ and $g_3 = -8$ (middle figure) and $\Delta < 0$, $g_2 = -20$ and $g_3 = 1$ (right figure)