

# On the stability of some exact solutions to the generalized convection-reaction-diffusion equation

*V. Vladimirov and Cz. Mączka*

Faculty of Applied Mathematics,  
AGH University of Science and Technology,  
Mickiewicz Avenue 30, 30-059 Kraków, PL  
Email address: vsevolod.vladimirov@gmail.com

**Abstract** Stability of a set of travelling wave solutions to the hyperbolic generalization of the convection-reaction-diffusion equation is studied by means of the qualitative methods and numerical simulation.

## 1 Introduction

In recent decades significant attention was paid to the study of the family of convection-reaction-diffusion equations

$$u_t = [\kappa(u) u_x]_x + a(u) u_x + f(u). \quad (1)$$

Equations belonging to this family describe a number of natural phenomena, such as transport in porous media, or the motion of a thin sheet of viscous liquid over the inclined plate (see [1] and the literature therein). This class also contains a nonlinear generalization of the Fokker-Plank equation [2] and a number of models encountered in the biological sciences [3]. Another source of inspiration for studying the convection-reaction-diffusion equations results from the fact that the equation (1) represents one of the simplest nonlinear models describing phenomena of patterns formation and evolution [4, 5]. It is, perhaps, the combination of relative simplicity and richness of physical contents, that made the family (1) the objective of numerous studies within the symmetry approach, purposed at constructing nontrivial exact solutions and finding out the conservation laws [6]–[10].

In this paper, we consider the following evolutionary equation (referred to as GBE):

$$\alpha u_{tt} + u_t + \mu u u_x - \kappa u_{xx} = f(u). \quad (2)$$

Here  $\mu, \kappa$  are positive constants,  $\alpha$  is nonnegative,  $f(u)$  is a smooth (polynomial) function, which will be specified later on. Equation (2) is a hyperbolic generalization of the convection-reaction-diffusion equation. Let us note, that the term  $\alpha u_{tt}$  appears when the memory effects are taken into account [11]–[14]. Equation (2), as well as its numerous modifications, were intensely studied in recent years within the generalized symmetry approach [10], [14]–[19]. Owing to these studies, the analytical description of a large variety of traveling wave (TW) solutions is actually available, including interacting traveling fronts, soliton-like solutions, periodic waves, compactons, shock fronts and many other. Undoubtedly, knowledge of exact solutions to a non-linear PDE

is a great advantage. At the same time, individual exact solution is interesting and important from the point of view of applications, if it is typical in some sense to the equation under consideration. In most noteworthy cases, self-similar exact solutions serve as the intermediate (or the true) asymptotics [20]–[22], manifesting attracting features.

The first stage towards the estimation of validity of the exact solution is a study of its stability, and this is the main topic of the present work. We formulate the conditions which guarantee the stability of some class of TW solutions to the equation (2), obtained in [16]. The structure of the study is following. In section 2 we present a family of exact TW solutions, satisfying under certain conditions equation (2) and formulate the conditions that guarantee the stability of some exact solutions in explicit form. In section 3 we construct the numerical scheme based on the Godunov method [23]–[25] and bring the results of numerical simulations, backing the qualitative study and partly completing it. In the last section we briefly summarize the results obtained and outline the ways of further investigations.

## 2 Stability analysis of the exact solutions to the equation (2)

### 2.1 Statement of the problem

Let us reformulate the results obtained in [16] for the equation (2), assuming that

$$f(u) = \nu (u - m_1) (u - m_2) (u - m_3),$$

where  $m_k$ ,  $k = 1, 2, 3$ , are constant parameters. We are looking for the TW solutions

$$u(t, x) = U(z) \equiv U(x - Vt), \quad (3)$$

where  $V$  is a constant velocity of the wave pack. After the formal substitution of the travelling wave ansatz (3) into the equation (2), one obtains a nonlinear second order ODE

$$(\alpha V^2 - \kappa) \ddot{U} + \dot{U} (\mu U - V) = \nu (U - m_1) (U - m_2) (U - m_3),$$

which is, generally speaking, non-integrable. In order to obtain the exact TW solutions, we employ a Hirota-like ansatz  $u(t, x) = \frac{\Psi'(\xi)}{\Psi(\xi)}$ , which, being substituted to (2), leads to the following third-order ODE:

$$\begin{aligned} \Psi^2 \left[ \Delta \Psi''' - V \Psi'' - \nu \sum_{i \neq j} m_i m_j \Psi' + \nu m_1 m_2 m_3 \Psi \right] + \\ + \Psi \Psi' \left[ (\mu - 3 \Delta) \Psi'' + \left( V + \nu \sum_{i=1}^3 m_i \right) \Psi' \right] + \\ + (2\Delta - \mu - \nu) (\Psi')^3 = 0, \end{aligned} \quad (4)$$

where  $\Delta = \alpha V^2 - \kappa$ . For physical reasons [27], we assume that  $\Delta > 0$ .

On first sight, the equation (4) is even more complicated than that obtained by the convenient TW ansatz (3). But as it easily seen, (4) reduces to the linear equation

$$\Delta \Psi''' - V \Psi'' - \nu [(m_1 m_2 + m_1 m_3 + m_2 m_3) \Psi' - m_1 m_2 m_3 \Psi] = 0, \quad (5)$$

provided that

$$\mu = 3 \Delta, \quad (6)$$

$$\nu = -\Delta, \quad (7)$$

$$V = (m_1 + m_2 + m_3) \Delta. \quad (8)$$

As it was shown in [16], the roots of the characteristic equation, corresponding to the linear equation (5), coincide with the parameters  $m_1, m_2, m_3$  when the restrictions (6)-(8) take place. This enables us to formulate the following result:

**Theorem 1.** Let the equalities (6)-(8) take place. Then, depending on the values of the parameters  $\{m_k\}_{k=1}^3$ , the equation (2) has the following exact solutions:

1.

$$u(t, x) = \frac{m_1 C_1 \exp [m_1 z] + m_2 C_2 \exp [m_2 z] + m_3 C_3 \exp [m_3 z]}{C_1 \exp [m_1 z] + C_2 \exp [m_2 z] + C_3 \exp [m_3 z]} \quad (9)$$

if  $m_1 \neq m_2 \neq m_3 \neq m_1$ ;

2.

$$u(t, x) = \frac{m_1 C_1 \exp [m_1 z] + \exp [m_2 z] [C_2 m_2 + C_3 + m_2 C_3 z]}{C_1 \exp [m_1 z] + \exp [m_2 z] [C_2 + z C_3]} \quad (10)$$

if  $m_1 \neq m_2 = m_3$ ;

3.

$$u(t, x) = m + \frac{C_2 + 2 z}{C_3 + z (C_2 + z)} \quad (11)$$

if  $m_1 = m_2 = m_3 = m$ ;

4.

$$u(t, x) = \frac{m_3 C_3 \exp [m_3 z] + 2 \exp [\alpha \xi] [\alpha \cos (\beta z) - \beta \sin \beta z]}{C_3 \exp [m_3 z] + 2 \exp [\alpha z] \cos (\beta z)}, \quad (12)$$

if  $m_3$  is real, while  $\bar{m}_2 = m_1 = \alpha + i \beta$ ,  $\alpha, \beta \in R$ .

**Remark.** Using (6)-(8), and the inequality  $\Delta > 0$ , one easily gets the following expression for the wave pack velocity:

$$V = \Delta \sum_{i=1}^3 m_i = \frac{1 + \sqrt{1 + 4 \alpha \kappa (\sum_{i=1}^3 m_i)^2}}{2 \alpha \sum_{i=1}^3 m_i}. \quad (13)$$

In order to study the stability of TV solutions, depending in fact on a single variable  $z = t - Vx$ , it is instructive to pass to new independent variables

$$\bar{t} = t, \quad \bar{z} = x - Vt,$$

in which the invariant solutions (9)–(12) become stationary. In the new variables the equation (2) reads as follows:

$$\alpha \left[ \frac{\partial}{\partial \bar{t}} - V \frac{\partial}{\partial \bar{z}} \right]^2 u + \left[ \frac{\partial}{\partial \bar{t}} - V \frac{\partial}{\partial \bar{z}} \right] u + \mu u \frac{\partial u}{\partial \bar{z}} - \kappa \frac{\partial^2 u}{\partial \bar{z}^2} = f(u) \quad (14)$$

(for simplicity, we omit the bars over the independent variables henceforth).

On studying the stability of stationary solutions, we proceed in the standard way, presenting the perturbed solution in the form

$$u(t, x) = U(z) + \epsilon \exp[-\lambda t] g(z), \quad (15)$$

where  $U(z)$  denotes one of the TW solutions described by the theorem 1. Up to  $O(\epsilon^2)$ , the function  $g(z)$  satisfies the equation

$$\Delta g''(z) + (\mu U(z) + 2\alpha V\lambda - V) g'(z) + \left[ \alpha \lambda^2 - \nu \sum_{i \neq j} m_i m_j + 2\nu U(z) \sum_{i=1}^3 m_i - 3\nu U^2(z) + \mu U'(z) \right] g(z).$$

For technical reasons, it is instructive to get rid of the terms proportional to  $g'(z)$ , and this can be done by the substitution

$$g(z) = h(z) \exp[\varphi(z)]. \quad (16)$$

One easily verifies by the direct inspection, that the following statement holds true:

**Lemma 1.** If

$$\varphi'(z) = -\frac{\mu U(z) + 2\alpha V\lambda - V}{2\Delta},$$

then the function  $h(z)$  satisfies the equation

$$\hat{L}[h(z), \lambda] = 4\alpha\kappa h(z)\lambda^2 - 4B(z)h(z)\lambda + \Delta^2 h(z)K(z) - 4\Delta^2 h''(z) = 0, \quad (17)$$

where

$$B(z) = \kappa - 3\alpha\Delta^2 U(z) \sum_{i=1}^3 m_i, \quad (18)$$

$$K(z) = \sum_{i=1}^3 m_i^2 - 2 \sum_{i \neq j} m_i m_j + 2U(z) \sum_{i=1}^3 m_i - 3U^2(z) - 6U'(z). \quad (19)$$

So, using the ansatz (15), followed by the substitution (16), we get the generalized eigenvalue problem (17). Evidently, the stability of the self-similar solution  $U(z)$  can be achieved, if all possible values of the parameter  $\lambda$  are positive.

In what follows, we'll restrict our consideration to a family of perturbations, vanishing beyond some compact set  $\langle -L, L \rangle$ . With such a restriction, we get the eigenvalue problem

$$\hat{L}[h(z), \lambda] = 0, \quad h(-L) = h(L) = 0. \quad (20)$$

Let us note, that in the parabolic case, i.e., when  $\alpha = 0$ , (20) reduces to the standard Sturm-Liouville Boundary Value Problem. Although the eigenvalue problem we deal with differs from the classical one, the main conclusions concerning the properties of the eigenvectors and eigenfunctions remain the same under quite general assumptions [26], satisfied with certain by the functions  $B(z)$  and  $K(z)$ .

In order to obtain the restrictions on the signs of the eigenvalues  $\lambda$ , we multiply the equation (17) by  $h(z)$  and then integrate the resulting equation over  $z$  from  $-L$  to  $L$ . As a result, we obtain the quadratic equation with respect to  $\lambda$ :

$$\lambda^2 - b\lambda + r = 0, \quad (21)$$

where

$$b = \frac{\int_{-L}^L B(z)h(z)^2 dz}{\alpha \kappa \|h\|^2}, \quad r = \frac{\Delta^2 \int_{-L}^L K(z)h(z)^2 dz + 4 \Delta^2 \|h'\|^2}{4 \alpha \kappa \|h\|^2},$$

$\|h\|^2 = \int_{-L}^L h(z)^2 dz$ ,  $\|h'\|^2 = \int_{-L}^L h'(z)^2 dz$ . From the above formulae, we get the following relations concerning the roots of the quadratic equations:

$$\lambda_1 + \lambda_2 = \frac{\int_{-L}^L B(z)h(z)^2 dz}{\alpha \kappa \|h\|^2}, \quad (22)$$

$$\lambda_1 \lambda_2 = \Delta^2 \frac{\int_{-L}^L K(z)h(z)^2 dz + 4 \|h'\|^2}{4 \alpha \kappa \|h\|^2}. \quad (23)$$

This immediately leads us to the statement:

**Proposition 1.** In order that the eigenvalues  $\lambda_k$ ,  $k = 1, 2$  be positive, it is sufficient that the functions  $B(z)$  and  $K(z)$ , restricted to the segment  $\langle -L, L \rangle$ , satisfy the following inequalities:

$$B(z) > 0, \quad K(z) \geq 0. \quad (24)$$

Below we pose the conditions that guarantee the fulfillment of the inequalities (24) for some exact invariant solutions to the equation (2).

## 2.2 Stability analysis of the solution (9)

We restrict our consideration to the real constants  $\{m_i\}_{i=1}^3$ . Without the loss of generality, we can assume that they are ordered as follows:  $0 \leq m_1 \leq m_2 \leq m_3$ . We assume in addition that the constant  $C_1$  is nonzero, and the solution (9) can be rewritten in the form

$$U(z) = \frac{\Psi'(z)}{\Psi(z)}, \quad \Psi(z) = \exp[m_1 z] + C_2 \exp[m_2 z] + C_3 \exp[m_3 z] \quad (25)$$

The solution (25) occurs to possess the following property:

**Lemma 2.** Function (25) is monotonic for any positive  $C_2$  and  $C_3$  and satisfies the inequalities

$$m_1 < U(z) < m_3. \quad (26)$$

*Proof.* Since the derivative of  $U(z)$  is expressed by the formula

$$U'(z) = \frac{\Psi''(z)\Psi(z) - \Psi'(z)^2}{\Psi(z)^2},$$

we concentrate upon the estimation of the sign of the numerator. After some algebraic manipulation, performed with the help of *Mathematica* package (and which can be easily verified manually), we get the inequality

$$\begin{aligned} \Psi''(z)\Psi(z) - \Psi'(z)^2 &= C_2 \exp[m_2 z] \{ \exp[m_1 z] (m_1 - m_2)^2 + \\ &+ C_3 \exp[m_3 z] (m_2 - m_3)^2 \} + C_3 \exp[(m_1 + m_3) z] (m_1 - m_3)^2 > 0. \end{aligned}$$

The validity of the inequalities (26) appear from the calculation of limits:

$$\lim_{z \rightarrow +\infty} U(z) = m_3, \quad \lim_{z \rightarrow -\infty} U(z) = m_1.$$

The above lemma can be used for the estimation of the signs of inequalities (24). We begin with the first one. The validity of the inequality  $B(z) > 0$  appears from the inequality

$$\kappa > \alpha \mu V \max_{z \in \mathbb{R}} U(z) = \alpha \mu V m_3 > \alpha \mu V U(z).$$

Taking into account the conditions (6)–(8), and expressing  $V$  by means of the formula (13) we can rewrite the inequality  $\kappa > \alpha \mu V m_3$  in the form

$$\kappa > 3 \alpha m_3 \frac{\left[ 1 + \sqrt{1 + 4 \alpha \kappa (m_1 + m_2 + m_3)^2} \right]^2}{4 \alpha^2 (m_1 + m_2 + m_3)^3},$$

or, what is the same,

$$\begin{aligned} &4 \alpha \kappa (m_1 + m_2 + m_3)^3 > \\ &> 3 m_3 \left\{ 2 + 4 \alpha \kappa (m_1 + m_2 + m_3)^2 + 2 \sqrt{1 + 4 \alpha \kappa (m_1 + m_2 + m_3)^2} \right\}. \end{aligned}$$

This, in turn, is equivalent to

$$\begin{aligned} & 4 \alpha \kappa (m_1 + m_2 + m_3)^2 [(m_1 - m_3) + (m_2 - m_3)] > \\ & > 6 m_3 \left\{ 1 + \sqrt{1 + 4 \alpha \kappa (m_1 + m_2 + m_3)^2} \right\}. \end{aligned} \quad (27)$$

It is evident, that under the above assumptions the inequality (27) cannot be fulfilled, so, following this way we cannot gain any useful information. It occurs to be possible, if we restrict the set of exact solutions described by the formula (25) by putting  $C_3 = 0$ :

$$U(z) = \frac{m_1 \exp [m_1 z] + C_2 m_2 \exp [m_2 z]}{\exp [m_1 z] + C_2 \exp [m_2 z]}. \quad (28)$$

In analogy with the lemma 2, one can check the validity of the following statement:

**Lemma 3.** if  $C_2 > 0$ , then the function (28) is monotonic and satisfies the inequalities

$$m_1 < U(z) < m_2.$$

So, now there is the condition  $\kappa - \alpha \mu V m_2 > 0$ , which guarantees the validity of the inequality  $B(z) > 0$  for the exact solution (28). Using (6)-(8), and their consequence (13), we get the inequality

$$\begin{aligned} & 4 \alpha \kappa (m_1 + m_2 + m_3)^2 [(m_1 - m_2) + (m_3 - m_2)] > \\ & > 6 m_2 \left\{ 1 + \sqrt{1 + 4 \alpha \kappa (m_1 + m_2 + m_3)^2} \right\}. \end{aligned} \quad (29)$$

Obviously, the inequality (29) does not hold for arbitrary values of the parameters. Yet if all the parameters, but  $m_3$  are fixed, then the LHS behaves as  $m_3^3$  while the RHS as  $m_3^1$ . Hence there exists the critical value  $m_3^{*1}$  such that for any  $m_3 \geq m_3^{*1}$  the inequality (29) does take place.

Now let us consider the condition

$$K(z) = \sum_{i=1}^3 m_i^2 - 2 \sum_{i \neq j} m_i m_j + 2U(z) \sum_{i=1}^3 m_i - 3U^2(z) - 6U'(z) \geq 0.$$

Below we shall use the following elementary statement:

**Lemma 4.** The derivative of the function (28) satisfies the inequality

$$0 < U'(z) \leq \frac{(m_1 - m_2)^2}{4}.$$

Using the above restrictions and the statements of the lemma 3, we get the estimation

$$\begin{aligned} K(z) & > \sum_{k=1}^3 m_k^2 - 2 \sum_{i \neq j} m_i m_j + \\ & + 2 m_1 \sum_{k=1}^3 m_k - 3 m_2^2 - 6 \frac{(m_2 - m_1)^2}{4} = K_1. \end{aligned} \quad (30)$$

Fixing all the parameters but  $m_3$ , we can treat  $K_1$  as the quadratic function:

$$K_1 = m_3^2 - 2m_2 m_3^1 + C(m_1, m_2).$$

So there exists a number  $m_3^{*2}$  such that for  $m_3 > m_3^{*2}$ ,  $K_1$  is positive. From this immediately follows the main result of this section:

**Theorem 2.** If  $m_3 > \max\{m_3^{*1}, m_3^{*2}\}$ , then, under the restrictions stated above, the TW solution (28) is stable.

### 3 Numerical study of the invariant TW solutions to the equation (2)

#### 3.1 Construction of the numerical scheme.

We base our numerical calculations on the Godunov method [23, 24]. It is not difficult to extend the construction of a numerical scheme upon somewhat more general equation

$$\alpha u_{tt} + u_t + \mu u u_x - \kappa [u^n u_x]_x = f(u), \quad (31)$$

containing nonlinear diffusive term. Introducing the new variable  $\Psi = u_t - \sqrt{\kappa \gamma u^n} u_x$ ,  $\gamma = \alpha^{-1}$ , we can rewrite the equation (31) in the form of the first order system:

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ \Psi \end{pmatrix} + \begin{pmatrix} -\sqrt{\kappa \gamma u^n} & 0 \\ \gamma u + \Psi \sqrt{\gamma \frac{n}{2} u^{n/2-1} + \kappa^{1/2} \gamma^{3/2} u^{n/2}} & \sqrt{\kappa \gamma u^n} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u \\ \Psi \end{pmatrix} = H, \quad (32)$$

where  $H = \{\Psi, \gamma [f(u) - \Psi]\}^{\text{tr}}$ , and  $(\cdot)^{\text{tr}}$  stands for the operation of transposition.

Let us consider the calculating cell  $abcd$  (see Fig. 1) lying between  $m - th$  and  $(m + 1) - th$  temporal layers of the uniform rectangular mesh. It is easy to see that the system (32) can be presented in the following vector form:

$$\frac{\partial F}{\partial t} + \frac{\partial G}{\partial x} = H, \quad (33)$$

with  $F = (u, \Psi)^{\text{tr}}$ ,

$$G = \left( -\sqrt{\gamma \kappa} \frac{u^{n/2+1}}{n/2+1}; \frac{\gamma}{2} \left[ u^2 + \frac{2\sqrt{\gamma \kappa}}{n/2+1} u^{n/2+1} \right] + \Psi \sqrt{\gamma \kappa u^2} \right)^{\text{tr}}.$$

From (33) arises the equality of integrals

$$\int \int_{\Omega} \left( \frac{\partial F}{\partial t} + \frac{\partial G}{\partial x} \right) dx dt = \int \int_{\Omega} H dx dt,$$

where  $\Omega$  is identified with the rectangle  $abcd$ . Due to the Gauss-Ostrogradsky theorem, integral in the LHS can be presented in the form

$$\int \int_{\Omega} \left( \frac{\partial F}{\partial t} + \frac{\partial G}{\partial x} \right) dx dt = \oint_{\partial \Omega} G dt - F dx. \quad (34)$$



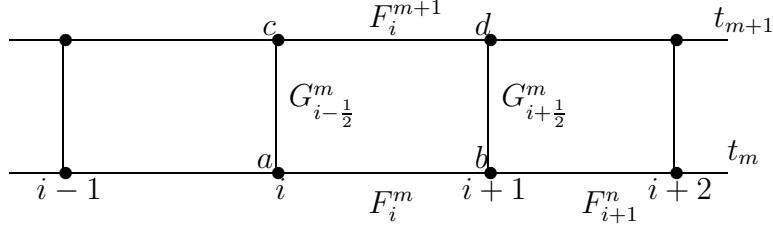


Figure 1: Scheme of the calculating cell

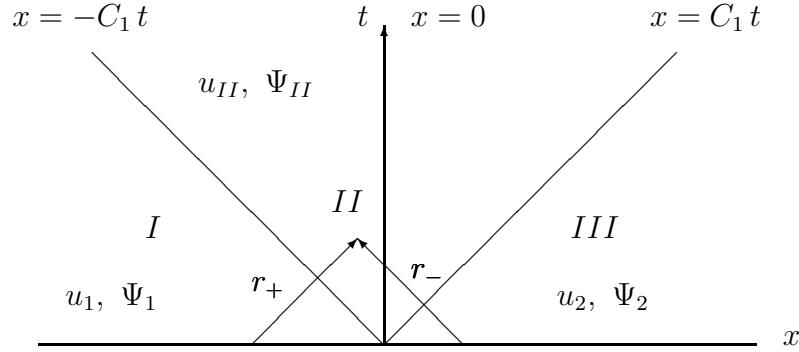


Figure 2: Scheme of solving the Riemann problem

Let us denote the distance between the  $i - th$  and  $(i + 1) - th$  nodes of the  $OX$  axis by  $\Delta x$  while the distance between the two adjacent temporal layers by  $\Delta t$ . Then, up to  $O(|\Delta x|^2, |\Delta t|^2)$ , we get from the equation (33) the following difference scheme:

$$(F_i^{m+1} - F_i^m) \Delta x + (G_{i+\frac{1}{2}}^m - G_{i-\frac{1}{2}}^m) \Delta t = H_i^m \Delta t \Delta x, \quad (35)$$

where  $G_{i+\frac{1}{2}}^m, G_{i-\frac{1}{2}}^m$  are the values of the vector-function  $G$  on the segments  $bd$  and  $ac$ , correspondingly. In the Godunov method these values are defined by solving the Riemann problem. Below we describe the procedure of their calculation.

In accordance with common practice, instead of dealing with the initial system (32), we look for the solution of the Riemann problem  $(u_1, \Psi_1)$  at  $x < 0$  and  $(u_2, \Psi_2)$  at  $x > 0$  to corresponding homogeneous system

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ \Psi \end{pmatrix} + \begin{pmatrix} -C_1 & 0 \\ C_2 & C_1 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u \\ \Psi \end{pmatrix} = 0, \quad (36)$$

where

$$C_1 = \sqrt{\gamma \kappa u_0^n},$$

$$C_2 = \gamma u_0 + \Psi_0 \sqrt{\gamma} \frac{n}{2} u_0^{n/2-1} + \kappa^{1/2} \gamma^{3/2} u_0^{n/2}.$$

Using the linearized system (36), it is easy to calculate the Riemann invariants

$$r_+ = C_2 u + 2 C_1 \Psi, \quad r_- = u,$$

corresponding to the characteristic velocities  $C_{\pm} = \pm C_1$ . Characteristics  $x = \pm C_1 t$  divide the half-plane  $t \geq 0$  into three sectors (see Fig. 2) and the problem is to find the values of the parameters in sector II, basing at the values  $(u_1, \Psi_1)$  and  $(u_2, \Psi_2)$ , which are assumed to be defined. The scheme of calculating the values  $u_{II}, \Psi_{II}$  is based on the property of the Riemann invariants to retain their values along the corresponding characteristics. From this we get the system of algebraic equations

$$\begin{aligned} C_2 u_1 + 2 C_1 \Psi_1 &= C_2 u_{II} + 2 C_1 \Psi_{II}, \\ u_2 &= u_{II}. \end{aligned}$$

So the values of the parameters  $u, \Psi$  in the sector  $-C_1 t < x < C_1 t$  are given by the formulae:

$$\begin{aligned} u_{II} &= u_2, \\ \Psi_{II} &= \Psi_1 + C_2 \frac{u_1 - u_2}{2 C_1}. \end{aligned} \tag{37}$$

Thus, the difference scheme for (31) takes the following form:

$$\begin{aligned} u_i^{m+1} &= u_i^m + \frac{\Delta t}{\Delta x} \left[ (G_1)_{i-\frac{i}{2}}^m - (G_1)_{i+\frac{i}{2}}^m \right] + \Delta t (H_1)_i^m, \\ \Psi_i^{m+1} &= \Psi_i^m + \frac{\Delta t}{\Delta x} \left[ (G_2)_{i-\frac{i}{2}}^m - (G_2)_{i+\frac{i}{2}}^m \right] + \Delta t (H_2)_i^m, \end{aligned}$$

where

$$\begin{aligned} (G_1)_{i-\frac{i}{2}}^m &= -\sqrt{\gamma \kappa} \frac{\left(u_{i-\frac{i}{2}}^m\right)^{n/2+1}}{n/2+1}, \quad i = 2, 3, \dots, N-1, \\ (G_2)_{i-\frac{i}{2}}^m &= \frac{\gamma}{2} \left[ \mu \left(u_{i-\frac{i}{2}}^m\right)^2 + \frac{2\sqrt{\gamma \kappa}}{n/2+1} \left(u_{i-\frac{i}{2}}^m\right)^{n/2+1} \right] + \left(\Psi_{i-\frac{i}{2}}^m\right) \sqrt{\gamma \kappa} \left(u_{i-\frac{i}{2}}^m\right)^{n/2}, \end{aligned}$$

$\left(u_{i-\frac{i}{2}}^m\right), \left(\Psi_{i-\frac{i}{2}}^m\right), i = 2, 3, \dots, N-1$ , are calculated by means of the formula (37), in which  $(u_1, \Psi_1)$  and  $(u_2, \Psi_2)$  are substituted, correspondingly, by  $(u_{i-1}^m, \Psi_{i-1}^m)$  and  $(u_i^m, \Psi_i^m)$ , while the constants  $C_k, k = 1, 2$  take the form

$$\begin{aligned} C_1 &= \sqrt{\gamma \varkappa} \left(u_{i-1}^m\right)^n, \\ C_2 &= \gamma \left(u_{i-1}^m\right) + \left(\Psi_{i-1}^m\right) \sqrt{\gamma} \frac{n}{2} \left(u_{i-1}^m\right)^{n/2-1} + \varkappa^{1/2} \gamma^{3/2} \left(u_{i-1}^m\right)^{n/2}. \end{aligned}$$

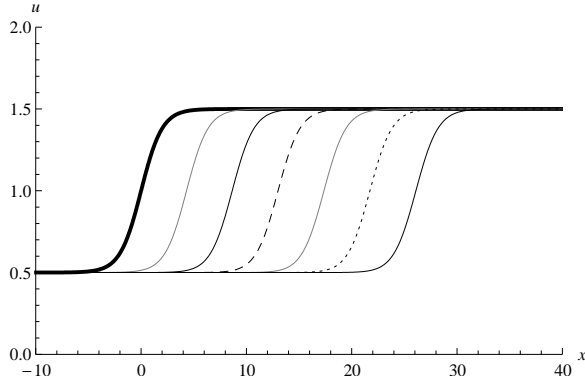


Figure 3: Numerical solution of the system (2) in case when the invariant kink-like solution (28) with  $\alpha = \kappa = 1$ ,  $m_1 = 0.5$ ,  $m_2 = 1.5$ ,  $m_3 = 5$ ,  $C_1 = C_2 = 1$ ,  $C_3 = 0$  is taken as the Cauchy data. Successive graphs of TW solution, moving from left to right, correspond to  $t_i = 4(i - 1)$ ,  $i = 1, \dots, 7$

### 3.2 Results of numerical simulation

Below we present the results of numerical solution of the Cauchy problem for system (2). In all numerical experiments the parameters  $\alpha$  and  $\kappa$  were taken to be equal to one, while the remaining parameters varied from one case to another. In the first series of the numerical experiments we put  $m_1 = 0.5$ ,  $m_2 = 1.5$ , and, since for this choice  $m_3^* \approx 4.4$ , we took  $m_3 = 5$  in order to satisfy the requirements of the theorem 2. As the Cauchy data we used the invariant solution described by the formula (28), and corresponding to  $t_0 = 0$ . Results of the numerical simulation are shown in Fig. 3. It is seen that the kink-like solution evolves for a long time in a stable self-similar mode. Figure 4 shows the graphs of the functions  $B(z)$  and  $K(z)$  for the above values of the parameters. It can be seen on this graphs that both of the functions are strictly positive in a vicinity of the front of the kink-like solution (28).

Next we performed the numerical experiments in which the full solution (9) was taken as the Cauchy data. We put in this case  $C_3 = 3$  and the rest of the parameters remained the same as in the previous case. The results of numerical simulation show that the self-similar solution is unstable, Fig. 5. The source of the instability is seen in Fig. 6, showing the graphs of the functions  $B(z)$  and  $K(z)$ . Both of these functions are negative in some vicinity of the origin. Besides, the function  $K(z)$  has a local minimum in this vicinity, in which it attains a sufficiently large negative value. Analysis of the formula (19) shows that the presence of such maximum can be attributed to the abruptness of the slope of the kink-like solution. Note that the graphs of the functions shown on Fig. 5 correspond to sufficiently large times  $t_i \geq 12$ , for which the effects of instability become evident. Therefore they do not coincide with initial profile described by the formula (9), which is quite sharp.

Let us briefly describe the results of some other numerical experiments. Fig. 7 shows the results of the numerical evolution of a step-like initial perturbation described by the formula (9). Fig. 8 shows the results of the numerical evolution of the initial

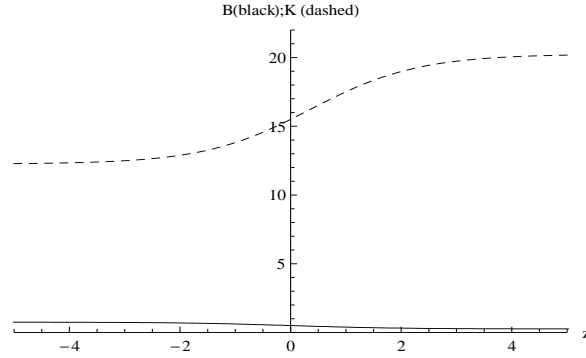


Figure 4: Graphs of the functions  $B(z)$  and  $K(z)$ , obtained for  $\alpha = \kappa = 1$ ,  $m_1 = 0.5$ ,  $m_2 = 1.5$ ,  $m_3 = 5$ ,  $C_1 = C_2 = 1$ , and  $C_3 = 0$

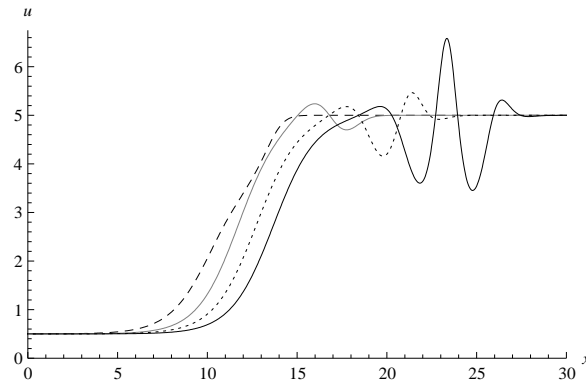


Figure 5: Numerical solution of the system (2) in case when the invariant kink-like solution (9) with  $\alpha = \kappa = 1$ ,  $m_1 = 0.5$ ,  $m_2 = 1.5$ ,  $m_3 = 5$ ,  $C_1 = C_2 = 1$ ,  $C_3 = 3$  is taken as the Cauchy data. Successive graphs correspond to  $t_i = 12 + 4(i - 1)$ ,  $i = 1, \dots, 4$

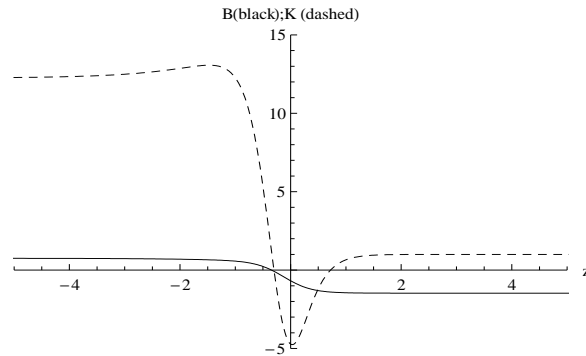


Figure 6: Graphs of the functions  $B(z)$  and  $K(z)$ , obtained for  $\alpha = \kappa = 1$ ,  $m_1 = 0.5$ ,  $m_2 = 1.5$ ,  $m_3 = 5$ ,  $C_1 = C_2 = 1$ , and  $C_3 = 3$

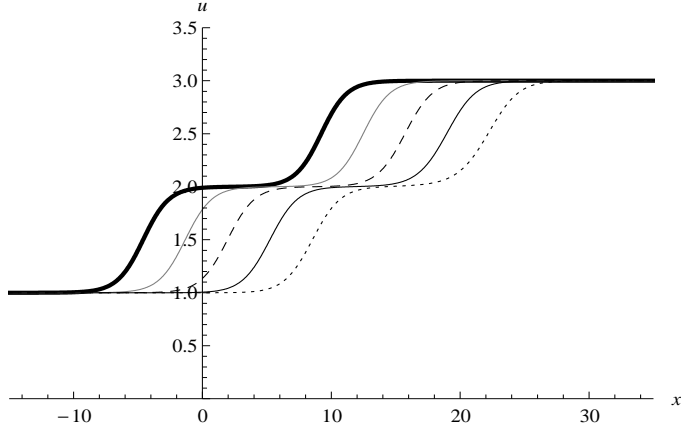


Figure 7: Numerical solution of the system (2) in case when the invariant kink-like solution (9) with  $\alpha = \kappa = 1$ ,  $m_1 = 1$ ,  $m_2 = 2$ ,  $m_3 = 3$ ,  $C_1 = 1$ ,  $C_2 = 100$ ,  $C_3 = 0.01$  is taken as the Cauchy data. Successive graphs of TW solution, moving from left to right, correspond to  $t_i = 3(i - 1)$ ,  $i = 1, \dots, 5$

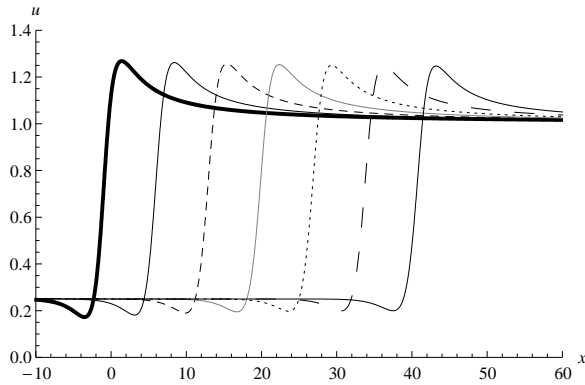


Figure 8: Numerical solution of the system (2) in case when the invariant kink-like solution (10) with  $\alpha = \kappa = 1$ ,  $m_1 = 0.25$ ,  $m_2 = m_3 = 1$ ,  $C_1 = C_2 = C_3 = 1$  is taken as the Cauchy data. Successive graphs of TW solution, moving from left to right, correspond to  $t_i = 5.5(i - 1)$ ,  $i = 1, \dots, 7$

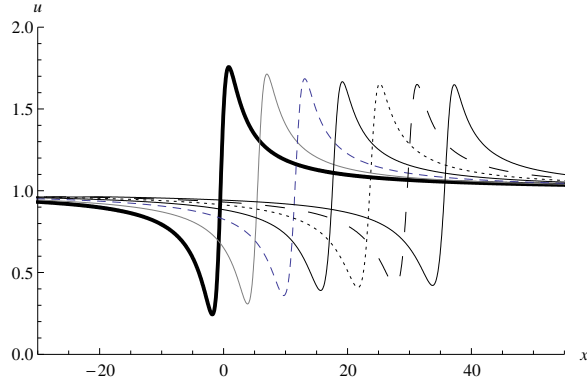


Figure 9: Numerical solution of the system (2) in case when the invariant kink-like solution (11) with  $\alpha = \kappa = 1$ ,  $m = 1$ ,  $C_2 = 1$ ,  $C_3 = 2$  is taken as the Cauchy data. Successive graphs of TW solution, moving from left to right, correspond to  $t_i = 3.75(i - 1)$ ,  $i = 1, \dots, 7$

perturbation described by the formula (10). Finally, the Fig. (9) describes the numerical evolution of the N-shaped soliton with the heavy "tail", described by the formula (11). Results of numerical simulation show that the first two wave patterns evolve without any drastic changes of their shapes. In the last case diminishing of the maximal and minimal amplitudes is evidently seen. Besides the wave instability, this effect can be caused by the numerical scheme's viscosity.

## 4 Summary

So in this paper stability of TW solutions, satisfying the equation (2) under some restrictions on the values of the parameters are analyzed, and sufficient conditions for the stability of some family of exact solutions are presented in explicit form. It is quite evident, that the stability conditions stated by the theorem 2 are sufficient, but not necessary. In fact, they are the most strong among all possible conditions of this sort. No wonder, thus, that for some invariant TW solutions which do not satisfy the inequalities (24), we succeeded to observe the stable self-similar evolution, as well. To gain the theoretical justification of the stability of the exact solutions differ from (28), an extra qualitative investigations, based on the more subtle methods are needed.

### Acknowledgements.

The authors express their gratitude to Prof. Petru Cojuchari for many fruitful discussions.

This research was supported by the AGH local grant.

## References

- [1] Gilding B. H., Kersner R., *Travelling Waves in Nonlinear Diffusion-Convection-Reaction*, Birkhauser, 2004.
- [2] Richards L. A., *Capillarity conduction of liquids through porous medium*, Physica, Vol. 1 (1931), 318–333.
- [3] Murray J. D., *Mathematical Biology*, Springer-Verlag, Berlin, 1989.
- [4] Glansdorff P., Prigogine I. R., *Thermodynamic Theory of Structure, Stability and Fluctuations*, Wiley Interscience, New York, 1971.
- [5] Haken H., *Synergetics: an Introduction*, Springer-Verlag, Berlin, New York, 1978.
- [6] Barannyk A., Yuryk I., *Construction of Exact Solutions of Diffusion Equation*, Proc. of the Institute of Mathematics of NAS of Ukraine, Vol. 50, Part I (2004), 29-33.
- [7] Ivanova N. M., *Exact Solutions of Diffusion-Convection Equations*, Dynamics of PDE, Vol. 5, No. 2 (2008), 139-171.
- [8] Tychynin V., Petrova O., *Nonlocal Symmetries and Generating of Solutions for Partial Differential Equations*, SIGMA, Vol. 3 (2007), Paper 19.
- [9] Cherniha R., Serov M., Rassokha I., *Lie Symmetries and Form-Preserving Transformations of Reaction-Diffusion-Convection Equations*, J. Math. Anal Appl., Vol. 342 (2008), 1363-1379.
- [10] Vladimirov V. and Kutafina E., *Exact Travelling Wave Solutions of Some Non-linear Evolutionary Equations*, Rep. Math. Physics, vol. 54 (2004), 261–271.
- [11] Joseph D.D., Preziosi, L., *Heat Waves*, Review of Modern Physics, vol. 61, no. 1 (1989), 41-73.
- [12] Makarenko A.S., *New Differential Equation Model for Hydrodynamics with Memory Effects*, Rep. Math. Physics, vol. 46, No. 1/2 (2000), 183-190.
- [13] Makarenko A.S., Moskalkov M., Levkov S., *On Blow-up Solutions in Turbulence*, Phys Lett., Vol. A23 (1997), 391-397.
- [14] Kar S., Banik S.K., Ray Sh., *Exact Solutions of Fisher and Burgers Equations with Finite Transport Memory*, Journal of Physics A: Mathematical and Theoretical, vol. 36, No. 11 (2003), 2771-2780.
- [15] Vladimirov V., Kutafina E. and Pudelko A., *Constructing Soliton and Kink Solutions of PDE Models in Transport and Biology*, SIGMA, Vol. 2 (2006), Paper 061.

- [16] Vladimirov V. and Kutafina E., *Analytical Description of the Coherent Structures within the Hyperbolic Generalization of Burgers Equation*, Rep. Math. Physics, Vol. 58 (2006), 465.
- [17] Vladimirov V. and Maczka Cz., *Exact Solutions of Generalized Burgers Equation, Describing Travelling Fronts and Their Interactions*, Rep. Math. Physics, Vol. 60 (2007), 317-328.
- [18] Fahmy E.S., Abdusalam h.A., Raslan K.R., *On the Solutions of the Time Delayed Burgers Equation*, Nonlinear Analysis, Vol. 69 (2008), 4475-4786.
- [19] Vladimirov, Mączka, *On the Localized wave patterns Supported by Convection-Reaction-Diffusion Equation*, Rep. Math. Physics, to appear.
- [20] Barenblatt G. I., *Similarity, Self-similarity and Intermediate Asymptotics*, Consultants Bureau, New York, 1979.
- [21] Kamin S., and Rosenau P., *Emergence of Wave Solutions in a Nonlinear Convection Reaction Diffusion Equation*, Advanced Nonlinear Studies, Vol. 4 (2004), 251-272.
- [22] Kamin S., and Rosenau P., *Convergence of Traveling Wave Solutions for a Nonlinear Reaction Diffusion Equation*, Rendiconti Math. Acc. Lincei, Vol. 15 (2004), 271-280.
- [23] Godunov S.K., *A Difference Scheme for Numerical Solution of Discontinuous solution of Hydrodynamic Equations*, Math. Sbornik, Vol. 47, 271-306, Translated US Joint Publ. Res. Service, JPRS 7226, 1969.
- [24] Rozhdestvenskij B., Yanenko N., *Systems of Quasilinear Equations and Their Applications to Gas Dynamics*, Transl. Math. Monographs. Vol. 55, A.M.S., Providence, RI, 1983.
- [25] Vladimirov V, *Compacton-like Solutions of the Hydrodynamic System Describing Relaxing Media*, Rep. Math. Physics, Vol. 61 (2008), 381-400.
- [26] Keldysh M.V., *O Sobstvennykh Znachenijach i Sobstvennykh Funktsijach Nekotorykh Klassov Nesamosopriazhennykh Uravnenij*, Soviet Doklady, Ser. Mathematics, Vol. 77, No. 1 (1951), 11-14 (in Russian).
- [27] Landau L. D., Lifshitz Ye. M., *Fluid Mechanics*, Nauka, Moscow, 1985.