On integrability of variable coefficient nonlinear Schrödinger equations

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Abstract

We apply Painlevé test to the most general variable coefficient nonlinear Schrödinger (VCNLS) equations as an attempt to identify integrable classes and compare our results versus those obtained by the use of other tools like group-theoretical approach and the Lax pairs technique of the soliton theory. We present explicit transformation formulae that can be used to generate new analytic solutions of VCNLS equations from those of the integrable NLS equation.

1 Introduction

Variable coefficient extensions of nonlinear evolution type equations tend to arise in cases when less idealized conditions such as inhomogeneities and variable topographies are assumed in their derivation. For example, variable coefficient Korteweg-de Vries and Kadomtsev-Petviashvili equations describe the propagation of waves in a fluid under the more realistic assumptions including non-uniformness of the depth and width, the compressibility of the fluid, the presence of vorticity and others. While these conditions lead to variable coefficient equations, all or some of the integrability properties of their standard counterparts, namely when the coefficients are set equal to constants, are in general destroyed. However, when the coefficients are

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appropriately related or have some specific form, the generalized equation can still be integrable as evidenced by the presence of Painlevé property, Lax pairs, symmetries and other attributes of integrability [1, 2]. This fact suggests us to look for a point transformation mapping VCNLS equation to the standard NLS one. It is thus our intention to examine the integrability conditions of a class of variable coefficient nonlinear Schrödinger (VCNLS) equation which can be thought of as a generalization of the standard integrable nonlinear Schrödinger (NLS) equation. More precisely, we ask the question: Under which conditions can the equation be transformed to its constant coefficient version which is notoriously integrable?

In this paper we consider the following class of equations

$$
\begin{aligned}
i\psi_t + f(x,t)\psi_{xx} + g(x,t)|\psi|^2\psi + h(x,t)\psi = 0,\\
f = f_1 + if_2, \quad g = g_1 + ig_2, \quad h = h_1 + ih_2,\\
f_j, g_j, h_j \in \mathbb{R}, \quad j = 1,2, \quad f_1 \neq 0, \quad g_1 \neq 0
\end{aligned}
\tag{1.1}
$$

and study its integrability and in particular reducibility to its standard form using the tools of singularity analysis and symmetry which are well-established approaches towards integrability. This class of equations model a large variety of physical phenomena and has been widely analyzed through various methods. The complex function $\psi(x,t)$ has different physical meanings in different physical contexts. An equation with an additional quintic nonlinearity when the coefficient functions mostly depend on time models the propagation of pulses in optic fibers and was studied in [3], where trigonometric type solutions were obtained through some transformations. [4] studies the equation with space and time coordinates switched, modeling propagation of pulses in optical fibers with distributed dispersion and nonlinearity and finds soliton-type solutions with a Darboux transformation, whereas elliptic-type solutions through various transformations [5], soliton-type solutions via Hirota method [6] and in terms of a double-Wronskian determinant [7], and self-similar solutions also exist in the literature [8]. In these works, the coefficients are considered as functions of a single variable. For another subclass of equations when f, g are real functions of t, reductions to a nonlinear ODE by an appropriate transformation is performed and solutions of hyperbolic, trigonometric and elliptic-type are obtained [9]. Besides, elliptic-type solutions by a direct-symmetry method are found in [10] for the same case. [11] considers the case with constant f, g and a real potential periodic in time, and presents soliton-type solutions. For $f = 1$ and g, h real, [12] uses similarity transformations to transform the equation into the stationary nonlinear Schrödinger equation and hence obtains soliton-type solutions of various characters.

For f, g real function of t and a real potential which is quadratic in x with time-dependent coefficients, [13] gives an integrability condition based on a Lax pair. [14] tries to transform this equation directly to the NLS equation, whereas [15] states that this equation and its extended version are equivalent to the NLS equation.

There is a vast amount of literature devoted to analyzing Painlevé property (P-property) of equations belonging to the class (1.1) as subcases (for example, see

[16]-[19]). Results of [20] and [21] belong to a wider class and the compatibility condition they find coincides with the exact integrability condition given by [13]. To our knowledge, [22] treats (1.1) in the most general case from Painlevé analysis. They obtain f and g as real functions of t and transform the equation to the NLS equation. However, we find that besides t, functions f, g depending on x can also be allowed, and the equation still can pass the WTC test. We thus recover results of [22] as a subcase. Finally, we want to mention Ref. [23], in which allowed transformations of (1.1) are found and used to do a complete symmetry group analysis of the equation.

The paper is organized as follows. Section 2 performs the Painlevé analysis of (1.1). Section 3 is devoted to the transformation of the equation into the NLS equation using allowed transformations. We end up by applying our results to a generalized Gross-Pitaevskii equation in a harmonic trap in Section 4.

2 Painlevé analysis of the equation

For convenience we write (1.1) together with its complex conjugate as the system

$$\begin{aligned} iu_t + f(x,t)u_{xx} + g(x,t)u^2v + h(x,t)u = 0, \\ -iv_t + p(x,t)v_{xx} + q(x,t)uv^2 + r(x,t)v = 0. \end{aligned} \tag{2.1}$$

Here u was employed instead of ψ and v denotes its complex conjugate, but in this context they are viewed as independent functions, whereas p, q, r are complex conjugates of f, g and h, respectively. This system will be subjected to the Painlevé test for partial differential equations. We shall look for solutions of the system (2.1) in the form of a Laurent expansion (known as Painlevé expansion) in the neighborhood of a non-characteristic, movable singularity surface (actually a curve in this case) defined by

$$\Phi(x,t) = 0. \tag{2.2}$$

Thus, we expand

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t)\Phi^{\alpha+j}(x,t), \quad v(x,t) = \sum_{j=0}^{\infty} v_j(x,t)\Phi^{\beta+j}(x,t), \tag{2.3}$$

where $u_0, v_0 \neq 0$ and $u_j, v_j, \Phi(x,t)$ are analytic functions. α and β are negative integers to be determined from the leading order analysis.

The partial differential equation (PDE) is said to pass the Painlevé test if the substitution of the above expansion into the equation leads to the correct number of arbitrary functions as required by the Cauchy-Kovalewski theorem given by the expansion coefficients, where $\Phi(x,t)$ should be one of the arbitrary functions. The coefficients in the expansion where the arbitrary functions figure are known as the resonances.

If a PDE passes this test, then it has a good chance of having the Painlevé property: solutions (2.3) are single valued about the singularity surface, and depends on sufficient number of arbitrary functions which are needed to satisfy arbitrary Cauchy data imposed for some $x = x_0$. We note that passing the Painlevé test is necessary, but not sufficient for having the Painlevé property. In that case, for integrable PDEs it is usually possible to construct auto-Bäcklund transformations which relate equations to themselves via differential substitions and also Lax pairs, which then ensures the sufficient condition of integrability. Application of the Painlevé expansion to nonintegrable PDEs can allow particular explicit solutions to be obtained by truncating the expansion which then imposes constraints on the arbitrary functions and the function Φ. This usually requires compatibility of an overdetermined PDE system. Let us mention that the method of truncated expansion has been successfully applied to many nonintegrable PDEs in constructing exact solutions.

For the determination of leading orders α and β, we substitute $u \sim u_0\Phi^\alpha$ and $v \sim v_0\Phi^\beta$ in (2.1) and see that by balancing the terms of smallest order

$$\alpha + \beta = -2 \tag{2.4}$$

must hold, which only allows the negative integers $\alpha = -1$ and $\beta = -1$. In addition, we obtain the relations

$$u_0(gu_0v_0 + 2f\Phi_x^2) = 0 \tag{2.5a}$$

$$v_0(qu_0v_0 + 2p\Phi_x^2) = 0 \tag{2.5b}$$

and it follows that

$$u_0v_0 = -2\Phi_x^2\frac{f}{g} = -2\Phi_x^2\frac{p}{q}. \tag{2.6}$$

From this it is seen that $\frac{f}{g}$ must be real, which requires that $f_1g_2 = f_2g_1$. We can say that, if one of f and g is real or pure imaginary, then so is the other.

After determination of the leading orders, we substitute (2.3) into (2.1). For $j = 0$, equating the coefficient of Φ^{-3} to zero we exactly obtain the relations (2.5). For $j \geq 1$, equating coefficient of Φ^{-3+j} to zero yields the system

$$Q(j)\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} (j^2-3j-2)f\Phi_x^2 & gu_0^2 \\ qv_0^2 & (j^2-3j-2)p\Phi_x^2 \end{pmatrix}\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} F_j \\ G_j \end{pmatrix}, \tag{2.7}$$

where

$$\begin{aligned} F_j =& -iu_{j-2,t} - i(-2+j)\phi_t u_{j-1} - fu_{j-2,xx} - 2(-2+j)f\Phi_x u_{j-1,x} \\ & -(-2+j)f\Phi_{xx}u_{j-1} - hu_{j-2} - gv_0\sum_{l=1}^{j-1}u_l u_{j-l} - g\sum_{k=1}^{j-1}\sum_{l=0}^{k}u_l u_{k-l}v_{j-k}, \\ G_j =& iv_{j-2,t} + i(-2+j)\phi_t v_{j-1} - pv_{j-2,xx} - 2(-2+j)p\Phi_x v_{j-1,x} \\ & -(-2+j)p\Phi_{xx}v_{j-1} - rv_{j-2} - qu_0\sum_{l=1}^{j-1}v_l v_{j-l} - q\sum_{k=1}^{j-1}\sum_{l=0}^{k}v_l v_{k-l}u_{j-k}. \end{aligned} \tag{2.8}$$

Here we have used the convention that $u_j = 0$ for $j < 0$ and sums which have upper indices that are less than the lower indices will be taken as zero. Notice that F_j and G_j can be obtained from each other by the interchange $i \leftrightarrow -i$, $u \leftrightarrow v$, $f \leftrightarrow p$, $g \leftrightarrow q$, $h \leftrightarrow r$. u_j and v_j are determined by the system (2.7) unless

$$\det Q(j) = |f|^2 \Phi_x^4 (j+1) j (j-3)(j-4) = 0. \tag{2.9}$$

Hence we find the resonance levels to be $j = -1, 0, 3, 4$. The resonance $j = -1$ corresponds to the arbitrariness of the singularity manifold Φ and the resonance $j = 0$ points out that there is one arbitrary function among u_0 and v_0, as it is verified by (2.6).

For $j = 1$ we solve

$$\begin{pmatrix} -4f\Phi_x^2 & gu_0^2 \\ qv_0^2 & -4p\Phi_x^2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} i\Phi_t u_0 + 2f\Phi_x u_{0,x} + f\Phi_{xx} u_0 \\ -i\Phi_t v_0 + 2p\Phi_x v_{0,x} + p\Phi_{xx} v_0 \end{pmatrix} \tag{2.10}$$

and find the expansion coefficients to be

$$\begin{aligned} u_1 &= -\frac{2f\Phi_x v_{0,x}}{gv_0^2} + \frac{4f_x\Phi_x}{3gv_0} - \frac{4fg_x\Phi_x}{3g^2v_0} + \frac{2i\Phi_t}{3gv_0} + \frac{i\Phi_t}{3qv_0} + \frac{3f\Phi_{xx}}{gv_0}, \\ v_1 &= -\frac{v_{0,x}}{\Phi_x} + \frac{f_x v_0}{3f\Phi_x} - \frac{g_x v_0}{3g\Phi_x} + \frac{i\Phi_t v_0}{6f\Phi_x^2} + \frac{ig\Phi_t v_0}{3fq\Phi_x^2} + \frac{\Phi_{xx} v_0}{2\Phi_x^2}; \end{aligned} \tag{2.11}$$

making use of the facts that $u_0 = -2\Phi_x^2 \frac{f}{gv_0}$ and $p = q\frac{f}{g}$. For $j = 2$ we solve

$$\begin{pmatrix} -4f\Phi_x^2 & gu_0^2 \\ qv_0^2 & -4p\Phi_x^2 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -hu_0 - gu_1^2 v_0 - 2gu_0u_1v_1 - iu_{0,t} - fu_{0,xx} \\ -rv_0 - qv_1^2 u_0 - 2qv_0v_1u_1 + iv_{0,t} - pv_{0,xx} \end{pmatrix} \tag{2.12}$$

and find the expansion coefficients u_2, v_2. However, we do not reproduce them here as they are so lengthy.

At the resonance level $j = 3$, we have a linear relation between u_3, v_3 and one of them is arbitrary. We see that the system

$$\begin{pmatrix} -2f\Phi_x^2 & gu_0^2 \\ qv_0^2 & -2p\Phi_x^2 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} F_3 \\ G_3 \end{pmatrix} \tag{2.13}$$

gives the compatibility condition at $j = 3$ as

$$qv_0 F_3 = gu_0 G_3. \tag{2.14}$$

Since the compatibility condition has to be satisfied for any choice of the manifold Φ, coefficients of $\Phi_t, \Phi_t^2, \Phi_t^3, \Phi_{tt}$ occurring in (2.14) must vanish identically. From the coefficient of Φ_{tt} we find that $g^2 = q^2$. It follows that $g = q$ or $g = -q$. The first is possible when $g = g_1$ and for the second we have $g = ig_2$. Since f and g are both real or pure imaginary, we have $f = f_1$ for the former and $f = if_2$ for the latter.

(i)The case $g = ig_2$, $f = if_2$. From the coefficient of Φ_t^2 we find that

$$2\frac{v_{0,x}}{v_0} + \frac{g_{2,x}}{g_2} - \frac{f_{2,x}}{f_2} - 2\frac{\Phi_{xx}}{\Phi_x} = 0, \tag{2.15}$$

which is a restriction on v_0 that contradicts its arbitrariness. We conclude that when f and g are pure imaginary, the equation cannot pass the Painlevé test.

(ii)The case $g = g_1$, $f = f_1$. From now on, we are going to drop the subscript 1 and assume that f and g are real functions. Equation (2.14) is satisfied if and only if

$$\left(\frac{f_x}{f}\right)^2 + \frac{f_x g_x}{fg} + 4\left(\frac{g_x}{g}\right)^2 - \frac{f_{xx}}{f} - 2\frac{g_{xx}}{g} = 0, \tag{2.16a}$$

$$h_2\frac{f_x}{f} + 2h_2\frac{g_x}{g} - 3h_{2,x} + \frac{3}{2}\frac{f_t f_x}{f^2} - \frac{1}{2}\frac{f_x g_t}{fg} + \frac{f_t g_x}{fg} - 2\frac{g_t g_x}{g^2} - \frac{f_{xt}}{f} + \frac{g_{xt}}{g} = 0. \tag{2.16b}$$

Next we proceed to the resonance level $j = 4$. From the system

$$\begin{pmatrix} 2f\Phi_x^2 & gu_0^2 \\ qv_0^2 & 2p\Phi_x^2 \end{pmatrix}\begin{pmatrix} u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} F_4 \\ G_4 \end{pmatrix} \tag{2.17}$$

we see that one of u_4 and v_4 should be arbitrary and it yields the compatibility condition

$$v_0 F_4 + u_0 G_4 = 0. \tag{2.18}$$

This relation contains the functions u_3, v_3 of the resonance $j = 3$. From (2.13) we substitute $u_3 = (gu_0^2 v_3 - F_3)/(2f\Phi_x^2)$ in (2.18). Similar to the previous resonance, from the coefficient of Φ_{tt} we get

$$\frac{f_x}{f} + 2\frac{g_x}{g} = 0, \tag{2.19}$$

which integrates to give

$$f(x,t)g^2(x,t) = K(t), \tag{2.20}$$

where $K(t)$ is an arbitrary integration constant. Notice that this condition is already satisfied for the special case when f and g are only functions of t. We set $f(x,t) = K(t)/g^2(x,t)$ and under this constraint see that (2.16a) is satisfied whereas (2.16b) simplifies as $h_{2,x} = (g_t/g)_x$ and determines the complex part of the potential

$$h_2(x,t) = \frac{g_t}{g} + \gamma(t), \tag{2.21}$$

where $\gamma(t)$ is an arbitrary function. Notice that h_2 may depend on x if g depends on x. There remains only one equation as to satisfy (2.18):

$$\begin{aligned} \frac{2K}{g}\left(\frac{h_{1,xx}}{g} - \frac{g_x h_{1,x}}{g^2}\right) = {} & 4\gamma^2 + 2\dot{\gamma} + 2\gamma\frac{\dot{K}}{K} + \frac{\ddot{K}}{K} - \left(\frac{\dot{K}}{K}\right)^2 + \frac{\dot{K}}{K}\frac{g_t}{g} - \frac{g_{tt}}{g} \\ & + K^2\left(-36\frac{g_x^4}{g^8} + 48\frac{g_x^2 g_{xx}}{g^7} - 6\frac{g_{xx}^2}{g^6} - 10\frac{g_x g_{xxx}}{g^6} + \frac{g_{xxxx}}{g^5}\right). \end{aligned} \tag{2.22}$$

It is possible to integrate this expression and find real part of the potential as

$$h_1(x,t) = \frac{\tau_0}{2}\left(\int g dx\right)^2 + \int g(\int \tau_1 dx)dx + \frac{K}{2}\left(\frac{g_{xx}}{g^3} - \frac{3}{2}\frac{g_x^2}{g^4}\right) + \delta_0 \int g dx + \delta_1, \quad (2.23)$$

where τ_0 and τ_1 denote

$$\tau_0 = \tau_0(t) = \frac{2\gamma^2}{K} + \frac{(\gamma K)_t}{K^2} + \frac{(\ln K)_{tt}}{2K}, \qquad \tau_1 = \tau_1(x,t) = -\frac{1}{2}\left(\frac{g_t}{K}\right)_t. \quad (2.24)$$

Here $\delta_0(t)$ and $\delta_1(t)$ are arbitrary functions. Note that integrals in (2.23) will be evaluated without additional integration constants.

We then consider a simpler case. We see from (2.19) that if one of f and g is only a function of t, then so is the other. Furthermore, we do not have a condition like (2.20) for f and g, they can be independent. For $f = f(t),\ g = g(t)$, (2.16a) holds identically and from (2.16b) we have $h_{2,x} = 0$, which means

$$h_2(x,t) = \beta(t). \quad (2.25)$$

Similarly, we are left with a single equation from(2.18):

$$2fh_{1,xx} = 4\beta^2 + 2\dot\beta + 2\beta\frac{\dot f}{f} - 4\beta\frac{\dot g}{g} - (\frac{\dot f}{f})^2 - \frac{\dot f}{f}\frac{\dot g}{g} + 2(\frac{\dot g}{g})^2 + \frac{\ddot f}{f} - \frac{\ddot g}{g}. \quad (2.26)$$

This relation also follows directly from (2.22) with the substitution $\gamma(t) = \beta(t) - \frac{\dot g}{g}$. We can write the right hand side as

$$2fh_{1,xx} = 4\beta^2 + 2\dot\beta + 2\beta\frac{\dot f}{f} - 4\beta\frac{\dot g}{g} + (\ln f)_{tt} + g(\frac{1}{g})_{tt} - (\ln f)_t(\ln g)_t, \quad (2.27)$$

where the subscript t denotes the derivative, and integrate to obtain $h_1(x,t)$:

$$h_1(x,t) = \frac{1}{4f}\{4\beta^2 + 2\dot\beta + 2\beta\frac{\dot f}{f} - 4\beta\frac{\dot g}{g} + (\ln f)_{tt} + g(\frac{1}{g})_{tt} - (\ln f)_t(\ln g)_t\}x^2 + H_1(t)x + H_2(t), \quad (2.28)$$

with $H_1(t)$ and $H_2(t)$ arbitrary. This coincides with the results of [20]-[22]. For $f = 1$ and $g = -2$ we have

$$h_1(x,t) = \frac{1}{2}(2\beta^2 + \dot\beta)x^2 + H_1(t)x + H_2(t) \quad (2.29)$$

and it is compatible with the results of [17].

3 Transformation to the standard NLS equation by allowed transformations

In [23] the authors give a classification of the symmetry algebras of (1.1) into canonical forms using the allowed (or equivalence) transformations of the form

$$\psi(x,t) = Q(x,t)\tilde{\psi}(\tilde{x},\tilde{t}), \qquad \tilde{x} = X(x,t), \qquad \tilde{t} = T(t), \tag{3.1}$$

which leaves form of (1.1) invariant, that is, the transformation does not add any term to the equation but may change the coefficient functions. Here $Q(x,t)$ is a complex function obeying the constraint

$$iQX_t + f(x,t)(X_{xx}Q + 2Q_xX_x) = 0. \tag{3.2}$$

If we set $Q(x,t) = R(x,t)e^{i\theta(x,t)}$, $R \geq 0, 0 \leq \theta < 2\pi$, the coefficient functions f, g and h map into

$$\tilde{f}(\tilde{x},\tilde{t}) = f(x,t)\frac{X_x^2}{\dot{T}}, \qquad \tilde{g}(\tilde{x},\tilde{t}) = g(x,t)\frac{R^2(x,t)}{\dot{T}(t)}, \tag{3.3}$$

$$\begin{aligned}\tilde{h}(\tilde{x},\tilde{t}) = \frac{1}{\dot{T}}\{&h_1(x,t) - \theta_t + f(x,t)\left(\frac{R_{xx}}{R} - \theta_x^2\right)\\ &+ i\left[h_2(x,t) + \frac{R_t}{R} + f(x,t)(\frac{2R_x\theta_x}{R} + \theta_{xx})\right]\}\end{aligned} \tag{3.4}$$

through this transformation. Note that here $f(x,t)$ is a complex-valued function. However, when writing $\tilde{h}$ we have not separated it into real and imaginary parts, since in the following it is going to be taken as a real function.

One of the main results of [23] is that any equation of the form (1.1) with a five-dimensional symmetry algebra is equivalent to the NLS equation with $\tilde{f} = 1$, $\tilde{g} = \epsilon + i\tilde{g}_2$, $\tilde{h} = 0$, where $\epsilon = \pm 1$ and $\tilde{g}_2 =$ constant. The algebra is solvable and has a basis

$$P_0 = \partial_{\tilde{t}}, \quad P_1 = \partial_{\tilde{x}}, \quad W = \partial_{\tilde{\omega}}, \quad B = \tilde{t}\partial_{\tilde{x}} + \frac{1}{2}\tilde{x}\partial_{\tilde{\omega}}, \quad D = \tilde{t}\partial_t + \frac{1}{2}\tilde{x}\partial_{\tilde{x}} - \frac{1}{2}\tilde{\rho}\partial_{\tilde{\rho}}, \tag{3.5}$$

which is isomorphic to the one-dimensional extended Galilei similitude algebra gs(1). Here ψ is expressed in terms of the modulus and the phase of the wave function: $\psi(x,t) = \rho(x,t)e^{i\omega(x,t)}$.

In the previous section we have been able to give the conditions for the equation to have the P-property. Now we ask, under these conditions, whether we can transform (1.1) to the standard NLS equation which is an integrable one from a group-theoretical point of view. This is equivalent to the condition that the equation under study has a five-dimensional symmetry algebra which is isomorphic to that of the standard NLS equation. Equivalence (or allowed) transformations will be used to produce the transformations mapping (1.1) to the NLS equation.

In Section 1, from the Painlevé test it turned out that f and g have to be real functions. When f is a real function, (3.2) is equivalent to

$$R^2(x,t) = \frac{R_0(t)}{X_x}, \qquad \theta_x = -\frac{1}{2f}\frac{X_t}{X_x}, \tag{3.6}$$

where $R_0(t)$ arbitrary. For the equation (1.1) to have a five-dimensional symmetry algebra, we only need to set

$$\tilde{f}(\tilde{x},\tilde{t}) = f(x,t)\frac{X_x^2}{\dot{T}} = 1, \tag{3.7a}$$

$$\tilde{g}(\tilde{x},\tilde{t}) = g(x,t)\frac{R^2(x,t)}{\dot{T}} = \epsilon, \quad \epsilon = \pm 1, \tag{3.7b}$$

$$\tilde{h}(\tilde{x},\tilde{t}) = 0. \tag{3.7c}$$

Notice that we have not included the constant $i\tilde{g}_2$ in the righthand side of (3.7b) since the functions occurring on the left are real. These equations are going to provide us the transformation functions R, θ, X and T.

We recall that we have succeeded in writing the potential h in terms of f and g for two (not distinct) cases.

(i) The case when $f(x,t)g^2(x,t) = K(t)$.

In the following calculations we used the substitution $f \to K/g^2$. First solving (3.7a) we find that

$$X_x = \epsilon_1\sqrt{\frac{\dot{T}}{K}}g, \qquad X(x,t) = \epsilon_1\sqrt{\frac{\dot{T}}{K}}\int g dx + \xi(t), \quad \epsilon_1 = \pm 1 \tag{3.8}$$

with an arbitrary ξ. From the second equality (3.7b) we easily find

$$R(x,t) = \left(\epsilon\frac{\dot{T}}{g}\right)^{1/2}. \tag{3.9}$$

Next we make use of the equations in (3.6). The first equation means that the product $R^2(x,t)X_x(x,t)$ should be a function of t, which is indeed the case. Integration of the second equation determines θ:

$$\theta(x,t) = -\frac{1}{8K}\left(\ln\frac{\dot{T}}{K}\right)_t \left(\int g dx\right)^2 - \frac{1}{2K}\int g\left(\int g_t dx\right)dx - \frac{\epsilon_1\dot{\xi}}{2\sqrt{\dot{T}K}}\int g dx + \eta(t), \tag{3.10}$$

where η is an arbitrary function. The remaining equation to be solved is (3.7c). We are going to make use of (2.21),(2.23),(3.9),(3.10) in the formula (3.4). From the real and imaginary part of (3.7c) we have, respectively,

$$\left(\delta_1 - \dot{\eta} - \frac{\dot{\xi}^2}{4\dot{T}}\right) + \left(\delta_0 + \frac{\epsilon_1 \ddot{\xi}}{2\sqrt{K\dot{T}}} - \frac{\epsilon_1 \dot{\xi}\ddot{T}}{2\sqrt{K}\dot{T}^{3/2}}\right)\int g dx$$
$$+\frac{1}{8K}\left(\frac{\dddot{T}}{\dot{T}} - \frac{3}{2}\left(\frac{\ddot{T}}{\dot{T}}\right)^2 + \frac{\ddot{K}}{K} - \frac{1}{2}\left(\frac{\dot{K}}{K}\right)^2 + 4\gamma\frac{\dot{K}}{K} + 8\gamma^2 + 4\dot{\gamma}\right)\left(\int g dx\right)^2 = 0, \tag{3.11}$$

$$4\gamma(t) + \frac{\dot{K}}{K} + \frac{\ddot{T}}{\dot{T}} = 0. \tag{3.12}$$

The second equation is easy to integrate and at once yields the time transformation

$$T(t) = T_1 \int \frac{e^{-4\int \gamma dt}}{K(t)} dt + T_2, \qquad T_1, T_2 \text{ constants.} \tag{3.13}$$

Once f, g and h are given by a specific case of (1.1), one can first check the P-property conditions and after that determine the function T, in principle. With this form of T the last term in (3.11) vanishes, but the first and the second ones do not. However, we can set them equal to zero by choosing ξ and η as solutions of the equations

$$\delta_0(t) + \frac{\epsilon_1 \ddot{\xi}}{2\sqrt{K\dot{T}}} - \frac{\epsilon_1 \dot{\xi}\ddot{T}}{2\sqrt{K}\dot{T}^{3/2}} = 0, \tag{3.14a}$$

$$\delta_1(t) - \dot{\eta} - \frac{\dot{\xi}^2}{4\dot{T}} = 0. \tag{3.14b}$$

Having found T we can arrange (3.14a) as

$$\delta_0(t) + \frac{\epsilon_1}{2}\sqrt{\frac{\dot{T}}{K}}\left(\frac{\dot{\xi}}{\dot{T}}\right)_t = 0, \tag{3.15}$$

of which integration twice yields

$$\xi(t) = -2\epsilon_1 \int \dot{T}(\int \sqrt{\frac{K}{\dot{T}}}\, \delta_0(t)dt)dt + \xi_0 T(t) + \xi_1, \tag{3.16}$$

where ξ_0, ξ_1 are constants. From (3.14b) follows directly

$$\eta(t) = \int \delta_1(t)dt - \int \dot{T}(\int \sqrt{\frac{K}{\dot{T}}}\delta_0 dt)^2 dt + \epsilon_1 \xi_0 \int \dot{T}(\int \sqrt{\frac{K}{\dot{T}}}\delta_0 dt)dt - \frac{\xi_0^2}{4}T + \theta_0, \tag{3.17}$$

with θ_0 constant. So far we have all the information to write the general transformation formula. T and R are given by (3.13) and (3.9), respectively. For the space variable transformation we have

$$X(x,t)=\epsilon_1\sqrt{\frac{\dot{T}}{K}}\int g dx-2\epsilon_1\int \dot{T}(\int\sqrt{\frac{K}{\dot{T}}}\delta_0(t)dt)dt+\xi_0 T(t)+\xi_1, \quad \epsilon_1=\pm 1. \tag{3.18}$$

The phase of Q is obtained in the form

$$\begin{aligned}\theta(x,t)=&-\frac{1}{8K}\left(\ln\frac{\dot{T}}{K}\right)_t(\int g dx)^2-\frac{1}{2K}\int g\,(\int g_t dx)dx\\&+\sqrt{\frac{\dot{T}}{K}}(\int\sqrt{\frac{K}{\dot{T}}}\delta_0 dt-\frac{\epsilon_1\xi_0}{2})\int g dx+\int\delta_1 dt\\&-\int\dot{T}(\int\sqrt{\frac{K}{\dot{T}}}\delta_0 dt)^2 dt+\epsilon_1\xi_0\int\dot{T}(\int\sqrt{\frac{K}{\dot{T}}}\delta_0 dt)dt-\frac{\xi_0^2}{4}T+\theta_0.\end{aligned} \tag{3.19}$$

Now we make use of the formula we have found for T and give the transformation functions together with (3.13) in its most explicit form. With $\Gamma(t)=2\int\gamma(t)dt$ we have

$$\begin{aligned}X(x,t)=&\,\epsilon_1\sqrt{T_1}\frac{e^{-\Gamma}}{K}\int g dx-2\epsilon_1\sqrt{T_1}\int\frac{e^{-2\Gamma}}{K}(\int e^{\Gamma}K\delta_0 dt)dt\\&+\xi_0 T_1\int\frac{e^{-2\Gamma}}{K}dt+\xi_1,\end{aligned} \tag{3.20}$$

$$R(x,t)=e^{-\Gamma}\left(\frac{\epsilon T_1}{gK}\right)^{1/2}, \tag{3.21}$$

$$\begin{aligned}\theta(x,t)=&\,\frac{1}{4K}(2\gamma+\frac{\dot{K}}{K})(\int g dx)^2-\frac{1}{2K}\int g\,(\int g_t dx)dx-\int\frac{e^{-2\Gamma}}{K}(\int e^{\Gamma}K\delta_0 dt)^2 dt\\&+\frac{e^{-\Gamma}}{K}(\int e^{\Gamma}K\delta_0 dt-\frac{\epsilon_1\xi_0\sqrt{T_1}}{2})\int g dx+\int\delta_1 dt-\frac{\xi_0^2 T_1}{4}\int\frac{e^{-2\Gamma}}{K}dt\\&+\epsilon_1\xi_0\sqrt{T_1}\int\frac{e^{-2\Gamma}}{K}(\int e^{\Gamma}K\delta_0 dt)dt+\theta_0.\end{aligned} \tag{3.22}$$

(ii) The case when $f(x,t)=f(t)$, $g(x,t)=g(t)$.

Surely this case is included in (i) but we want to present the results since a wide literature is devoted to the study of equations with this special form of the coefficient functions. In this case, eqs. (3.14) are not imposed but rather satisfied identically. Transformations can be directly obtained from (3.13),(3.20)-(3.22) via the substitution

$$K(t)=f(t)g^2(t), \quad \gamma(t)=\beta(t)-\frac{\dot{g}(t)}{g(t)}, \quad \delta_0(t)=\frac{H_1(t)}{g(t)}, \quad \delta_1(t)=H_2(t). \tag{3.23}$$

Allowed transformations are found to be, with $B(t) = 2\int \beta(t)dt$,

$$
\begin{aligned}
T(t) =&T_1 \int \frac{g^2}{f} e^{-2B} dt + T_2, && (3.24)\\
X(x,t) =&\epsilon_1 \sqrt{T_1} \frac{g}{f} e^{-B} x - 2\epsilon_1 \sqrt{T_1} \int \frac{g^2}{f} e^{-2B} (\int \frac{f}{g} e^{B} H_1 dt) dt \\
&+ \xi_0 T_1 \int \frac{g^2}{f} e^{-2B} dt + \xi_1, && (3.25)\\
R(x,t) =&e^{-B} \left(\epsilon T_1 \frac{g}{f}\right)^{1/2}, && (3.26)\\
\theta(x,t) =&\frac{1}{4f}(2\beta + (\ln \frac{f}{g})_t) x^2 + \frac{g}{f} e^{-B} (\int \frac{f}{g} e^{B} H_1 dt - \frac{\epsilon_1 \xi_0 \sqrt{T_1}}{2}) x \\
&- \int \frac{g^2}{f} e^{-2B} (\int \frac{f}{g} e^{B} H_1 dt)^2 dt - \frac{\xi_0^2 T_1}{4} \int \frac{g^2}{f} e^{-2B} dt \\
&+ \epsilon_1 \xi_0 \sqrt{T_1} \int \frac{g^2}{f} e^{-2B} (\int \frac{f}{g} e^{B} H_1 dt) dt + \int H_2 dt + \theta_0. && (3.27)
\end{aligned}
$$

Remark. For $f = f_1 + if_2$, one can set $X_x^2 = \dot{T}/f_1$ and this normalizes f through (3.3) into $\tilde{f}(x,t) = 1 + i\tilde{f}_2(x,t)$ [23]. Thus, without losing any generality, we could equivalently investigate the equation

$$i\psi_t + (1 + if_2)\psi_{xx} + g(x,t)|\psi|^2\psi + h(x,t)\psi = 0. \tag{3.28}$$

From Painlevé analysis of Section 2 would immediately follow $f_2 = 0$ and $g = g(t)$ as a real-valued function and the potential components

$$h_1(x,t) = (\beta^2 + \frac{\dot{\beta}}{2} - \beta\frac{\dot{g}}{g} + \frac{g}{4}(\frac{1}{g})_{tt})x^2 + H_1(t)x + H_2(t), \quad h_2(x,t) = \beta(t). \tag{3.29}$$

Transformation formulae would considerably simplify in this setting. Yet, our results will remain useful when one begins with an equation with $f \neq 1$.

4 Integrability of a Gross-Pitaevskii equation

We apply our results obtained in previous sections to the equation

$$i\psi_t + \psi_{xx} + g(x,t)|\psi|^2\psi + kx^2\psi = 0, \tag{4.1}$$

where $g \in \mathbb{C}$ and $k \in \mathbb{R}$. This equation governs the dynamics of a Bose-Einstein condensate and its integrability was studied in [19] through the Painlevé analysis. The equation was shown to pass the Painlevé test for PDEs (WTC test) when the coefficient g satisfies

$$g\,\ddot{g} - 2\dot{g}^2 + 4kg^2 = 0 \tag{4.2}$$

and has the special form

$$g(x,t) = g(t) = \frac{2g_0 e^{\pm 2\sqrt{k}t}}{Ae^{\pm 4\sqrt{k}t} - B}, \tag{4.3}$$

where A, B, g_0 are arbitrary constants. Subsequently, by a proper transformation the equation is transformed to standard integrable nonlinear Schrödinger equation.

We can recover the equation (4.2) using our results of Section 1. Then, without considering results of the Painlevé analysis, we will try to transform the equation into the standard NLS equation based on its group-theoretical properties. To this end, we shall require that the equation under study has a five-dimensional symmetry algebra. We then show that this invariance requirement will force g to have exactly the same form obtained by the Painlevé approach.

Identifying (4.1) with (1.1), we see that for the Gross-Pitaevskii equation $f_1 = 1$, $f_2 = 0$, coefficient of the cubic term is the same complex function $g(x,t)$ and $h_1(x,t) = kx^2$, $h_2 = 0$. Since f is a constant (which is equal to 1), g is necessarily a function of t only. We make use of (2.26) and obtain the same equation (4.2) for g, under which equation (4.1) does have the P-property. If g is taken as a solution to (4.2), the GP equation transforms to standard NLS equation through the transformation formulae (3.24)-(3.27).

Existence of a transformation into standard NLS equation naturally motivates the question whether (4.1) can have a five-dimensional symmetry algebra for some form of coefficient function $g(x,t)$, in particular the one given in (4.3). We try to transform (4.1) directly to the NLS equation via (3.6) and (3.7). From (3.7a) we have

$$X(x,t) = \epsilon_1 \sqrt{\dot{T}} x + \xi(t) \tag{4.4}$$

and (3.6) implies $R(x,t) = R(t)$ and

$$\theta(x,t) = -\frac{1}{8}\frac{\ddot{T}}{\dot{T}} x^2 - \frac{\epsilon_1 \dot{\xi}(t)}{2\sqrt{\dot{T}}} x + \eta(t). \tag{4.5}$$

(3.7c) has real and imaginary parts, both to be set to zero. From the imaginary part we have $\frac{\dot{R}}{R} + \theta_{xx} = 0$, which integrates to give

$$R(t) = R_0 [\dot{T}(t)]^{1/4}, \qquad R_0 = \text{constant}. \tag{4.6}$$

From the real part we have $\theta_x^2 + \theta_t - kx^2 = 0$ and this gives differential conditions on $T(t), \xi(t)$ and $\eta(t)$:

$$\frac{\dddot{T}}{\dot{T}} - \frac{3}{2}\left(\frac{\ddot{T}}{\dot{T}}\right)^2 + 8k = 0, \tag{4.7a}$$

$$\ddot{\xi}\dot{T} - \dot{\xi}\ddot{T} = 0, \tag{4.7b}$$

$$\dot{\xi}^2 + 4\dot{\eta}\dot{T} = 0. \tag{4.7c}$$

The second equation yields

$$\xi(t) = \xi_0 T(t) + \xi_1 \tag{4.8}$$

with ξ_0, ξ_1 constants, while from the third it follows that

$$\eta(t) = -\frac{1}{4}\xi_0^2\, T(t) + \theta_0$$

with the constant θ_0. These determine θ in terms of T:

$$\theta(x,t) = -\frac{1}{8}\frac{\ddot{T}(t)}{\dot{T}(t)}x^2 - \frac{\epsilon_1\xi_0}{2}\dot{T}^{1/2}(t)\, x - \frac{\xi_0^2}{4}T(t) + \theta_0 \tag{4.9}$$

There remains the solution of (4.7a), which is recognizable to be a Schwarzian differential equation which we rewrite as

$$\{T, t\} + 8k = 0, \tag{4.10}$$

where $\{T, t\}$ denotes the Schwarzian derivative of T with respect to the variable t. By the Schwarz theorem it is well-known that its solution is given by a ratio $T(t) = \frac{\sigma_1(t)}{\sigma_2(t)}$, where σ_1 and σ_2 are linearly independent solutions of

$$\ddot{\sigma}(t) - 4k\sigma(t) = 0. \tag{4.11}$$

For a derivation of this result, the reader is referred to [24]. An alternative formulation based on a unimodular group invariance can be found in [25]. Furthermore, if the constants $T_1, ..., T_4$ satisfy $T_2T_3 - T_1T_4 \neq 0$, $T_1\sigma_1(t) + T_2\sigma_2(t)$ and $T_3\sigma_1(t) + T_4\sigma_2(t)$ are also linearly independent. Hence

$$T(t) = \frac{T_1\sigma_1(t) + T_2\sigma_2(t)}{T_3\sigma_1(t) + T_4\sigma_2(t)} \tag{4.12}$$

is a general solution to (4.10). Linearly independent solutions of (4.11) have constant Wronskian, so, without loss of generality, we can arrange to have

$$W[T_1\sigma_1(t) + T_2\sigma_2(t), T_3\sigma_1(t) + T_4\sigma_2(t)] = (T_1T_4 - T_2T_3)W[\sigma_1(t), \sigma_2(t)] = -1, \tag{4.13}$$

which provides T as a solution with three arbitrary constants to the third-order equation (4.10). After further calculations, we are going to see that this choice of the Wronskian will enable us to obtain the function $g(x,t)$ in its simplest form.

Let $k > 0$. From (4.11) we have $\sigma_1(t) = \cosh 2\sqrt{k}t$ and $\sigma_2(t) = \sinh 2\sqrt{k}t$ and (4.13) imposes $T_2T_3 - T_1T_4 = \frac{1}{2\sqrt{k}}$. We write

$$T(t) = \frac{T_1 \cosh 2\sqrt{k}t + T_2 \sinh 2\sqrt{k}t}{T_3 \cosh 2\sqrt{k}t + T_4 \sinh 2\sqrt{k}t} \tag{4.14}$$

and

$$\dot{T}(t) = [T_3 \cosh 2\sqrt{k}t + T_4 \sinh 2\sqrt{k}t]^{-2}. \tag{4.15}$$

For $k < 0$, we choose solutions of (4.11) as $\sigma_1(t) = \cos 2\sqrt{-k}t$ and $\sigma_2(t) = \sin 2\sqrt{-k}t$ and write

$$T(t) = \frac{T_1 \cos 2\sqrt{-k}t + T_2 \sin 2\sqrt{-k}t}{T_3 \cos 2\sqrt{-k}t + T_4 \sin 2\sqrt{-k}t} \tag{4.16}$$

and

$$\dot{T}(t) = [T_3 \cos 2\sqrt{-k}t + T_4 \sin 2\sqrt{-k}t]^{-2}, \tag{4.17}$$

with the similar condition $T_2T_3 - T_1T_4 = \frac{1}{2\sqrt{-k}}$.

Now we have determined all the unknown functions of the equivalence transformation. Our aim was to determine $g(x,t)$ so that (4.1) has a five-dimensional symmetry algebra. From (4.6) and (3.7b) we find

$$g(x,t) = g(t) = \frac{\epsilon}{R_0^2}\, \dot{T}^{1/2}. \tag{4.18}$$

The following summarizes possible forms of $g(x,t)$:

$$g_1(t) = \frac{\epsilon}{R_0^2}\, [T_3 \cosh 2\sqrt{k}t + T_4 \sinh 2\sqrt{k}t]^{-1}, \qquad k > 0, \tag{4.19}$$

$$g_2(t) = \frac{\epsilon}{R_0^2}\, [T_3 \cos 2\sqrt{-k}t + T_4 \sin 2\sqrt{-k}t]^{-1}, \qquad k < 0. \tag{4.20}$$

Now we would like to compare the results obtained by symmetry consideration with those given in [19]. In a search for the function $g(x,t)$ through Painlevé analysis, authors of [19] obtained the differential equation (4.2) and write (4.3) as its solution. Setting

$$\dot{T} = Lg^2, \quad L = \text{constant} \tag{4.21}$$

taking the relation (4.18) between T and g into account, and plugging it in (4.7a), we obtain exactly the equation (4.2) for g. This implies that (4.3) ought to be contained in $g(x,t)$ obtained by our approach. Indeed, if we choose

$$T_3 = \frac{R_0^2}{\epsilon}\frac{A-B}{2g_0} \quad \text{and} \quad T_4 = \pm\frac{R_0^2}{\epsilon}\frac{A+B}{2g_0}$$

in (4.19), we see that it transforms to (4.3). We note that the solution (4.3) does not include the case when $k < 0$. The possibility for g given by (4.20) should be added as an integrable case.

In fact, (4.7a) is nothing but a special case of the third term which vanished identically in (3.11) because at the beginning of Section 3 we assumed that the equation have the P-property; i.e., (2.23) holds. But now, since we have not taken the

P-property into consideration, it is obtained as an equation that has to be satisfied. We show that it is equivalent to the condition that the equation has the P-property.

To illustrate our results in a tidy form we introduce the functions

$$U(t) = \begin{cases} T_1 \cosh 2\sqrt{k}t + T_2 \sinh 2\sqrt{k}t, & \text{for } k > 0 \\ T_1 \cos 2\sqrt{-k}t + T_2 \sin 2\sqrt{-k}t, & \text{for } k < 0, \end{cases} \tag{4.22}$$

$$V(t) = \begin{cases} T_3 \cosh 2\sqrt{k}t + T_4 \sinh 2\sqrt{k}t, & \text{for } k > 0 \\ T_3 \cos 2\sqrt{-k}t + T_t \sin 2\sqrt{-k}t, & \text{for } k < 0. \end{cases} \tag{4.23}$$

In conclusion, we have shown that, if $g(x,t)$ of (4.1) is one of $g_j(t),\ j = 1, 2$, using the transformation

$$\begin{aligned} \tilde{t}(t) &= T(t) = \frac{U}{V}, \\ \tilde{x}(x,t) &= X(x,t) = \frac{\epsilon_1}{V}\, x + \xi_0 \frac{U}{V} + \xi_1, \\ R(x,t) &= \frac{R_0}{V^{1/2}}, \\ \theta(x,t) &= \frac{\dot{V}}{4V}\, x^2 - \frac{\epsilon_1 \xi_0}{V}\, x - \frac{\xi_0^2}{4} \frac{U}{V} + \theta_0, \end{aligned} \tag{4.24}$$

we can convert (4.1) to the equation

$$i\tilde{\psi}_{\tilde{t}} + \tilde{\psi}_{\tilde{x}\tilde{x}} + \epsilon |\tilde{\psi}|^2 \tilde{\psi} = 0. \tag{4.25}$$

Thus, if $\tilde{\psi}(\tilde{x}, \tilde{t})$ is any solution of (4.25), then for $g_j(t)$, $j = 1, 2$

$$\psi(x,t) = R(x,t) e^{i\theta(x,t)}\ \tilde{\psi}\left(X(x,t), T(t)\right) \tag{4.26}$$

is a solution of (4.1).

As a by-product, we can transform the symmetry algebra (3.5) by the equivalence transformations to obtain a basis for the symmetry algebra of (4.1) as follows:

$$\begin{aligned} X_1 &= V^2 \partial_t + (-\xi_0 V + V\dot{V} x)\partial_x - \frac{1}{2}\rho V \dot{V} \partial_\rho + \left(\frac{\xi_0^2}{4} - \frac{\xi_0}{2}\dot{V} x + \frac{1}{4}(\dot{V}^2 + V\ddot{V})x^2 \right) \partial_\omega, \\ X_2 &= V\partial_x + \frac{1}{2}(-\xi_0 + \dot{V} x)\partial_\omega, \\ X_3 &= \partial_\omega, \\ X_4 &= U\partial_x + \frac{1}{2}(\xi_1 + \frac{1 + U\dot{V}}{V} x)\partial_\omega, \\ X_5 &= UV\partial_t + \left(-\frac{\xi_0}{2} U + \frac{\xi_1}{2} V + (\frac{1}{2} + U\dot{V})x \right) \partial_x - \frac{1}{2}\rho(1 + U\dot{V})\partial_\rho \\ &+ \left(-\frac{\xi_0 \xi_1}{4} + (-\frac{\xi_0}{4}\frac{1}{V} + \frac{\xi_1}{4}\dot{V} - \frac{\xi_0}{4}\frac{U\dot{V}}{V})x + \frac{1}{4}(\frac{\dot{V}}{V} + \frac{U\dot{V}^2}{V} + U\ddot{V})x^2 \right) \partial_\omega. \end{aligned} \tag{4.27}$$

5 Summary

We took the approach of Painlevé test combined with the symmetry properties as a preliminary attempt towards integrability of the most general variable coefficient NLS equation. The other tool which plays a central role in the analysis is the notion of equivalence transformations relating the VCNLS equation to its canonical form. As confirmed by its various special cases studied in literature, equations passing Painlevé test possess a Lax pair which is usually considered as an essential characteristic of complete integrability. We also expect that this is also the case for the coefficients singled out by the Painlevé test.

The results of this paper can be used to obtain exact (soliton) solutions of the VCNLS equation, when the coefficients have some specific form, from those of the integrable NLS equation by point transformations whose precise forms are presented throughout the paper.

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