Orthogonal Main Effect Plans on blocks of small size.

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Abstract

In this paper we define the concept of orthogonality between two factors "through another factor". Exploiting this property we have been able to obtain orthogonal main effect plans (OMEP) on non-orthogonal blocks requiring considerably smaller number of blocks than the existing methods.

We have also constructed saturated partially orthogonal main effect plans (MEPs) for (i) an $n^4.2^3$ experiment and (ii) an $n^4.2.3$ experiment both on $4n$ runs. Here n is an integer $\geq 3, n \neq 4$.

As particular cases, we have been able to accomodate four six-level factors on 8 blocks of size 4 each using the first method and on 24 runs using the second.

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1 Introduction

Main effect plans (MEPs) with mixed levels are often required for industrial experiments. The orthogonal main effect plans (OMEP) are, of course, the best option. However, due to the divisibility conditions, an OMEP for an asymmetrical experiments often require a large number of runs. For this reason, considerable attention has been devoted in the recent past to find main effect plans with the orthogonality condition relaxed to an extent. Such plans were first proposed in Wang and Wu (1992). Subsequently many others like Nguyen (1996), Ma, Fang and Liski (2000), Huang, Wu and Yen (2002) and Xu (2002) proposed and studied what they term "nearly orhogonal" arrays, concentrating mostly on two- or three-level factors. In many of these plans, there are factors non-orthogonal to three or more factors. As a result, in spite of the elegant combinatorial properties, the precision of the estimates go down.

In our earlier paper [Bagchi (2006)] we have presented "inter-class orthogonal" MEPs, where a factor is non-orthogonal only to factors in its own class - to at most two factors in many of the plans. In this paper we continue the search for efficient MEPs with run size not too big.

Study of an orthogonal main effect plan with possibly non-orthogonal blocking was initiated in Mukerjee, Dey and Chatterjee (2001). They derived sufficient conditions for an OMEP to be universally optimal and also suggested a construction procedure for obtaining optimal OMEPs. In Bose and Bagchi (2007) we came up with a new set of suficient conditions for OMEP on non-orthogonal blocks of size two, requiring smaller number of blocks than Mukerjee, Dey and Chatterjee (2001). In this paper we generalise this idea. We define the concept of orthogonality between two factors "through another factor" (which may or may not be a blocking factor). Using this concept we have constructed an OMEP for a $4.n^3$ experiment on n blocks of size 4 each, for $n \geq 5$. Treating the block factor as another treatment factor, one would obtain a $4.n^4$ experiment on $4n$ runs. Although the main effects of this additional factor are not estimated with high precision, [see (3.14)], there is considerable amount of reduction in number of runs. [We may recall that an OMEP for an n^4 experiment requires at least n^2 runs]. The four-level factor may also be replaced by three two-level factors or one three and one two-level factor.

We have presented another method showing that an OMEP for an s^m experiment on bk non-orthogonal blocks of size $k \geq 2$ exists provided a connected binary block design with parameters (b, k, s) exist. Here m is the maximum number of constraints of an orthogonal array of strength two, k^2 runs and k symbols. In particular, we have obtained an OMEP for a 6^4 experiment on 8 blocks of size 4 each.

2 An alternative to an existing MEP.

We begin with a new main effect plan for a $3^4 2^3$ experiment and compare it with the existing plan. Before that we need a few notation.

Notation 2.1 *A main effect (MEP) plan with a set F of m factors $\{A, B \cdots\}$ and n runs will be represented by an array $\rho(n, m; s_A \times s_B \cdots)$, termed* **mixed array**. *In ρ rows will represent factors (in natural order) and S_A (respectively s_A) will denote the set (respectively number) of levels of the factor $A \in F$. Often, different rows have the same number of symbols and we may denote the plan as $\rho(n, m; \prod_{i=1}^{k} (s_i)^{m_i})$, where $\sum_{i=1}^{k} m_i = m$.*

The **vector of unknown effects** *of the levels of factor A will be denoted by the $s_A \times 1$ vector α. The* **replication vector** *of the factor A is the vector of replication numbers of its levels in the natural order and will be denoted by r^A. R^A will denote the diagonal matrix with entries as that of r^A in the same order. C_A will denote the* **coefficient matrix (C-matrix)** *in the reduced normal equation obtained by eliminating the effects of all factors other than A.*

For two factors A,B, the **incidence matrix** *$N^{A,B}$ is an $s_A \times s_B$ matrix with the (k, l)th entry as the number of occurences of the level k of A and l of B together. $N^{A,B} = R^A$, when $B = A$.*

For a plan represented by a mixed array $\rho(n, m; s_A \times s_B \cdots)$ let Y denote the vector of yields. Then, our model is

$$E(Y) = \mu 1_n + \sum_{A \in F} X_A \alpha, \tag{2.1}$$

where μ is the general effect, α is as stated above and X_A is a 0-1 matrix as described below. For $A \in F$, the $(i, j)$the entry of X_A is 1 if the ith column of ρ contains $j \in S_A$ in the row coresponding to factor A and 0 otherwise.

We present the well-known definition of an orthogonal array for the sake of completeness.

Definition 2.1 *An orthogonal array $OA(n, m, s_1 \times \cdots \times s_m, t)$, having $m(\geq 2)$ rows, n columns, $s_1, \ldots, s_m(\geq 2)$ symbols and strength $t(\leq m)$, is an $m \times n$ array, with elements in the ith row from a set of s_i distinct symbols $(1 \leq i \leq m)$, in which all t-tuples of symbols appear equally often (say λ times) as columns in every $t \times n$ subarray.*

If $s_i = s, 1 \leq i \leq m$, the OA is denoted by $OA(n, m, s, t)$. λ is called the index of the OA. Let us recall the nearly orthogonal array $L'_{12}(3^4 2^3)$ of Wang and Wu (1992), referred to here as $A_{WW}(12)$.

We shall suggest an MEP $A_1(12)$ for a $3^4 2^3$ experiment described as follows.

$$\text{Let } U_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \qquad (2.2)$$

Let V denote the array obtained by adding a row of all zeros to U_1. Then,

$$A_1(12) = \begin{bmatrix} V & V+1 & V+2 \\ U_1 & U_1 & U_1 \end{bmatrix}.$$

[Here addition is modulo 3].

Comparing the MEPs $A_{WW}(12)$ and $A_1(12)$, we find the following.

1. In both the plans, each two-level factor is orthogonal to every other factor.

2. In both the plans, no pair of three-level factors satisfy proportional frequency condition.

3. The C-matrices of the three-level factors for the plan $A_{WW}(12)$ are as follows.

$$C_Q = (7/3)K_3,\ Q = A, B, C, D. \qquad (2.3)$$

[Here $K_n = I_n - (1/n)J_n$, J is the matrix of all-ones.]

Those for $A_N(12)$ are as follows.

$$C_Q = 3K_3, Q = A, B, C;\ Q_D = (4/3)K_3.$$

We observe that $A_1(12)$ provides more information to factors A,B,C than $A_{WW}(12)$ but less to D. Further, $\sum\limits_{Q=A,B,C,D} C_Q$ is bigger for $A_1(12)$, so that the total information is more for this plan. Thus, $A_1(12)$ is useful in situations where all the factors are not equally important.

Now, we go to the deeper question. We note that in $A_{WW}(12)$, for each pair $P \neq Q, P, Q \in \{A, B, C, D\}$, N_{PQ} is the incidence matrix of a balanced block design (BBD), the "best" possible incidence matrix. On the other hand, for $A_1(12)$, neither of $N_{PQ}, P \neq Q$ is a "good" incidence matrix. How is then, C_A, for instance, is bigger for $A_1(12)$?

Before going to the investigation of this mystry, we present two plans which are obtained by modifying $A_1(12)$ slightly.

$$\text{Let } U_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \text{ and } U_3 = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}. \qquad (2.4)$$

Then,

$$A_i(12) = \begin{bmatrix} V & V+1 & V+2 \\ U_i & U_i & U_i \end{bmatrix}, i = 2, 3.$$

$A_2(12)$ and $A_3(12)$ are an MEPs for a $3^5.2$ and a $3^4.4$ experiment. The C-matrices of the first four (three-level) factors are, of course, same as those for $A_1(12)$. The C-matrices of the new three-level factor and the two-level factor of $A_2(12)$ are $3K_3$ and $3K_2$ respectively. The C-matrix of the four-level factor in $A_3(12)$ is $3K_4$. [K_n is as in the statement below (2.3).

To go into the mystry of "bigger" C-matrices of A,B,C in $A_i(12)$, we need some more notation.

Notation 2.2 *For $L \subseteq F; U, V \in F \setminus L$, $C_{U,V;L}$ will denote the (U,V)th submatrix of the coefficient matrix (C-matrix) in the reduced normal equation obtained by eliminating the effects of the member(s) of L. $C_{U,U;L}$ will be denoted by simply $C_{U;L}$. When $L = S \setminus U$, $C_{U;L}$ will be denoted simply by C_U, to be consistent with Notation 2.1.*

Definition 2.2 *If in a mixed array the factors A,B, and C satisfies the following condition, then we say that factors A and B are mutually* **orthogonal through C**.

$$N^{A,B} = N^{A,C}(R^C)^{-1}N^{C,B} \qquad (2.5)$$

The following lemma is immediate from the definition above.

Lemma 2.1 *If in a mixed array a pair of factors A and B are orthogonal through C, then $C_{A,B;C} = 0$.*

Lemma 2.2 *Suppose in a mixed array ρ two factors A and B satisfy the following condition.*

For every $Q \in \tilde{F} = F \setminus \{A, B\}$, A and Q are mutually orthogonal through B. Then,

$$C_A = C_{A;B}. \qquad (2.6)$$

Proof : We know that

$$C_A = C_{A;B} - E_{A;B}(H_{A;B})^{-}(E_{A;B})^T, \qquad (2.7)$$

where $E_{A;B} = ((C_{A,Q;B}))_{Q\in\tilde{F}}$ and $H_{A;B} = ((C_{P,Q;B}))_{P\neq Q,\ P,Q\in\tilde{F}}$.

From the hypothesis, $E_{A:B} = 0$. Hence the result. $\Box$.

Let us now look at the plans $A_i(12), i = 1, 2, 3$. We observe the following property.

Lemma 2.3 *For each of the plans $A_i(12), i = 1, 2, 3$, the C-matrices of factors A,B,C satisfy*

$$C_P = C_{P;D}, P \in \{A, B, C\}. \qquad (2.8)$$

Proof : It is easy to verify that each pair of factors $(P,Q), P,Q \in \{A,B,C\}$, are mutually orthogonal through D. Hence the result follows from Lemma 2.2. $\square$

Lemma 2.3 is the clue of the mystry of the "bigger" C-matrix in spite of the "poor" incidence matrices. We note that for $A_{WW}(12)$ $E_{P:D} \neq 0$ for each $P \in \{A,B,C\}$ and so, $C_P < C_{P;D}$. As a result, even though $C_{P;D}$ is bigger for $A_{WW}(12)$ than that for $A_1(12)$, C_P becomes smaller.

However, no such facility is available for the factor D. Further, the incidence matrices of D with the other factors are not "good". This is why C_D is smaller (than that in $A_{WW}(12)$.

Before going to the general construction in the next section, we present another example of a plan with two factors "orthogonal through another". The plan is for a 3^3 experiment on 8 runs.

$$A_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that for A_8 the condition obtained by interchanging A and C in (2.5) is satisfied, so that factors B and C are orthogonal through A. As a result,

$$C_B = C_{B;A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \qquad (2.9)$$

We note that C_B has spectrum $0^1.1^1.3^1$, which is only marginally "smaller" than the spectrum $0^1.(9/4)^1.3^1$ of the C-matrix of a three-level factor with the hypothetical OMEP with best possible replication vector.

Factor C also has the same C-matrix.

Since A is non-orthogonal to two others, C_A is "smaller", as shown below.

$$C_A = (1/6) \begin{bmatrix} 4 & -2 & -2 \\ -2 & 7 & -5 \\ -2 & -5 & 7 \end{bmatrix}. \qquad (2.10)$$

The spectrum of C_A is $0^1.1^1 2^1$.

3 Two series of orthogonal main effect plans on blocks of small size

Notation 3.1 *An MEP for an $\prod_{i=1}^m s_i$ experiment laid out on b blocks of size $k_1.k_2, \cdots k_b$ each will be represented by a mixed array $\rho(n, m+1; s_A \times s_B \cdots \times b;$ the last row of ρ represting the block factor. The set of factors will be represented by $F \cup \{bl\}$, bl denoting the block factor. R^{bl} will denote the diagonal matrix with entries as $k_1.k_2, \cdots k_b$.*

For the sake of easy reference, we write down the definition for orthogonality through the block factor for an equi-block-sized plan in a combinatorial form.

Let us consider an MEP on b blocks of size k each blocks represented by a mixed array $\rho(n, m+1; s_A \times s_B \cdots \times b)$. Fix two factors A and B. For $i \in S_A, j \in S_B$, let $U_{i,j}$ denote the number of columns in ρ with A at level i and B at level j. Let $B_{i,j}$ denote the number of times the level combination (i,j) of A,B appear in the same block of ρ (need not be in the same column).

Definition 3.1 *Consider a mixed array ρ including a blocking factor of equal frequency k. Two factors A and B of ρ are said to be mutually orthogonal through the block factor if*

$$B_{i,j} = kU_{i,j}, \ \forall i \in S_A, j \in S_B. \tag{3.11}$$

Note that $U_{i,j}$ is nothing but the (i,j)th entry of $N^{A,B}$, while $B_{i,j} = N^{A,bl}N^{bl,B}$. These together with the fact that $R^{bl} = kI_b$ implies that (3.11) is a special case of (2.5) and hence Definition 3.1 is a special case of Definition 2.2.

Remark : The case $k = 2$ is considered in Theorem 3.1 of Bose and Bagchi (2007).

We now generalise the arrays $A_i(12), i = 1, 2, 3$ to get an infinite series of MEPs.

Theorem 3.1 *(a) For an integer $n \geq 5$, there exist saturated MEPs for the following asymmetrical experiments on n blocks of size 4 each.*

(i) $n^3.2^3$, (ii) $n^3.2.3$ and (iii) $n^3.4$.

(b) Plans (i) and (iii) are orthogonal while (ii) is almost orthogonal. Specifically, in all these plans, the two, three and four-level factors are orthogonal to each of the n-level factors as well as the block factor. In plan (i) the two-level factors are orthogonal among each other. In plan (ii) the two and three-level factors are non-orthogonal among each other.

(c) The two-level factors have C-matrix $(2n)K_2$ in Plan (i) and nK_2 in Plan (ii). The C-matrices of the three-level factor in Plan (ii) and the four-level factor in Plan (iii) are given by nK_3 and nK_4 respectively.

(d) The C-matrices of three n-level factors A,B and C are as given below.

$$C_Q = \left(\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \end{pmatrix}\right), \ Q \in G = \{A, B, C\}.$$

Proof : (a) : Recall the arrays V and $U_i, i = 1, 2, 3$ described in (2.2) and (2.4 . The following arrays represent the plans for the experiments (i), (ii) and (iii) respectively. Here the last row represent the block factor. [Recall Notation 3.1].

$$A_i(4n) = \begin{bmatrix} U_i & U_i & \cdots & U_i \\ V & V+1 & \cdots & V+n-1 \end{bmatrix}, i = 1, 2, 3$$

(b) : It can be seen that each of the n-level factors satisfies proportional frequency condition with the two, three and four-level factors.

(c) : These can be checked by straight computation.

(d) : We note the following. Let the n-level factors be named as A,B,C in that order. Let $G = \{A, B, C\}$. Then, the incidence matrices between them are circulant matrices as given below.

$$N^{PQ} = L, P \neq Q, P, Q \in G, \qquad (3.12)$$

L is as given below.

$$L = ((\begin{array}{cccccc} 2 & 1 & 0 & \cdots & 0 & 1 \end{array})) \ . \qquad (3.13)$$

Further, one can verify that the pair of factors P and Q, $P \neq Q$, $P, Q \in G$, are mutually orthogonal through the block factor. Now, it follows from Lemma 2.1 that

$$C_Q = C_{Q;bl}, Q \in G.$$

Now, using (3.12) it is easy to verify that $C_Q, Q \in G$ is as given in the statement. □

Remark 1: In the plans constructed in Theorem 3.1, the blocking factor may also be considered as a treatment factor named D say. Since D is non-orthogonal to each of A,B,C, it's efficiency would be low. Still it may be useful : particularly for the case $n = 6$ as using 24 runs we can accomodate four six-level factors, while on 36 runs one can accomodate at most three mutually orthogonal six-level factors.

The C-matrix of D can be obtained as follows.

$$C_D = 4I_4 - E_D(H_D)^{-}(E_D)^T, \qquad (3.14)$$

where $E_D = \left[\begin{array}{ccc} M & M & M \end{array}\right]$ and $H_D = \left[\begin{array}{ccc} 4I_4 & L & L \\ L & 4I_4 & L \\ L & L & 4I_4 \end{array}\right]$.

Here M is the circulat matrices described below and L is as in (3.13).

$$M = 2((\begin{array}{ccccc} 1 & 1 & 0 & \cdots & 0 \end{array})) \qquad (3.15)$$

We now present another construction.

Theorem 3.2 *Suppose there exist a connected binary block design d with b blocks of size k each and v treatments. Let m denote the maximum number of constraints of an orthogonal array of strength two, k^2 runs, k symbols and index 1.*

(a) Then, $\exists$ an OMEP for a v^{m-1} experiment on bk blocks of size k each.

(b) Further, the C-matrix C_P of every factor P is kC_d.

Proof: Let us take the j th block of d consisting of the treatment set T_j say. Let A_j denote the $OA(k^2, m, k, 2)$ with T_j as the set of symbols for the first $m-1$ rows and the set $\{jk+1, jk+2, \cdots, (j+1)k\}$ for the last row. Then the array

$$A = \left[\begin{array}{cccc} A_1 & A_2 & \cdots & A_b \end{array} \right].$$

is the mixed array $\rho(bk^2, m; v^{m-1}.bk)$ representing the required MEP.

It is easy to verify that condition (3.11) is satisfied by every pair of factor. Thus A represent an OMEP. Part (b) follows easily.

Examples : 1. For a given v, there are many connected binary block designs with b blocks of size k each. Clearly, it is desirable that k is a prime or a prime power, so that one can accomodate k factors. Again, for the sake of economy, b should not be too large. Given these conditions, one possibility is to take $b = 2$ and k the smallest prime power $\geq [v/2]$. A binary connected design satisfying the above conditions always exist. Thus, Theorem 3.2 implies the existence of the OMEPs for the following experiments.

(i) 4^3 and 5^3 on 6 blocks of size 3 each, (ii) 6^4 and 7^4 on 8 blocks of size 4 each, (iii)8^5 and 9^5 on 10 blocks of size 5 each and so on.

2. In the examples above the block design d is not equireplicate and hence the factors would not have equal frequency. To achieve equal frequency, of course, number of runs has to be bigger, particularly when v is prime. [In that case the method of Mukerjee, Dey and Chatterjee (2001) might be useful]. We list the following connected and equireplicate block designs with composite v to be used in Theorem 3.2 together with the OMEP obtained.

(a) $v = 6 : b = 3, k = 4.$ $d :$

Block 1	Block 2	Block 3
1 2 3 4	1 2 5 6	3 4 5 6

OMEP : 6^4 experiment on 12 blocks of size 4 each.

(b) $v = 8 : b = k = 4.$ $d :$

Block 1	Block 2	Block 3	Block 4
1 2 3 4	5 6 7 8	1 2 5 6	3 4 7 8

OMEP : 8^4 experiment on 16 blocks of size 4 each.

(c) $v = 10 : b = 4, k = 5.$ $d :$

Block 1	Block 2	Block 3	Block 4
1 2 3 4 5	6 7 8 9 10	1 2 3 6 7	4 5 8 9 10

OMEP : 10^5 experiment on 20 blocks of size 5 each.

(d) $v = 12 : b = 3, k = 8.$ $d :$ $\left[\begin{array}{ccc} \text{Block 1} & \text{Block 2} & \text{Block 3} \\ \hline 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8 & 1\ 2\ 3\ 4\ 9\ 10\ 11\ 12 & 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12 \\ \hline \end{array}\right]$

OMEP : 12^8 experiment on 24 blocks of size 8 each.

Remark 3 : It is interesting to find that the performance of the OMEPs derived from the block designs (a), (b), (c) and (d) above are quite good in the following sense. For each factor in each plan, the BLUEs of all the main effect contrsts except two have the same variance with the hypothetical (real in the case $v = 8$) OMEP without the blocking factor on the same number of runs. The varinces of the BLUEs of the two remaining contrasts are, of course bigger.

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