

# ON THE EFFECT OF RANDOM ALTERNATING PERTURBATIONS ON HAZARD RATES\*

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## Abstract

We consider a model for systems perturbed by dichotomous noise, in which the hazard rate function of a random lifetime is subject to additive time-alternating perturbations described by the telegraph process. This leads us to define a real-valued continuous-time stochastic process of alternating type expressed in terms of the integrated telegraph process for which we obtain the probability distribution, mean and variance. An application to survival analysis and reliability data sets based on confidence bands for estimated hazard rate functions is also provided.

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*Key words and phrases:* Hazard rates; telegraph process; dichotomous noise; estimated hazard rates; confidence bands.

## 1 Introduction

Large attention has been given recently in the physical and mathematical literature to stochastic systems perturbed by dichotomous noise, such as those related to Brownian motion or Brownian motors (see, for instance, Bena *et al.* [5], [6] and Porrà *et al.* [21]). The relevance of Markovian dichotomous noise in biological systems has been also pointed out. See, for instance, Laing and Longtin [14] where the benefic role of noise is investigated within the context of mathematical neuroscience. In this paper we aim to study the effect of dichotomous noise to the hazard rate function of random lifetimes.

It is well known that in many fields related to survival analysis and reliability theory the study of hazard rates plays a very important role (see the classical book of Barlow and Proschan [3], for instance). Specifically, here we are interested to hazard rate functions that are realizations of stochastic processes. Such types of hazard rates arise in some models that were recently proposed to describe doubly random phenomena such as lifetimes of devices operating in random environments. See, for instance, Kebir [13] and the included references for a model in which the hazard rate is a functional of a stochastic process that describes the random variability of the environment. A similar model is treated in Di Crescenzo and Pellerey [9], where the hazard function is the realization of a non-decreasing stochastic process with independent increments. Other models characterized by stochastic hazard rate functions were also discussed by Ěžov and Jadrenko [11] and by Yadrenko and Zhegrii [28]. Furthermore, a stochastic model in which a constant hazard rate (describing

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the fractional dissolution rate of a drug potion) is corrupted by a white noise was recently analyzed by Lánský and Lánská [16] and by Lánský and Weiss [17] within the theory of drug dissolution. Other applications of stochastic hazard rates were proposed in the actuarial literature and in studies of population biology and human aging (see Milevsky and Promislow [19] and references therein). All these stochastic models are suitable to describe phenomena characterized by an intrinsic randomness, which is evidenced at the hazard rate level.

Along the line traced by the above mentioned papers, herewith we propose a new model to describe systems characterized by failure rates which are subject to random perturbations expressed as dichotomous noise. We aim to discuss a model that incorporates a more realistic kind of noise characterized by a constant intensity, and that at the same time presents the feature of noise with alternating behavior that is frequently found in biological systems. For instance, see the papers by Buonocore *et al.* [7] and Di Crescenzo *et al.* [8], dealing with neuronal dynamics subject to alternating inputs. Stochastic models characterized by alternating behaviour are also used to describe the cell motility on one-dimensional and two-dimensional state-spaces (see, for instance, Lutscher [18], Stadje [25], [26], and references therein).

Suppose that (non-negative) lifetime  $T$  has distribution function

$$(1) \quad F(t) = 1 - \exp \left\{ - \int_0^t r(s) ds \right\}, \quad t \geq 0,$$

so that its hazard rate function is

$$(2) \quad r(t) = \lim_{\Delta t \rightarrow 0^+} \frac{\mathbb{P}(T \leq t + \Delta t | T > t)}{\Delta t}, \quad t \geq 0.$$

We assume that  $r(t) > c$ , where  $c$  is a positive constant, and that  $r(t)$  is subject to additive time-alternating perturbations described by the well-known telegraph process (see, for instance, Orsingher [20] or Beghin *et al.* [4]). This assumption consists of applying the substitution

$$r(t) \rightarrow r(t) + V(0)(-1)^{N(t)}, \quad t \geq 0$$

in the right-hand-side of (1), where  $V(0) \in \{-c, c\}$  and  $\{N(t); t \geq 0\}$  is a Poisson process independent of  $T$ . We are thus led to defining a real-valued continuous-time stochastic process  $\{X(t); t \geq 0\}$ , where  $X(t)$  is expressed in terms of the integrated telegraph process:

$$(3) \quad X(t) = 1 - \exp \left\{ - \int_0^t [r(s) + V(0)(-1)^{N(s)}] ds \right\}, \quad t \geq 0.$$

Here, we present some basic results on process (3). First of all, in Section 2 we obtain the moment generating function of the integrated telegraph process, which is involved in the expression of mean and variance of  $X(t)$ . Such moments and the probability distribution of  $X(t)$  are given in Section 3. This distribution has a discrete as well as an absolutely continuous component. Finally, in Section 4 we use an asymptotic confidence band for estimated hazard rates in two case-studies where our model provides adequate fit to two data sets taken from the survival analysis and reliability theory literature.

We point out that the results obtained in this paper can be extended to the case where there are two different values  $V(0) \in \{-c_2, c_1\}$  and  $N(t)$  is an alternating Poisson process characterized by two rates  $\lambda_1$  and  $\lambda_2$ . Finally, we point out that stochastic processes obtained by transformations of the integrated telegraph process are of interest in stochastic modelling literature. For example, Di Crescenzo and Pellerey [10] use a geometric telegraph process to model the price dynamics of risky assets.

## 2 Moment generating function of integrated telegraph<sup>3</sup> process

Let  $\{W(t); t \geq 0\}$  denote the well-known integrated telegraph process, defined by

$$(4) \quad W(t) = V(0) \int_0^t (-1)^{N(s)} ds, \quad t \geq 0,$$

where  $\{N(t); t \geq 0\}$  is a homogeneous Poisson process with intensity  $\lambda$  and  $V(0)$  is a random variable independent from  $N(t)$  such that

$$(5) \quad P\{V(0) = \pm c\} = \frac{1}{2},$$

with  $c > 0$ . The distribution of (4) is characterized by discrete components concentrated at  $\pm ct$  and by an absolutely component on  $(-ct, ct)$ . Orsingher [20], shows that the discrete component satisfies

$$(6) \quad P\{W(t) = ct\} = P\{W(t) = -ct\} = \frac{1}{2} e^{-\lambda t}, \quad t \geq 0,$$

and that, for  $t \geq 0$ ,  $-ct < x < ct$ , the continuous component has density

$$(7) \quad \frac{\partial}{\partial x} P\{W(t) \leq x\} = \frac{1}{2c} e^{-\lambda t} \left[ \lambda I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right) + \frac{\partial}{\partial t} I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right) \right],$$

where

$$(8) \quad I_0(x) = \sum_{k=0}^{+\infty} \frac{(x/2)^{2k}}{(k!)^2}$$

denotes the modified Bessel function of order 0.

**Proposition 2.1** *For all  $s \in \mathbb{R}$  and  $t \geq 0$  the moment generating function of  $W(t)$  is*

$$(9) \quad \begin{aligned} M(s, t) &:= E \left[ e^{s W(t)} \right] \\ &= e^{-\lambda t} \left[ \cosh \left( t \sqrt{\lambda^2 + s^2 c^2} \right) + \frac{\lambda}{\sqrt{\lambda^2 + s^2 c^2}} \sinh \left( t \sqrt{\lambda^2 + s^2 c^2} \right) \right]. \end{aligned}$$

**Proof.** From Eqs. (6) and (7) it follows

$$(10) \quad \begin{aligned} M(s, t) &= \frac{e^{-\lambda t}}{2} (e^{sct} + e^{-sct}) \\ &\quad + \frac{e^{-\lambda t}}{2c} \int_{-ct}^{ct} e^{sx} \left[ \lambda I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right) + \frac{\partial}{\partial t} I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right) \right] dx \\ &= \frac{e^{-\lambda t}}{2c} \left[ \lambda Q(s, t) + \frac{\partial}{\partial t} Q(s, t) \right], \quad s \in \mathbb{R}, t \geq 0, \end{aligned}$$

where we have set

$$Q(s, t) := \int_{-ct}^{ct} e^{sx} I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right) dx.$$

Making use of Eq. (25), from Orsingher [20] we obtain:

$$\int_{-ct}^{ct} e^{sx} \frac{\partial^2}{\partial t^2} I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right) dx = \int_{-ct}^{ct} e^{sx} c^2 \frac{\partial^2}{\partial x^2} I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right) dx + \lambda^2 Q(s, t).$$

A two-fold integration by parts shows that

$$\frac{d^2}{dt^2} Q(s, t) = (\lambda^2 + s^2 c^2) Q(s, t).$$

Solving this equation with initial conditions  $Q(s, 0) = 0$  and  $\frac{d}{dt} Q(s, t)|_{t=0} = 2c$  we have:

$$Q(s, t) = \frac{c}{\sqrt{\lambda^2 + s^2 c^2}} \left[ e^{t\sqrt{\lambda^2 + s^2 c^2}} - e^{-t\sqrt{\lambda^2 + s^2 c^2}} \right], \quad s \in \mathbb{R}, t \geq 0.$$

Using this formula in the right-hand-side of (10), expression (9) finally follows.  $\square$

It should be noticed that (9) could also be obtained from the initial-value problem for the telegraph equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} p + 2\lambda \frac{\partial}{\partial t} p = c^2 \frac{\partial^2}{\partial x^2} p \\ p(x, 0) = \delta(x) \\ \frac{\partial}{\partial t} p(x, t)|_{t=0} = 0, \end{cases}$$

where  $\delta(x)$  is the Dirac delta function. Indeed,  $M$  is solution of

$$\begin{cases} \frac{d^2}{dt^2} M + 2\lambda \frac{d}{dt} M = s^2 c^2 M \\ M(s, 0) = 1 \\ \frac{\partial}{\partial t} M(s, t)|_{t=0} = 0. \end{cases}$$

### 3 Results

Let  $T$  be a non-negative random variable, with absolutely continuous distribution function (1), hazard rate function (2), and support

$$\mathcal{S} := [0, \ell), \quad \text{where } \ell = \sup\{t \geq 0 : F(t) < 1\} \in (0, +\infty].$$

Throughout the paper we will assume that  $r(t) > c$  for all  $t \in \mathcal{S}$ , where  $c$  is a positive constant. Hence,

$$F(0) = 0, \quad F(t) > 1 - e^{-ct} \quad \text{for all } t > 0,$$

so that

$$T \leq_{\text{st}} \text{Exp}(c),$$

where ' $\leq_{\text{st}}$ ' denotes the usual stochastic order (see, for instance, Shaked and Shantikumar [23]) and  $\text{Exp}(c)$  denotes an exponentially distributed random variable with mean  $c^{-1}$ .

Let us now consider the stochastic process  $\{X(t); t \in \mathcal{S}\}$ , with  $X(0) = 0$  and

$$(11) \quad X(t) = 1 - \overline{F}(t) e^{-W(t)}, \quad t \in \mathcal{S},$$

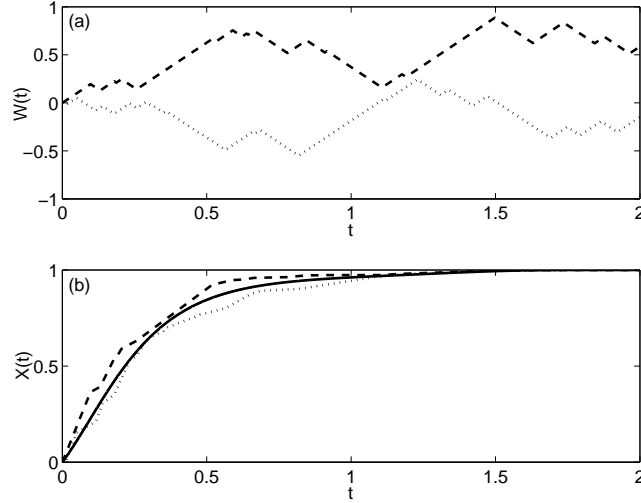


Figure 1: (a) Two sample-paths of process (4) for  $c = 2$  and  $\lambda = 15$ ; (b) the corresponding sample-paths of (11), with  $F(t)$  and  $r(t)$  defined in (12) and (13), respectively, with  $\alpha = 15$  and  $\beta = 0.001$ ; the solid line shows  $F(t)$ .

where  $W(t)$  has been defined in (4) and where  $\bar{F}(t) = 1 - F(t)$  denotes the survival function of  $T$ . Note that process (11) is equivalent to process (3). Hence, for any fixed  $t$ ,  $X(t)$  describes the probability that a random lifetime is not larger than  $t$ . The system hazard rate is random and is given by  $r(t) + V(0)(-1)^{N(t)}$ , where the distribution of  $V(0)$  is specified in (5). As pointed out in Section 1, this hazard rate accounts for random perturbations on  $r(t)$  that occur according to a telegraph process. The sample-paths of  $X(t)$  are absolutely continuous distribution functions that approach (1) as  $c$  goes to 0. As an example, Figure 1 shows (a) two sample-paths of  $W(t)$ , and (b) the corresponding sample-paths of  $X(t)$ , characterized by distribution function

$$(12) \quad F(t) = 1 - \exp \left\{ - \left[ \frac{\alpha}{4} t^4 - \frac{2\alpha}{3} t^3 + \frac{\alpha}{2} t^2 + (c + \beta) t \right] \right\}, \quad t \geq 0,$$

where

$$(13) \quad r(t) = \alpha t(t-1)^2 + c + \beta, \quad t \geq 0, \quad (\alpha, \beta > 0).$$

Next, we derive the probability distribution of  $X(t)$ . Since  $\mathbb{P}\{-ct \leq W(t) \leq ct\} = 1$ ,  $t \geq 0$ , from Eq. (11) we have:

$$\mathbb{P}\{a(t) \leq X(t) \leq b(t)\} = 1, \quad t \in \mathcal{S},$$

and

$$a(t) < F(t) < b(t) \quad \text{for all } t \in (0, \ell),$$

where

$$(14) \quad a(t) = 1 - \bar{F}(t) e^{ct}, \quad b(t) = 1 - \bar{F}(t) e^{-ct}, \quad t \in \mathcal{S}.$$

From (14), we note that  $a(t)$  and  $b(t)$  are distribution functions with hazard functions  $r(t) - c$  and  $r(t) + c$ , respectively. Furthermore, there results:

$$\lim_{t \rightarrow \ell} a(t) = 1 - e^{-\nu}, \quad \lim_{t \rightarrow \ell} b(t) = 1,$$

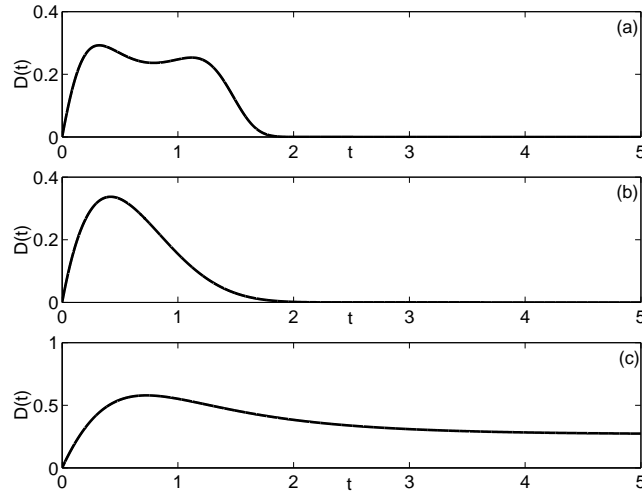


Figure 2: Plots of  $D(t)$  in 3 different cases: (a)  $r(t)$  given by Eq. (13), with  $c = 1$ ,  $\lambda = \alpha = 15$  and  $\beta = 0.001$ ; (b)  $r(t) = 1 + e^t$ ,  $t \geq 0$ , with  $c = \lambda = 1$ ; (c)  $r(t) = 1 + 3(1 - e^{-t})/(e^t + e^{-t})$ ,  $t \geq 0$ , with  $c = \lambda = 1$ .

where

$$\nu = \int_0^\ell [r(s) - c] ds.$$

We note that  $\nu = +\infty$  when  $\ell < +\infty$ , whereas  $\nu \leq +\infty$  if  $\ell = +\infty$ . Moreover, by setting  $D(t) := b(t) - a(t)$ , from (14) we obtain:

$$D(t) = 2\bar{F}(t) \sinh(ct), \quad t \in \mathcal{S},$$

with  $D(0) = 0$  and  $\lim_{t \rightarrow \ell} D(t) = e^{-\nu}$ , with  $D(t)$  non-decreasing in  $t$  if

$$r(t) \leq c \coth(ct).$$

For instance,  $D(t)$  is shown in Figure 2 for three different choices of  $r(t)$ . The first choice shows a case in which  $D(t)$  is bimodal. In the third choice we have  $\lim_{t \rightarrow +\infty} D(t) = 0.268$ , whereas such limit vanishes in the first two choices.

We shall now determine the probability distribution of  $X(t)$ . Like as for the integrated telegraph process, the distribution of  $X(t)$  consists of a discrete component on  $a(t)$  and  $b(t)$  and an absolutely continuous component inside of  $(a(t), b(t))$ .

**Proposition 3.1** *For all  $t \in \mathcal{S}$  the discrete component of the distribution of  $X(t)$  is:*

$$(15) \quad \mathbb{P}\{X(t) = a(t)\} = \mathbb{P}\{X(t) = b(t)\} = \frac{1}{2} e^{-\lambda t}.$$

Moreover, for all  $x \in (a(t), b(t))$  and  $t \in \mathcal{S}$  the continuous component has density

$$(16) \quad f(x, t) = \frac{1}{2c} \frac{1}{1-x} e^{-\lambda t} \left[ \lambda I_0 \left( \frac{\lambda}{c} \sqrt{u(x, t)} \right) + \frac{\partial}{\partial t} I_0 \left( \frac{\lambda}{c} \sqrt{u(x, t)} \right) \right],$$

where

$$(17) \quad \begin{aligned} u(x, t) &:= c^2 t^2 - \ln^2 \frac{\bar{F}(t)}{1-x} \\ &= \ln \left( \frac{1-a(t)}{1-x} \right) \ln \left( \frac{1-x}{1-b(t)} \right), \end{aligned}$$

with  $a(t)$  and  $b(t)$  defined in (14).

**Proof.** Due to (11), the distribution function of  $X(t)$  can be expressed as

$$\mathrm{P}\{X(t) \leq x\} = \mathrm{P}\left\{W(t) \leq \ln \frac{\bar{F}(t)}{1-x}\right\}, \quad t \in \mathcal{S}, \quad a(t) \leq x \leq b(t).$$

Hence, recalling (6) and (7), Eqs. (15) and (16) follow.  $\square$

The expression of density (16) is similar to that of the underlying integrated telegraph process. Unlike the latter process, however, the support of  $f(x, t)$  is bounded when  $t$  increases. Moreover, the expression for  $f(x, t)$  reflects the form of interval  $(a(t), b(t))$ . Indeed, from (17) it is not hard to note that  $u(x, t) = 0$  when  $x = a(t)$  and  $x = b(t)$ , whereas  $u(x, t) > 0$  when  $x \in (a(t), b(t))$ . Furthermore, recalling (14) and (8), for  $t \in \mathcal{S}$  we have:

$$\lim_{x \downarrow a(t)} f(x, t) = \frac{\lambda e^{-(\lambda+c)t}}{2c \bar{F}(t)} \left(1 + \frac{\lambda t}{2}\right), \quad \lim_{x \uparrow b(t)} f(x, t) = \frac{\lambda e^{-(\lambda-c)t}}{2c \bar{F}(t)} \left(1 + \frac{\lambda t}{2}\right).$$

Figure 3 shows for instance  $f(x, t)$  for three different choices of  $t$ .

**Proposition 3.2** For all  $t \geq 0$  mean and variance of  $X(t)$  are:

$$(18) \quad \mathrm{E}[X(t)] = 1 - \bar{F}(t) M(-1, t),$$

$$(19) \quad \mathrm{Var}[X(t)] = \bar{F}^2(t) \{M(-2, t) - [M(-1, t)]^2\},$$

where  $M(s, t)$  is given in (9).

**Proof.** Recalling (11), we have

$$\mathrm{E}[X(t)] = 1 - \bar{F}(t) \mathrm{E}\left(e^{-W(t)}\right),$$

from which Eq. (18) immediately follows. Moreover, due to

$$\begin{aligned} \mathrm{E}[X^2(t)] &= \mathrm{E}\left(1 - 2\bar{F}(t)e^{-W(t)} + \bar{F}^2(t)e^{-2W(t)}\right) \\ &= 1 - 2\bar{F}(t)\mathrm{E}\left(e^{-W(t)}\right) + \bar{F}^2(t)\mathrm{E}\left(e^{-2W(t)}\right), \end{aligned}$$

we obtain:

$$\begin{aligned} \mathrm{Var}[X(t)] &= 1 - 2\bar{F}(t)\mathrm{E}\left(e^{-W(t)}\right) + \bar{F}^2(t)\mathrm{E}\left(e^{-2W(t)}\right) \\ &\quad - 1 + 2\bar{F}(t)\mathrm{E}\left(e^{-W(t)}\right) - \left[\mathrm{E}\left(e^{-W(t)}\right)\right]^2 \bar{F}^2(t) \\ &= \bar{F}^2(t) \left\{ \mathrm{E}\left(e^{-2W(t)}\right) - \left[\mathrm{E}\left(e^{-W(t)}\right)\right]^2 \right\}, \end{aligned}$$

and thus Eq. (19).  $\square$

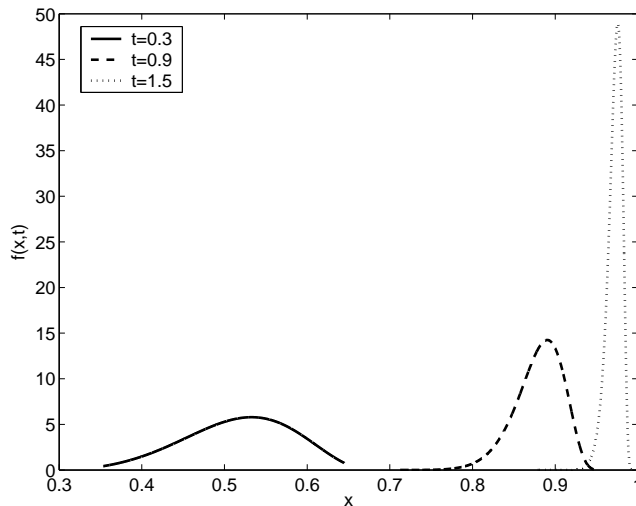


Figure 3: Plot of density (16) for three choices of  $t$  and for  $x$  ranging within  $(0, 1)$ ;  $r(t)$  and all parameters are chosen as in case (a) of Figure 2, i.e.  $c = 1$ ,  $\lambda = \alpha = 15$ ,  $\beta = 0.001$  and  $r(t)$  is given by Eq. (13).

**Proposition 3.3** *The mean of  $X(t)$  is increasing in  $t \in \mathcal{S}$ , and is decreasing in  $c$ . Moreover,*

$$(20) \quad \lim_{t \rightarrow 0} \mathbb{E}[X(t)] = 0, \quad \lim_{t \rightarrow 0} \text{Var}[X(t)] = 0;$$

$$(21) \quad \lim_{t \rightarrow \ell} \mathbb{E}[X(t)] = 1, \quad \lim_{t \rightarrow \ell} \text{Var}[X(t)] = 0.$$

**Proof.** Using Eqs. (9) and (1),

$$\begin{aligned} & \frac{d}{dt} [\bar{F}(t)M(-1, t)] \\ &= -\frac{r(t) + \lambda - \sqrt{\lambda^2 + c^2}}{2} \left(1 + \frac{\lambda}{\sqrt{\lambda^2 + c^2}}\right) \exp \left\{ -\int_0^t r(u)du - \lambda t + \sqrt{\lambda^2 + c^2} t \right\} \\ & \quad - \frac{r(t) + \lambda + \sqrt{\lambda^2 + c^2}}{2} \left(1 - \frac{\lambda}{\sqrt{\lambda^2 + c^2}}\right) \exp \left\{ -\int_0^t r(u)du - \lambda t - \sqrt{\lambda^2 + c^2} t \right\}. \end{aligned}$$

Hence, recalling  $r(t) > c$  for all  $t \in \mathcal{S}$  we have  $\frac{d}{dt} [\bar{F}(t)M(-1, t)] < 0$ . By virtue of (18) the mean of  $X(t)$  is thus increasing in  $t$ . Moreover, from Eq. (9) it is not hard to see that  $M(s, t)$  is increasing in  $c \geq 0$ . Hence,  $\mathbb{E}[X(t)]$  is decreasing in  $c$  due to (18). Limits (20) easily follow from (18) and (19). Recalling Eqs. (18) and (9) we have:

$$\lim_{t \rightarrow \ell} \mathbb{E}[X(t)] = 1 - \frac{1}{2} \left(1 + \frac{\lambda}{\sqrt{\lambda^2 + c^2}}\right) \lim_{t \rightarrow \ell} \exp \left\{ -\int_0^t r(u)du - \lambda t + \sqrt{\lambda^2 + c^2} t \right\},$$



where

$$(22) \quad \begin{aligned} & \lim_{t \rightarrow \ell} \exp \left\{ - \int_0^t r(u) du - \lambda t + \sqrt{\lambda^2 + c^2} t \right\} \\ &= \lim_{t \rightarrow \ell} \exp \left\{ - \int_0^t [r(u) - c] du \right\} \exp \left\{ - \int_0^t [c + \lambda - \sqrt{\lambda^2 + c^2}] du \right\} = 0. \end{aligned}$$

The first of (21) thus follows. Moreover, due to Eqs. (19) and (9), we have:

$$\begin{aligned} \lim_{t \rightarrow \ell} \text{Var} [X(t)] &= \frac{1}{2} \left( 1 + \frac{\lambda}{\sqrt{\lambda^2 + 4c^2}} \right) \\ &\quad \times \lim_{t \rightarrow \ell} \exp \left\{ -2 \int_0^t [r(u) - c] du \right\} \exp \left\{ - \int_0^t [2c + \lambda - \sqrt{\lambda^2 + 4c^2}] du \right\} \\ &\quad - \frac{1}{4} \left( 1 + \frac{\lambda}{\sqrt{\lambda^2 + c^2}} \right)^2 \\ &\quad \times \lim_{t \rightarrow \ell} \exp \left\{ -2 \int_0^t [r(u) - c] du \right\} \exp \left\{ -2 \int_0^t [c + \lambda - \sqrt{\lambda^2 + c^2}] du \right\}. \end{aligned}$$

Finally, noting that

$$\lim_{t \rightarrow \ell} \exp \left\{ -2 \int_0^t [r(u) - c] du \right\} \exp \left\{ - \int_0^t [2c + \lambda - \sqrt{\lambda^2 + 4c^2}] du \right\} = 0$$

and recalling Eq. (22), the second of (21) holds.  $\square$

Figure 4 shows mean and variance of  $X(t)$  for an example involving distribution function (12) and hazard rate (13).

An immediate consequence of Proposition 3.3 is the following

**Proposition 3.4** *Process  $X(t)$  converges in probability to 1 as  $t \rightarrow \ell$ .*

Finally, denoting by  $V(t) = V(0) (-1)^{N(t)}$  the telegraph process that describes the perturbing noise, making use of a well-known result (see, for instance, Theorem 3.4.4 of Ross [22]) the following asymptotic probabilities follow:

$$\lim_{t \rightarrow +\infty} \text{P}\{V(t) = c\} = \lim_{t \rightarrow +\infty} \text{P}\{V(t) = -c\} = \frac{1}{2}.$$

## 4 An application

In this section we indicate a statistical procedure based on an asymptotic confidence band for estimated hazard rates, which is useful to assess the validity of the proposed stochastic model.

Let  $T_1, T_2, \dots, T_n$  be iid absolutely continuous random variables describing a random sample of failure times having density  $f(t)$  and distribution function  $F(t)$ . The usual kernel estimator of  $f(t)$  is

$$(23) \quad \hat{f}(t) = \frac{1}{nh} \sum_{i=1}^n k \left( \frac{t - T_i}{h} \right),$$

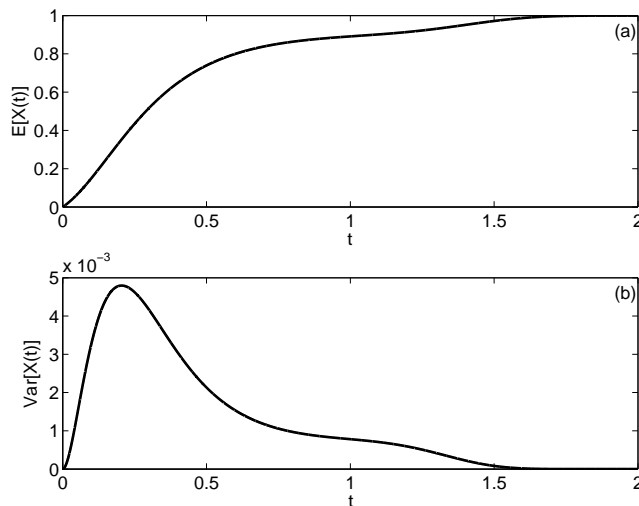


Figure 4: (a) Mean and (b) variance of  $X(t)$ ;  $r(t)$  and all parameters are as in case (a) of Figure 2, i.e.  $c = 1$ ,  $\lambda = \alpha = 15$ ,  $\beta = 0.001$  and  $r(t)$  is given by Eq. (13).

where  $k(\cdot)$  is a bounded even density function, and  $h$  is the bandwidth. The corresponding estimator of  $F(t)$  is

$$(24) \quad \widehat{F}(t) = \int_{-\infty}^t \widehat{f}(u) du = \frac{1}{n} \sum_{i=1}^n K\left(\frac{t - T_i}{h}\right),$$

where  $K(t) = \int_{-\infty}^t k(u) du$ . From (23) and (24) we can build up the following estimator for the hazard rate function:

$$(25) \quad \widehat{r}(t) = \frac{\widehat{f}(t)}{1 - \widehat{F}(t)}.$$

Taking into account the asymptotic normality of (23) and (24), with  $\text{Var}(\widehat{f}) = (nh)^{-1}\mathcal{K}f$ , where  $\mathcal{K} = \int_{-\infty}^{+\infty} k^2(t) dt$ , and  $\text{Var}(\widehat{F}) = O(n^{-1})$ , it follows that also  $\widehat{r}(t)$  is asymptotically normal, with variance  $\text{Var}(\widehat{r}) = (nh)^{-1}\mathcal{K}r^2/f$  (see Hall *et al.* [12]). We can thus consider the following confidence bands for  $r(t)$ , having nominal coverage  $1 - 2\alpha$ ,

$$(26) \quad \widehat{r}^-(t) = \widehat{r}(t) - \left[ \frac{\mathcal{K}}{nh \widehat{f}(t)} \right]^{1/2} \widehat{r}(t) z_\alpha, \quad \widehat{r}^+(t) = \widehat{r}(t) + \left[ \frac{\mathcal{K}}{nh \widehat{f}(t)} \right]^{1/2} \widehat{r}(t) z_\alpha,$$

where  $z_\alpha$  is the  $\alpha$ -level critical point of the standard normal distribution.

We note that a random hazard rate of type  $r(t) + V(0)(-1)^{N(t)}$ , related to process (3), alternates randomly between functions  $r(t) - c$  and  $r(t) + c$ , thus being contained inside the strip  $r(t) \pm c$ . This suggests the following procedure to test if the random sample  $(T_1, T_2, \dots, T_n)$  can be viewed as drawn from the (random) distribution function  $X(t)$ : If, for a fixed  $\alpha$ , the strip  $r(t) \pm c$  is contained within a realization of the confidence band (26) obtained from observed data, the model defined by (3) is defensible for the given data set.

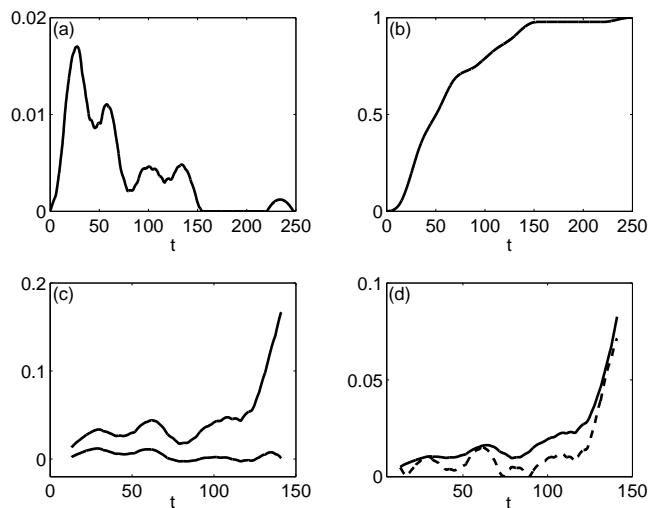


Figure 5: For the sample data of Application 4.1, with  $r(t) = 0.0125$ ,  $t \geq 0$ ,  $\alpha = 0.025$ ,  $c = 0.0004$ ,  $h = 6$  and  $k(t)$  given in (28), it is shown: (a) the estimated density obtained by Eq. (23), (b) the estimated distribution function obtained by Eq. (24), (c) the realization of the confidence band (26), (d) the realization of the functions appearing in the left-hand-side (dashed line) and in the right-hand-side (solid line) of Eq. (27).

In other terms, with a  $(1 - 2\alpha)$ -level confidence, we can adopt model  $X(t)$  if

$$(27) \quad |r(t) - \hat{r}(t)| \leq \left[ \frac{\mathcal{K}}{nh \hat{f}(t)} \right]^{1/2} \hat{r}(t) z_\alpha - c \quad \text{for all } t \in (t_{(1:n)}, t_{(n-1:n)}),$$

where  $t_{(j:n)}$  denotes the value of the  $j$ -th order statistic in the observed random sample. We stress that, due to (27), this procedure is effective to reject “large” values of  $c$ .

**Application 4.1 – A case-study with constant hazard rate function.**

Let us consider the following set of 46 sample data given in Ahmad [1]:

$$\{13, 14, 19, 19, 20, 21, 23, 23, 25, 26, 26, 27, 27, 31, 32, 34, 34, \\ 37, 38, 38, 46, 46, 50, 53, 54, 57, 58, 59, 60, 65, 65, 66, 70, 85, \\ 90, 98, 102, 103, 110, 118, 124, 130, 136, 138, 141, 234\}.$$

These represent the survival times of certain patients in a melanoma study conducted by the Central Oncology Group (see Susarla and Van Ryzin [27]). Figure 5(a) shows the estimated density obtained by use of (23), where  $k(t)$  is the Epanechnikov kernel (see Silverman [24])

$$(28) \quad k(t) = \begin{cases} \frac{3}{4\sqrt{5}} \left(1 - \frac{t^2}{5}\right) & \text{if } -\sqrt{5} \leq t \leq \sqrt{5} \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding estimated distribution function is given in Figure 5(b). A bandwidth depending on the sample standard deviation was proposed in Azzalini [2], where it is also

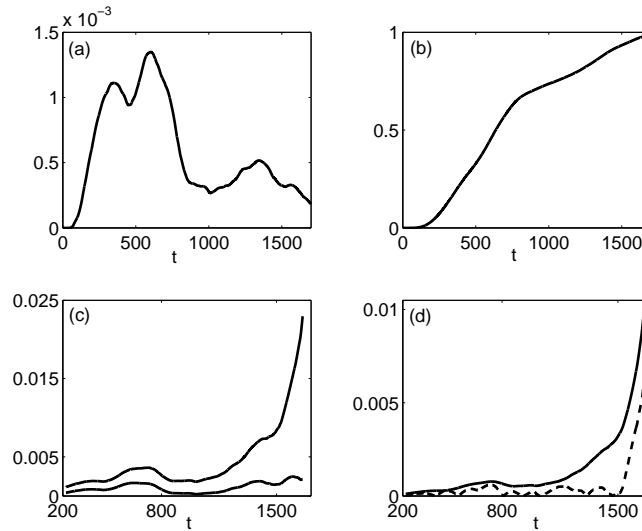


Figure 6: For the sample data of Application 4.2, with  $r(t)$  given in (29),  $\alpha = 0.025$ ,  $c = 0.00025$ ,  $h = 75$  and  $k(t)$  given in (28), we show the same functions appearing in Figure 5.

mentioned that these results are hardly affected if the standard deviation is estimated from the sample data. Hence, we have empirically used the fixed bandwidth  $h = 6$ ; this choice has been also motivated by the need of obtaining a sufficiently smooth estimated density with small tails. We consider a baseline constant hazard rate function  $r(t) = 0.0125$ ,  $t \geq 0$ . The realization of the confidence band (26) obtained from the above mentioned data is plotted in Figure 5(c) for  $\alpha = 0.025$ . With a 0.95-level confidence the strip  $r(t) \pm c$ , with  $c = 0.0004$ , falls within the confidence band, i.e. condition (27) is fulfilled (this is graphically shown in Fig. 5(d)). In conclusion, stochastic model (3) is defensible for the observed data when  $r(t) = 0.0125$ ,  $t \geq 0$ , and  $c \leq 0.0004$ .

#### Application 4.2 – A case-study with varying hazard rate function.

Hereafter we analyze the following set of  $n = 86$  sample data, taken from Table 1 of Langseth and Lindqvist [15]:

{220, 233, 234, 240, 265, 270, 273, 279, 285, 287, 294, 295, 300, 325, 328,  
 333, 365, 368, 369, 381, 417, 418, 429, 460, 470, 474, 475, 476, 508, 522, 523,  
 535, 542, 570, 580, 604, 612, 613, 614, 615, 634, 636, 637, 638, 651, 657, 660,  
 666, 668, 680, 681, 684, 691, 693, 705, 717, 834, 837, 841, 843, 845, 875, 972,  
 1037, 1084, 1091, 1109, 1117, 1197, 1258, 1269, 1297, 1309, 1322, 1346, 1349,  
 1359, 1363, 1448, 1476, 1481, 1557, 1606, 1610, 1642, 1659}.

They describes the service time of a single component in a reliability study. Figures 6(a) and 6(b) show the estimated density obtained from (23), with  $k(t)$  given in (28), and the corresponding estimated distribution function. Here, the bandwidth appearing in (23) has been empirically fixed as  $h = 75$ , in agreement with the remarks in Application 4.1. We

choose the following baseline non-monotonic hazard rate function:

$$(29) \quad r(t) = \begin{cases} 3.5 \cdot 10^{-6} t & \text{if } 0 \leq t \leq 650, \\ -4.07143 \cdot 10^{-6} t + 0.00492143 & \text{if } 650 < t \leq 1000, \\ 8 \cdot 10^{-6} t - 0.00715 & \text{if } t > 1000. \end{cases}$$

Fig. 6(c) shows a realization of the confidence band (26) obtained from the above data for  $\alpha = 0.025$ . For  $c = 0.00025$  the strip  $r(t) \pm c$  falls within the confidence band, with a 0.95-level confidence. Condition (27) is thus fulfilled (see Fig. 6(d)). Hence, we are finally led to consider model (3) as defensible for the observed data, with  $r(t)$  given in (29) and  $c \leq 0.00025$ .

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