EXTREMAL PROBABILISTIC PROBLEMS AND HOTELLING'S T^2 TEST UNDER SYMMETRY CONDITION[1]

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We consider Hotelling's T^2 statistic for an arbitrary d-dimensional sample. If the sampling is not too deterministic or inhomogeneous, then under zero means hypothesis, T^2 tends to χ_d^2 in distribution. We are showing that a test for the orthant symmetry condition introduced by Efron can be constructed which does not essentially differ from the one based on χ_d^2 and at the same time is applicable not only for large random homogeneous samples but for all multidimensional samples without exceptions. The main assertions have the form of inequalities, not that of limit theorems; these inequalities are exact representing the solutions to certain extremal problems. Let us also mention an auxiliary result which itself may be of interest: $\chi_d - (d-1)^{\frac{1}{2}}$ decreases in distribution in d to its limit $N(0, \frac{1}{2})$.

1. Introduction

Efron (1969), Eaton and Efron (1970) discovered that the Hotelling's statistic

$$T^2 = \overline{X} C^{-1} \overline{X}^T \tag{1.1}$$

possesses a strong conservativeness property; here and further $\overline{X}$ is the sample mean, C is the sample covariance matrix.

To describe it, put

$$R^2 = \frac{T^2}{1+T^2}. \tag{1.2}$$

Under some natural conditions, including zero means, both nT^2 and nR^2 tend in distribution to χ_d^2 as $n \to \infty$, where n is the volume of the sample, d is the dimension of the sample space.

Efron (1969), for $d = 1$, Eaton and Efron (1970), for all d, proved that

$$\mathbf{E} f(n^{\frac{1}{2}} R) \leq \mathbf{E} f(\chi_d) \tag{1.3}$$

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for $f(u) = u^{2m}$, $m = 1, 2, \ldots$; the only requirement they put on the sample was the so-called orthant symmetry condition (we should not like to discuss now extensions of definition (1.1) applicable to an arbitrary, not necessarily normal sample when C^{-1} can not exist).

Our main result here in the paper is that (1.3) holds for every convex function f belonging to the class C^2_{conv} of all even functions having convex second derivative; besides, the inequality

$$\mathbf{P}(n^{\frac{1}{2}} R \geq x) < c \cdot \mathbf{P}(\chi_d \geq x), \quad x \geq 0, \tag{1.4}$$

is being extracted from (1.3), with the best possible constant $c = 2e^3/9$.

The factor $2e^3/9$ can be found in Eaton (1974). Earlier, it was proved in Eaton (1970) that

$$\mathbf{E} f(S_n) \leq \mathbf{E} f(\chi_1), \tag{1.5}$$

where $S_n = \epsilon_1 x_1 + \cdots + \epsilon_n x_n$, $x_1^2 + \cdots + x_n^2 = 1$, $\epsilon_1, \ldots, \epsilon_n$ are independent identically distributed (i.i.d.) random variables (r.v.'s) with $\mathbf{P}(\epsilon_i = 1) = \mathbf{P}(\epsilon_i = -1) = \frac{1}{2}$, $i = 1, \ldots, n$; f belongs to a certain subclass of the class F of all differentiable even functions f such that $[f'(t+\Delta) + f'(t-\Delta)]/t$ is non-decreasing in $t > 0$ for each real Δ. Then, based on (1.5), Eaton (1974) obtained the inequality

$$\mathbf{P}(S_n \geq x) \leq \frac{2e^3}{9} \frac{\varphi(x)}{x} e^{-\frac{9}{2x^2}} \left(1 - \frac{3}{x^2}\right)^{-4}, \quad x > 3^{\frac{1}{2}},$$

with $\varphi(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}$, and stated the conjecture that

$$\mathbf{P}(S_n \geq x) \leq \frac{2e^3}{9} \frac{\varphi(x)}{x}, \quad x > 2^{\frac{1}{2}}.$$

In this paper, we are proving that (1.5) holds for all f of the class F (which actually coincides with C^2_{conv}) and that

$$\mathbf{P}(S_n \geq x) < \frac{2e^3}{9}(1 - \phi(x)) < \frac{2e^3}{9} \frac{\varphi(x)}{x} \tag{1.6}$$

for all $x > 0$, where $\phi(x) = \int_{-\infty}^{x} \varphi(u) du$.

Moreover, we give analogues of (1.5), (1.6) for definite quadratic forms in $\epsilon_1, \ldots, \epsilon_n$ (instead of linear forms such as S_n) to which (1.3), (1.4) are simple corollaries.

In section 2 below, we give our variants of strict definitions and also representations of T^2 and R^2 for any multidimensional sample.

In section 3, we reproduce the orthant symmetry condition by Efron.

In section 4, we present probabilistic problems related to (1.5), (1.6).

In section 5, based on the monotonicity of the likelihood ratio, it is shown that

$\chi_d - (d-1)^{\frac{1}{2}}$ decreases in distribution to its limit $N\left(0, \frac{1}{2}\right)$ when $d \uparrow \infty$.

In section 6, the results of the preceding sections are used to obtain (1.3), (1.4); the size of the corresponding confidence region is discussing.

Section 7, Appendix, contains some proofs.

2. Hotelling's T^2 statistic for an arbitrary distributed multidimensional sample.

Let $X_1, \ldots, X_n$ be a sample such as in the title of the section, $X_i \in \mathbb{R}^d$; we identify $\mathbb{R}^d$ with M_{1d}; M_{nd} denotes the set of all $n \times d$ real matrices. Consider

$$\overline{X} = \sum_{i=1}^{n} X_i/n, \;\; S = \sum_{i=1}^{n} X_i^T X_i/n, \;\; C = S - \overline{X}^T \overline{X},$$

i.e., the sample mean, the matrix of second sample moments and the sample covariance matrix, resp.; ^{T} means transposition.

If e.g. the sample is normally distributed, then C^{-1} exists with probability (w.p.) 1, and T^2 statistic is defined as

$$(n-1)\overline{X} C^{-1} \overline{X}^T. \tag{2.1}$$

In general, it may be defined as in Eaton and Efron (1970):

$$T^2 = \cot^2 \Theta, \tag{2.2}$$

where Θ is the angle between vector

$$\nu = (1, \ldots, 1)^T \in M_{n1} \tag{2.3}$$

and the linear hull $L(X)$ of the columns of the matrix

$$X = (X_1^T \ldots X_n^T)^T,$$

ith line of which is X_i (it is convenient to omit the factor $n-1$ in (2.1), which corresponds to (1.1)). It is also reasonable to define another statistic,

$$R^2 = \cos^2 \Theta, \tag{2.4}$$

i.e.,

$$R^2 = \frac{T^2}{1+T^2}, \tag{2.5}$$

assuming that $R^2 = 1 \Leftrightarrow T^2 = \infty$ (to specify (1.2)).

We suggest a somewhat different approach to the general-case definition of T^2 statistic, closer to (1.1). It leads to the same notions as in (2.2), (2.4). Namely, put

$$T^2 = \lim_{\epsilon \downarrow 0} \overline{X}(C+\epsilon I)^{-1}\overline{X}^T, \tag{2.6}$$

$$R^2 = \lim_{\epsilon \downarrow 0} \overline{X}(S+\epsilon I)^{-1}\overline{X}^T, \tag{2.7}$$

where I is the unit matrix; the limits here always exist, finite or infinite. (Replacing here I by any strictly positively definite symmetric matrix (p.d.m.), one would obtain the equivalent definitions since the matrix function $A \to A^{-1}$ is monotone on the set of all p.d.m.'s; see Marshall and Olkin (1979)). Indeed, considering

$$T_\epsilon^2 = \overline{X}C_\epsilon^{-1}\overline{X}^T, \;\; R_\epsilon^2 = \overline{X}S_\epsilon^{-1}\overline{X}^T, \;\; C_\epsilon = C+\epsilon I, \;\; S_\epsilon = S+\epsilon I,$$

we have

$$\begin{aligned} T_\epsilon^2 R_\epsilon^2 &= \overline{X}C_\epsilon^{-1}\overline{X}^T\overline{X}S_\epsilon^{-1}\overline{X}^T \\ &= \overline{X}C_\epsilon^{-1}(S_\epsilon - C_\epsilon)S_\epsilon^{-1}\overline{X}^T = T_\epsilon^2 - R_\epsilon^2, \end{aligned}$$

which implies (2.5) when using definitions (2.6), (2.7).

Consider the matrix

$$P = X(X^TX)^-X^T \tag{2.8}$$

of the orthoprojector from M_{n1} onto $L(X)$; A^- denotes a g-inverse matrix for A (Rao (1965)). Taking into account equalities $\overline{X} = \nu^TX/n$, $S = X^TX/n$ and "quasispectral" representation for X (Rao (1965), Ch. 1, appendix 6.1), it can be seen that

$$R^2 = \overline{X}S^-\overline{X}^T, \tag{2.9}$$

which is equivalent to

$$nR^2 = \nu^T P\nu, \tag{2.10}$$

i.e., definition (2.7) coincides with (2.4), and, in view of (2.5), definitions (2.2) and (2.6) are equivalent too.

In contrast to (2.9), the equality $T^2 = \overline{X}C^-\overline{X}^T$ is not always true; the simplest reason is that the right-hand side of it cannot be infinite.

3. Orthant symmetry.

This condition was defined in Efron (1969), Eaton and Efron (1970) as

$$(X_1, \ldots, X_n) \stackrel{D}{=} (\epsilon_1 X_1, \ldots, \epsilon_n X_n) \tag{3.1}$$

or, equivalently,

$$X \stackrel{D}{=} \Delta_\epsilon X, \tag{3.2}$$

where $\stackrel{D}{=}$ means equality in distribution, $\Delta_\epsilon = diag\{\epsilon_1, \ldots, \epsilon_n\}$ is the diagonal matrix with $\epsilon_1, \ldots, \epsilon_n$ on its diagonal, $\epsilon_1, \ldots, \epsilon_n$ are i.i.d.r.v.'s independent of X

$$\mathbf{P}(\epsilon_i = \pm 1) = \frac{1}{2}, \ \ i = 1, \ldots, n.$$

It was mentioned in Eaton and Efron (1970) that under the orthant symmetry condition,

$$nR^2 \stackrel{D}{=} \epsilon^T P\epsilon, \tag{3.3}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T$. It may be also deduced from (2.8), (2.10), (3.2) and the equalities $\epsilon = \Delta_\epsilon \nu$, $\Delta_\epsilon^2 = I$.

4. Some extremal probability problems.

Let f be a locally bounded Borel even function on $\mathbb{R}$; $A \in M_{nn}$; $A \geq 0$, i.e., A is a nonnegatively definite matrix; $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$, $\xi_0, \xi_1, \ldots, \xi_n$ are independent r.v.'s; $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T$ as above; $\xi = (\xi_1, \ldots, \xi_n)^T$; ϵ_i and ξ_i are symmetrically distributed, $\mathbf{P}(\epsilon_i = 1) = \frac{1}{2}$, $\mathbf{E}\xi_i^2 = 1$, $i = 0, 1, \ldots, n$; ξ_0 does not coincide in distribution with ϵ_0; $x = (x_1, \ldots, x_n)^T \in M_{n1}$.

In section 1, there were mentioned the class C^2_{conv} of all real even functions f on $\mathbb{R}$ having finite and convex (and hence continuous) second derivative f'' and the class F of all real differentiable even functions f on $\mathbb{R}$ such that $[f'(t+\Delta)+f'(t-\Delta)]/t$ is non-decreasing in $t>0$ for each $\Delta\in\mathbb{R}$.

Theorem 4.1. The following statements are mutually equivalent.

(i) $\mathbf{E}f((\epsilon^T A\epsilon)^{\frac{1}{2}}) \le \mathbf{E}f((\xi^T A\xi)^{\frac{1}{2}})\forall A,\xi;$

(ii) $\mathbf{E}f(\epsilon_1 x_1+\cdots+\epsilon_n x_n) \le \mathbf{E}f(\xi_1 x_1+\cdots+\xi_n x_n)\forall x,\xi;$

(iii) $f\in C^2_{conv}$;

(iv) $f\in F$;

(v) $g_{c,f}\in C^2_{conv}\forall c\ge 0$, where $g_{c,f}(u)=f((u^2+c)^{\frac{1}{2}}))$;

(vi) $\exists\xi_0$ bounded w.p. 1 $\forall a,\ \ b\in\mathbb{R}$

$$\mathbf{E}f(a+b\epsilon_0)\le\mathbf{E}f(a+b\xi_0); \tag{4.1}$$

(vii) $\forall\xi_0\forall a,b\in\mathbb{R}$ (4.1) holds;

(viii) $\exists\xi_0$ bounded w.p.1

$$\mathbf{E}f(\eta_0+b\epsilon_0)\le\mathbf{E}f(\eta_0+b\xi_0) \tag{4.2}$$

for every random variable η_0 bounded w.p.1 and independent of ϵ_0,ξ_0.

(Boundedness of ξ_i is required nowhere but in (vi), (viii). Of course, some expectations may be infinite.)

Proof. See Appendix.

Define the class $C^2_{\uparrow,conv}$ of all the functions $f\in C^2_{conv}$ which are convex on $[0,\infty)$ or, equivalently, are non-decreasing on $[0,\infty)$ and, besides, are nonnegative; in other words, $f\in C^2_{conv}$ and $f(0)\ge 0, f''(0)\ge 0$; the additional requirement of being nonnegative does not diminish the generality of following corollaries 4.2 - 4.4, but we do need it in corollaries 4.5, 4.6 and in some places further.

Corollary 4.2. If $f\in C^2_{\uparrow,conv}$, then $\forall A,\xi,x$

$$\mathbf{E}f((\eta^T A\eta)^{\frac{1}{2}}) \le \mathbf{E}f((\xi^T A\xi)^{\frac{1}{2}}),$$
$$\mathbf{E}f(\eta_1 x_1+\cdots+\eta_n x_n) \le \mathbf{E}f(\xi_1 x_1+\cdots+\xi_n x_n),$$

where $\eta = (\eta_1, \ldots, \eta_n)^T$, η_i are independent r.v.'s, $\mathbf{E}\eta_i = 0$, $|\eta_i| \le 1$ w.p.1.

Proof. See Appendix.

Corollary 4.3. If P is an orthoprojector matrix, then

$$\mathbf{E}f((\epsilon^T P \epsilon)^{\frac{1}{2}}) \le \mathbf{E}f(\chi_r),$$

where $r = \text{rank } P$, $\chi_r = (\chi_r^2)^{\frac{1}{2}}$, χ_r^2 is a r.v. with χ_r^2 distribution, $f \in C^2_{conv}$. If, moreover, $f \in C^2_{\uparrow,conv}$, then one can substitute here η for ϵ.

Proof. It suffices to note that $\xi^T P \xi \overset{D}{=} \chi_r^2$ if $\xi_i \sim N(0,1)$.

This corollary in the case $f(u) = u^{2m}$, $m = 1, 2, \ldots$, was proved, in essential, in (Eaton and Efron (1970), corollary 6.1).

Corollary 4.4. If ξ_1 is $N(0,1)$ r.v. and $x_1^2 + \cdots + x_n^2 = 1$, then $\forall f \in C^2_{conv}$

$$\mathbf{E}f(\epsilon_1 x_1 + \cdots + \epsilon_n x_n) \le \mathbf{E}f(\xi_1).$$

If, moreover, $f \in C^2_{\uparrow,conv}$, then we can substitute η for ϵ.

Proof. Put $P = xx^T$ in corollary 4.3.

This corollary was proved in Eaton (1970, 1974) for $f \in F$ (and so, in view of the equivalence (iii) $\Leftrightarrow$ (iv) of theorem 4.1, for $f \in C^2_{conv}$) but under the additional restriction $\mathbf{E}|f(T_n)|^{1+\delta} \le M$, $\delta > 0$, $T_n = \epsilon^T x$ or $\eta^T x$, resp.

Corollary 4.5. Under the conditions of corollary 4.3,

$$\mathbf{P}(\eta^T P \eta \ge u^2) \le Q_r(u) \tag{4.3}$$

$$< \frac{2e^3}{9}\mathbf{P}(-\chi_r \ge u), \quad u \ge 0, \tag{4.4}$$

where

$$Q_r(u) = inf\{\mathbf{E}f(\chi_r)/f(u) : \ f \in C^2_{\uparrow,conv}, \ f(u) > 0\}.$$

Proof. See corollary 4.3 for (4.3) and (4.13) below for (4.4).

Corollary 4.6. Under the conditions of corollary 4.4,

$$\mathbf{P}(\eta_1 x_1 + \cdots + \eta_n x_n \ge u) \le Q_1(u)/2 \tag{4.5}$$

$$< \frac{2e^3}{9}\mathbf{P}(\xi_1 \ge u), \quad u \ge 0. \tag{4.6}$$

Proof. Put $P = xx^T$ in corollary 4.5.

A statement close to (4.5) was given in Eaton (1974); (4.6) is an improvement of corollaries 1, 2 in Eaton (1974) and of the conjecture following those corollaries therein.

Let us provide further information on $Q_r(u)$ which, in particular, contains (4.4), (4.6).

Proposition 4.7.

$$Q_r(u) = \min[1, r/u^2, W_r(u)] \tag{4.7}$$

$$= \begin{cases} 1 & \text{if} \quad 0 \le u \le r^{\frac{1}{2}}, \\ r/u^2 & \text{if} \quad r^{\frac{1}{2}} \le u \le \mu_r, \\ W_r(u) & \text{if} \quad u \ge \mu_r, \end{cases} \tag{4.8}$$

where

$$\begin{aligned} W_r(u) &= inf\{(u-t)^{-3}\mathbf{E}(\chi_r - t)_+^3 : \ t \in (0,u)\}, \\ \mu_r &= \mathbf{E}\chi_r^3/\mathbf{E}\chi_r^2; \end{aligned} \tag{4.9}$$

besides,

$$\mu_r \in ((r+1)^{\frac{1}{2}}, \ (r+2)^{\frac{1}{2}}). \tag{4.10}$$

Proof. See Appendix.

Proposition 4.8. Constant $2e^3/9 = 4.463\ldots$ is the best possible in (4.4), (4.6) in the sense that for each r

$$Q_r(u) \sim \frac{2e^3}{9}\mathbf{P}(\chi_r \ge u), \ \ u \to \infty; \tag{4.11}$$

here and in what follows $a \sim b$ means $a/b \to 1$.

Proof. See Appendix.

Consider the ratio

$$\Lambda_r(u) = Q_r(u)/\mathbf{P}(\chi_r \ge u),$$

and define

$$q = q(u) = q_r(u) = \int_u^\infty s^{r-1}e^{-s^2/2}I\{s > 0\}ds$$

so that

$$\mathbf{P}(\chi_r \geq u) = q(u)/q(0);$$

here and further, $I\{\mathcal{A}\} = 1$ when $\mathcal{A}$ is true, $I\{\mathcal{A}\} = 0$ otherwise.

Proposition 4.9.

$$\Lambda_r(u) < (2e^3/9) + 3[\mathcal{J}(a_u) - \mathcal{J}(3)], \;\; u \geq \mu_r, \tag{4.12}$$

$$\Lambda_r(u) < 2e^3/9, \;\; u \geq 0, \tag{4.13}$$

where

$$a_u = 3q(u)q''(u)/q'(u)^2 > 0,$$
$$\mathcal{J}(a) = 6a^{-4}(e^a - 1 - a - a^2/2 - a^3/6); \tag{4.14}$$

$$a_u \uparrow 3(u \geq \mu_r, \;\; u \uparrow \infty); \tag{4.15}$$

$$\mathcal{J}(a) \quad \text{increases in} \quad a > 0 \tag{4.16}$$

(actually in all a, with $\mathcal{J}(0) = \frac{1}{4}$, but we need not this improvement).

Proof. See Appendix.

Proposition 4.10. Function $\Lambda_1(u)$ increases in $u \geq \mu_1$.

Proof. See Appendix.

The most non-trivial of propositions 4.7 - 4.10 is of course proposition 4.9. It shows that $\Lambda_r(u) < \tilde{\Lambda}_r(u)$, where $\tilde{\Lambda}_r(u) \left[= \frac{2e^3}{9} + 3(\mathcal{J}(a_u) - \mathcal{J}(3))\right] \uparrow \frac{2e^3}{9}, \;\; u \geq \mu_r$. Apparently, $\Lambda_r(u)$ itself increases in $u \geq \mu_r$ for each r (to $2e^3/9$, in view of proposition 4.8), but we can prove this fact only for $r = 1$ (proposition 4.10); a scheme of proof like that of proposition 4.10 may do for all r though it becomes too complicated in the general case. Nevertheless, proposition 4.9 given here may often be more exact and useful than that hypothetic qualitative result.

5. The monotonicity of a likelihood ratio and a stochastic majorization.

Consider the family $\chi_r - (r-1)^{\frac{1}{2}}$, where r is any real number in $[1,\infty)$, χ_r has the density

$$p_r(u) = C_r u^{r-1} e^{-u^2/2} I\{u > 0\}, \tag{5.1}$$

C_r depends only on r; one can see that $(r-1)^{\frac{1}{2}}$ is the mode of χ_r.

We shall show that this family has monotone likelihood ratio and hence is stochastically monotone. Let $(E, \leq)$ be any partially ordered set.

We say that a family $(\xi_r : \ r \in E)$ of r.v.'s having densities $(p_r : \ r \in E)$ has monotone likelihood ratio (MLR) if the implication

$$r \leq d, \ \ r \in E, \ \ d \in E, \ \ -\infty < s < t < \infty$$

$$\Rightarrow p_r(t) p_d(s) \geq p_d(t) p_r(s) \tag{5.2}$$

is true. In the case $p_r(t) > 0 \forall r, t$, this definition just means that p_d/p_r is non-increasing on $\mathbb{R}$ when $r \leq d$.

We say that a family of r.v.'s $(\xi_r : \ r \in E)$ with the tails $F_r(t) = \mathbf{P}(\xi_r \geq t)$ has monotone tail ratio (MTR) if the implication

$$r \leq d, \ \ r \in E, \ \ d \in E, \ \ -\infty < s < t < \infty$$

$$\Rightarrow F_r(t) F_d(s) \geq F_d(t) F_r(s) \tag{5.3}$$

is true.

A family $(\xi_r : \ r \in E)$ of r.v.'s is called stochastically monotone (SM) if

$$r \leq d, \ \ r \in E, \ \ d \in E, \ \ t \in \mathbb{R} \Rightarrow F_r(t) \geq F_d(t); \tag{5.4}$$

this definition of the stochastical monotonicity or, in other words, monotonicity in distribution, is generally accepted (see, e.g., Marshall and Olkin (1979)).

Proposition 5.1. If (ξ_r) has MLR, then it has MTR.

Proof. If (ξ_r) has MLR, and $-\infty < s < t < \infty$, then

$$F_r(t)[F_d(s) - F_d(t)] = \iint\limits_{s \leq u < t \leq v} p_r(v) p_d(u) dv du \geq$$

$$\geq \iint\limits_{s \leq u < t \leq v} p_d(v) p_r(u) dv du = F_d(t)[F_r(s) - F_r(t)],$$

and so $F_r(t)F_d(s) \geq F_d(t)F_r(s) (r \leq d)$.

Proposition 5.2. If (ξ_r) has MTR, then it is SM.

Proof. In (5.3), tend s to $-\infty$.

Theorem 5.3. The family $\{\chi_r - (r-1)^{\frac{1}{2}} : \ r \geq 1\}$ has MLR.

Proof. Take $d \geq r \geq 1$ and put $a = (r-1)^{\frac{1}{2}}, \ b = (d-1)^{\frac{1}{2}}$. Then

$$(\log[p_d(u)/p_r(u)])' = (r-d)u^2/(a+b)(u+a)(u+b) \leq 0$$

if $u > -a$, hence $p_r(t)p_d(s) \geq p_d(t)p_r(s)$ when $-a \leq s < t$; if $s < -a$, then $p_r(s) = 0$, so this case is trivial.

Corollary 5.4. This family has MTR.

Proof. See proposition 5.1.

Corollary 5.5. This family, $\{\chi_r - (r-1)^{\frac{1}{2}} : \ r \geq 1\}$, is SM.

Proof. See proposition 5.2.

Remark 5.6. Consider $\xi_r = \chi_r - (r-1)^{\frac{1}{2}}, \ 1 \leq r < \infty$; let ξ_∞ have $N\left(0, \frac{1}{2}\right)$ distribution. Then $\xi_r \to \xi_\infty$ in distribution when $r \to \infty$, and one may supplement the family in statements 5.3 - 5.5 by ξ_∞.

Remark 5.7. If required additionally that inequality (5.2) should be strict for some $s, \ r \leq d, \ r \neq d$ on a positive-measure set of values t such that $t > s$, then the conclusions of propositions 5.1, 5.2 can be also improved so that inequalities (5.3), (5.4) become strict for those values of s, r, d. In particular, statements 5.3 - 5.6 can be improved so that inequalities (5.3), (5.4) become strict whenever $1 \leq r < d \leq \infty$, $-(d-1)^{\frac{1}{2}} \leq s < t < \infty$, for $\xi_r = \chi_r - (r-1)^{\frac{1}{2}}$, $1 \leq r < \infty$, and ξ_∞ having $N\left(0, \frac{1}{2}\right)$ distribution. E.g., for all $u \in \mathbb{R}, \ d \geq 1$,

$$\mathbf{P}(\chi_d - (d-1)^{\frac{1}{2}} \geq u) > 1 - \phi(2^{\frac{1}{2}} u), \tag{5.5}$$

where ϕ is $N(0,1)$ distribution function; this is an improvement of the central limit theorem for χ_r.

6. Inequalities for the distribution of T^2 under the orthant symmetry condition.

It is known that if $X_1, \dots, X_n, \dots$ are i.i.d. random vectors with zero means and a finite non-degenerate matrix of second moments, then the distribution of T^2 (and hence that of R^2), is close to that of χ_d^2/n when $n \to \infty$. Similarly to Efron (1969), Eaton and Efron (1970), we are showing that some kind of conservativeness holds under the orthant symmetry condition only; this conservativeness is stronger than that in Efron (1969), Eaton and Efron (1970).

We permanently suppose that the orthant symmetry condition takes place.

Theorem 6.1. If $f \in C^2_{\uparrow,conv}$, then

$$\mathbf{E} f(n^{\frac{1}{2}} R) \le \mathbf{E} f(\chi_d),$$

where $R = (R^2)^{\frac{1}{2}}$. (When $f(u) = u^{2m}$, $m = 1, 2, \dots$, this inequality was proved in Eaton and Efron (1970)).

Proof. See (3.3) and corollary 4.3.

The class $C^2_{\uparrow,conv}$ is sufficiently large to make it possible to obtain the following, main result of this paper.

Theorem 6.2. For all $u \ge 0$

$$\mathbf{P}(n^{\frac{1}{2}} R \ge u) \le Q_d(u) < \frac{2e^3}{9} \mathbf{P}(\chi_d \ge u). \tag{6.1}$$

Proof. See (3.3) and corollary 4.5.

One can find further information about $Q_d(u)$ in propositions 4.7 - 4.10.

Consider now δ-quantiles $\tilde{x}_d(\delta)$ and $x_d(\delta)$, $0 < \delta < 1$, for $n^{\frac{1}{2}} R$ and χ_d, resp., i.e.,

$$\mathbf{P}(x_d \ge x_d(\delta)) = \delta, \;\; \tilde{x}_d(\delta) = inf\{x \in \mathbb{R} : \mathbf{P}(n^{\frac{1}{2}} R \ge x) \le \delta\}.$$

In particular, $\mathbf{P}(n^{\frac{1}{2}} R \ge \tilde{x}_d(\delta)) \ge \delta > \mathbf{P}(n^{\frac{1}{2}} R > \tilde{x}_d(\delta))$.

Theorem 6.3. If $\delta \le 0.5$, then $x_d(\delta) > (d-1)^{\frac{1}{2}}$ and

$$\tilde{x}_d(\delta) < x_d(\delta/c) \tag{6.2}$$

$$< x_d(\delta) + [x_d(\delta) - (d-1)/x_d(\delta)]^{-1} \log c \tag{6.3}$$

$$< x_d(\delta) + [x_d(\delta) - (d-1)^{\frac{1}{2}}]^{-1} \log c \tag{6.4}$$

$$< x_d(\delta) + o(1) \tag{6.5}$$

$$< x_d(\delta)(1 + d^{-\frac{1}{2}} \cdot o(1)) \tag{6.6}$$

$$< x_d(\delta)(1 + o(1)), \tag{6.7}$$

where $o(1) \to 0$ uniformly in $d, n, X_1, \ldots, X_n$ when $\delta \downarrow 0$; $c = 2e^3/9$.

Proof. See Appendix.

Thus this theorem means that the size $\tilde{x}_d(\delta)$ of Hotelling's criterion under the orthant symmetry condition can exceed the limit size $x_d(\delta)$, if can, only by some negligible value. It is just that conservativeness property we spoke about.

Moreover, this theorem implies that the greater the dimension is, the even better the situation becomes (the same tendency takes place when δ decreases). This is illustrated by the following numerical data for $\delta = 0.05$, where x_δ stands for $x_d(\delta)$, $z_\delta = x_\delta + [x_\delta - (d-1)/x_\delta]^{-1} \cdot \log c$.

d	1	2	5	10	20	50	∞
x_δ	1.96	2.45	3.33	4.28	5.61	8.22	$d^{\frac{1}{2}} + 1.16$
$x_{\delta/c}$	2.54	3.00	3.85	4.78	6.10	8.69	$d^{\frac{1}{2}} + 1.61$
z_δ	2.72	3.18	4.03	4.97	6.28	8.88	$d^{\frac{1}{2}} + 1.80$

7. Appendix.

Proof of theorem 4.1. It consists in checking the following implications:

(vii) $\Rightarrow$ (vi) $\Rightarrow$ (viii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (vii),

(iii) $\Leftrightarrow$ (v) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (vii).

Let us proceed.

(vii) $\Rightarrow$ (vi). It is trivial.

(vi) $\Rightarrow$ (viii). Inequality (4.2) is an "average" of (4.1).

(viii) $\Rightarrow$ (iii). For the beginning, suppose that f has continuous 4th derivative $f^{(4)}$. Then Taylor's expansion gives

$$(\mathbf{E}f(a + b\xi_0) - f(a) - f''(a)b^2/2) \cdot 24b^{-4} \to f^{(4)}(a) \cdot \mathbf{E}\xi_0^4$$

when $b \to 0$. Using the analogous expansion with ϵ_0 instead of ξ_0, one can see that $f^{(4)}(a)(\mathbf{E}\xi_0^4 - 1) \geq 0$. But $\mathbf{E}\xi_0^4 > 1$ since, as it was fixed, ξ_0 does not coincide with ϵ_0 in distribution and $\mathbf{E}\xi_0^2 = 1$. Thus $f^{(4)}(a) \geq 0$, and so f'' is convex. To check now the general case, put $f_m(u) = \mathbf{E}f(u + \eta_0/m), m = 1, 2, \ldots$, where η_0 is a bounded r.v. with a sufficiently smooth density. By what has been already proved, $f_m^{(4)}(u) \geq 0 \ \forall u \epsilon \mathbb{R}$. Consider the operator $\Delta^3 f(u) = f(u+3) - 3f(u+2) + 3f(u+1) - f(u)$. Then $\forall u \geq 0 \exists \Theta \in (0,3) \Delta^3 f_m(u) = f_m'''(u + \Theta) \geq f_m'''(u) \geq f_m'''(0) = 0$ since $f_m^{(4)} \geq 0$ and therefore f_m'''

is non-decreasing. Note that $f_m(u) \to f(u)$ almost everywhere (a.e.). Hence, $\Delta^3 f_m(u)$ is bounded a.e., and so is $f_m'''(u)$. By Helly's theorem, $f_m''' \to h$ weakly on each compact in $\mathbb{R}$, for some subsequence of the values of $m \to \infty$ and some non-decreasing finite function h. Integrating by parts 3 times or, more exactly, using Fubini theorem, it is easy to see that

$$f \in C^2_{conv} \Leftrightarrow f(u) = a + bu^2/2 + \int\limits_{t \geq 0} (|u| - t)^3_+ \nu_f(dt)/6 \tag{7.1}$$

for some real a, b and a σ-finite nonnegative measure ν_f, where $u_+ = \max(u, 0)$; besides, if $f \in C^2_{conv}$, then $a = f(0), b = f''(0), \forall u \geq 0 \nu_f([0, u]) = f'''(u + 0)$, where $f'''(u + 0)$ is the right derivative of convex function $f''(u)$; (7.1) is but a kind of Taylor's expansion. Further,

$$\int_{t \geq 0} (|u| - t)^3_+ df_m'''(t) \to \int_{t \geq 0} (|u| - t)^3_t dh(t)$$

since $f_m''' \to h$ weakly on compacts. Hence, $\exists a, b$

$$f_m(0) \to a, \;\; f_m''(0) \to b,$$

$$f(u) = a + bu^2/2 + \int_{t \geq 0} (|u| - t)^3_+ dh(t);$$

thus, $f \in C^2_{conv}$.

(iii) $\Rightarrow$ (iv). Essentially, it was proved in (Eaton (1974), Lemma 1); we need only to substitute $f'''(u + 0)$ for $f'''(u)$ in Eaton (1974).

(iv) $\Rightarrow$ (vii). Put $g(x) = f(a + bx^{\frac{1}{2}}) + f(a - bx^{\frac{1}{2}}), \; x \geq 0$. Then, $f \in F \Leftrightarrow g'$ is non-decreasing in $x > 0$. Therefore, g is convex on $[0, \infty)$, and

$$2\mathbf{E}f(a + b\epsilon_0) = g(1) \leq \mathbf{E}g(\xi_0^2) = 2\mathbf{E}f(a + b\xi_0),$$

by Jensen's inequality.

(v) $\Rightarrow$ (iii). Take $c = 0$.

(iii) $\Rightarrow$ (v). In view of (7.1), it is sufficient to prove that $\forall t \geq 0 \forall c \geq 0$ the function $g(u) = (z^{\frac{1}{2}} - t)^3_+$, $z = u^2 + c$, belongs to C^2_{conv}. Calculations show:

1) $z = t^2 \Leftrightarrow u = u_+$ or $u = u_-$, where $u_\pm = \pm(t^2 - c)^{\frac{1}{2}}$; here it is necessary that $t^2 \geq c$; further,

$$g'''(u_\pm + 0) - g'''(u_\pm - 0) = 6t^{-3}(t^2 - c)^{\frac{3}{2}} \geq 0$$

is $t > 0$; $u_\pm = 0$, $g'''(0 + 0) - g'''(0 - 0) = 12 > 0$ when $t = 0$ (and, hence, $c = 0$);

2) $z < t^2 \Rightarrow g^{(4)}(u) = 0$;

3) $z > t^2 \Rightarrow g^{(4)}(u) = 9cz^{-\frac{7}{2}}[(z-5t^2)c + 4t^2 z] \geq 9cz^{-\frac{5}{2}} \cdot \min(z-t^2, 4t^2) \geq 0$ since $0 \leq c \leq z$. Thus $g'''(u+0)$ is non-decreasing in $u \in \mathbb{R}$, $g \in C^2_{conv}$.

(v) $\Rightarrow$ (i). Note that $\epsilon^T A \epsilon = (\alpha\epsilon_1 + \beta)^2 + c$, where $c \geq 0, \alpha, \beta$ do not depend on ϵ_1. Function

$$
\begin{aligned}
h(u) &= f(((\alpha u + \beta)^2 + c)^{\frac{1}{2}}) + f(((-\alpha u + \beta)^2 + c)^{\frac{1}{2}}) \\
&= g_{c,f}(\alpha u + \beta) + g_{c,f}(-\alpha u + \beta)
\end{aligned}
$$

belongs to C^2_{conv} as $g_{c,f}(u)$ does. Thus

$$
2\mathbf{E} f((\epsilon^T A \epsilon)^{\frac{1}{2}}) = \mathbf{E} h(\epsilon_1) \leq \mathbf{E} h(\xi_1) = 2\mathbf{E} f((\tilde{\epsilon}^t A \tilde{\epsilon})^{\frac{1}{2}}),
$$

where $\tilde{\epsilon} = (\xi_1, \epsilon_2, \ldots, \epsilon_n)^T$, because, as we have already proved, (v) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (vii). Successively replacing the remaining ϵ_i's by ξ_i's, we are coming to (i).

(i) $\Rightarrow$ (ii). Put $A = xx^T$.

(ii) $\Rightarrow$ (vii). Take $n = 2$ and note that

$$
\mathbf{E} f(a + b\xi_0) = \mathbf{E} f(-a + b\xi_0) = \mathbf{E} f(a\epsilon_1 + b\xi_0).
$$

The theorem is proved.

Proof of corollary 4.2. If $f(u) = bu^2,\ b \geq 0$, then the inequalities we are proving are evident. In view of (7.1), it suffices to consider the case $f(u) = (|u|) - t)^3_+$, $t \geq 0$. In this case, calculations show that (cf. with the proof of statement (iii)$\Rightarrow$ (v) of theorem 4.1) $g_{c,f}$ is convex on $\mathbb{R}$. Therefore, the function h in the proof of the implication (v)$\Rightarrow$ (i) of theorem 4.1 is convex. It remains to use the following inequality due to Hunt (1955), see also Eaton (1974):

$$
\mathbf{E} G(\eta_1, \ldots, \eta_n) \leq \mathbf{E} G(\epsilon_1, \ldots, \epsilon_n),
$$

if G is a function convex in each its argument.

In the following proofs we need some auxiliary results. Put

$$
\begin{aligned}
\gamma = \gamma(u) = \gamma_r(u) = \mathbf{E}(\chi_r - u)^3_+ / C_r &= \int_u^\infty (s-u)^3 p_r(s) ds / C_r \\
&= \int_u^\infty (s-u)^3 s^{r-1} e^{-s^2/2} I\{s > 0\} ds,
\end{aligned}
$$

where p_r is defined by (5.1), and, as in section 4,

$$q = q(u) = q_r(u) = \int_u^\infty p_r(s)ds/C_r$$
$$= \int_u^\infty s^{r-1}e^{-s^2/2}I\{s > 0\}ds.$$

Lemma 7.1. For all $r > 0$, $j = 0, 1, 2, 3, 4$

$$(-1)^j\gamma^{(j)}(u) \geq 0, \;\; u \in \mathbb{R}, \tag{7.2}$$

$$(-1)^j\gamma^{(j)}(u) \sim 6u^{r-5+j}e^{-u^2/2}, \;\; u \to \infty \tag{7.3}$$

$$\gamma^{(3)}(u) = -6q(u), \tag{7.4}$$

$$\gamma^{(4)}(u) = 6u^{r-1}e^{-u^2/2}I\{u \geq 0\} = -6q'(u), \;\; u \neq 0, \tag{7.5}$$

$$\begin{aligned}\gamma^{(5)}(u) = -6q''(u) = 6[u - (r-1)/u]q'(u) = \\ = -[u - (r-1)/u]\gamma^{(4)}(u), \;\; u > 0,\end{aligned} \tag{7.6}$$

where $\gamma^{(j)}(u) = (d^j/du^j)\gamma(u)$, $j = 1, 2, \ldots, \gamma^{(0)}(u) = \gamma(u)$.

Proof. Equalities (7.4) - (7.6) are the results of simple calculations. By L'Hospital's rule,

$$\gamma(u)/(6u^{r-5}e^{-u^2/2}) \sim \gamma'(u)/(6[(r-5)u^{r-6} - u^{r-4}]e^{-u^2/2})$$

$$\sim -\gamma'(u)/(6u^{r-4}e^{-u^2/2}) \sim \cdots \sim \gamma^{(4)}(u)/(6u^{r-1}e^{-u^2/2}) = 1(u \to \infty),$$

and so (7.3) is proved. In particular, $\gamma^{(j)}(u) \to 0(u \to \infty)$. This, together with (7.5), implies (7.2).

Lemma 7.2. For each $r \geq 1$

$$q(u)q''(u)/q'(u)^2 \uparrow 1 \quad \text{when} \quad u \geq (r-1)^{\frac{1}{2}}, \;\; u \uparrow \infty, \tag{7.7}$$

or, equivalently,

$$(u - (r-1)/u)^{-1} > -q(u)/q'(u) \tag{7.8}$$

(7.9) $$> (u - (r-1)/u)/[(u-(r-1)/u)^2 + 1 + (r-1)/u^2], \;\; u > (r-1)^{\frac{1}{2}}.$$

Proof. In view of (7.6), inequality (7.9) means that $f(u) = q(u)q''(u)/q'(u)^2$ increases in $u \geq (r-1)^{\frac{1}{2}}$, and (7.8) means that $f(u) < 1$. On account of (7.3), (7.4), we see that $f(u) \to 1 (u \to \infty)$, and so

$$(7.7) \Leftrightarrow (7.8) \;\&\; (7.9) \Leftrightarrow (7.9).$$

Write (7.9) as $g(u) > 0$, $u > (r-1)^{\frac{1}{2}}$, where we put

$$g(u) = q(u) + q'(u)(u-(r-1)/u)/[(u-(r-1)/u)^2 + 1 + (r-1)/u^2].$$

Taking into account (7.3), (7.6), we see that $g(u) \to 0$ as $u \to \infty$ and

$$\begin{aligned} g'(u) = q'(u)&[(1+(r-1)/u^2)^2 + (u-(r-1)/u)(r-1)/u^3] \\ &\times [(u-(r-1)/u)^2 + 1 + (r-1)/u^2]^{-2} < 0, \;\; u > (r-1)^{\frac{1}{2}}. \end{aligned}$$

Hence $g(u) > 0 \;\; \forall u > (r-1)^{\frac{1}{2}}$, and the proof is completed.

Proof of proposition 4.7. Using (7.1) and arguments similar to (2.8), (2.9) of Eaton (1974), one can obtain (4.7).

Integration by parts gives, in view of (5.1),

(7.10) $$\mathbf{E}\chi_r^j = (r+j-2)\mathbf{E}\chi_r^{j-2}, \;\; j > 2-r.$$

Thus, by Schwartz inequality,

$$\begin{aligned} \mu_r &= \mathbf{E}\chi_r^3/\mathbf{E}\chi_r^2 > (\mathbf{E}\chi_r^3/\mathbf{E}\chi_r)^{\frac{1}{2}} = (r+1)^{\frac{1}{2}}, \\ \mu_r &< (\mathbf{E}\chi_r^4/\mathbf{E}\chi_r^2)^{\frac{1}{2}} = (r+2)^{\frac{1}{2}}, \end{aligned}$$

so (4.10) is proved.

Consider the function

$$\mu(t) = t - 3\gamma(t)/\gamma'(t).$$

We have

$$\begin{aligned} \mu'(t) &= (3\gamma(t)\gamma''(t) - 2\gamma'(t)^2)/\gamma'(t)^2 \\ &= 2(\beta_3\beta_1 - \beta_2^2)/\beta_2^2 > 0, \end{aligned}$$

by Schwartz inequality, where $\beta_i = \mathbf{E}(\chi_r - t)_+^i$. In view of (7.3), $\mu(t) \to \infty$ when $t \to \infty$, and so

$$t \leftrightarrow u = \mu(t)$$

is one-to-one increasing correspondence, under which, in particular, numbers $t \geq 0$ correspond to $u \geq \mu(0) = \mu_r$ (see (4.9)), and vice versa. Put

$$F(t,u) = \mathbf{E}(\chi_r - t)_+^3/(u-t)^3 = C_r\gamma(t)/(u-t)^3.$$

Then

$$\begin{aligned}(\partial/\partial t)F(t,u) &= C_r(u-t)^{-4}(3\gamma(t) + (u-t)\gamma'(t)) \\ &= C_r(u-t)^{-4}\gamma'(t)(u-\mu(t)),\ \ t < u,\end{aligned}$$

and, taking into account that $\gamma'(t) < 0$, we deduce

$$\begin{aligned}W_r(u) &= F(\mu^{-1}(u), u) \\ &= \min\{F(t,u):\ \ t \in (-\infty, u)\},\ \ u \geq \mu_r.\end{aligned} \tag{7.11}$$

In particular, $\forall u \geq \mu_r$

$$\begin{aligned}W_r(u) \leq F(0,u) &= u^{-3}\mathbf{E}\chi_r^3 \leq u^{-2}\mathbf{E}\chi_r^3/\mu_r = r/u^2 \\ &\leq r/\mu_r^2 < 1,\end{aligned}$$

in view of (4.10), already proved, which implies (4.8) when $u \geq \mu_r$. If, conversely, $u \leq \mu_r$, then

$$W_r(u) = F(0,u) \geq u^{-2}\mathbf{E}\chi_r^3/\mu_r = r/u^2,$$

and so, by (4.7), $Q_r(u) = \min[1, r/u^2]$. Thus (4.8) is completely proved, and so is the proposition.

Proof of proposition 4.8. By proposition 4.7, $Q_r(u) = W_r(u)$ for $u \geq \mu_r$. And so (see (7.11))

$$Q_r(u) = F(\mu^{-1}(u), u) = -C_r\gamma'(\mu^{-1}(u))^3/(27\gamma(\mu^{-1}(u))^2),\ \ u \geq \mu_r.$$

Using (7.3), we see that for each r

$$\begin{aligned}Q_r(u)/\mathbf{P}(\chi_r \geq u) &\sim (2/9)(\mu^{-1}(u)^{r-2}/u^{r-2})\exp\left\{\frac{u^2}{2} - \frac{\mu^{-1}(u)^2}{2}\right\} \\ &\sim (2/9)\exp\{u(u-\mu^{-1}(u))\} \sim 2e^3/9,\ \ u \to \infty.\end{aligned}$$

Proof of proposition 4.9. First, note that assertion (4.15) coincides with (7.7); (4.16) is a consequence of Taylor's expansion.

Then, take for the beginning $u \geq \mu_r$. In view of (7.11),

$$Q_r(u) \leq F(\tau, u), \tag{7.12}$$

where

$$\tau = \tau(u) = u + 3q/q',$$

and, as above, $q = q(u)$, $q' = q'(u)$. Taylor's expansion gives

$$\begin{aligned}\gamma(\tau) &= \gamma(u) - (u-\tau)\gamma'(u) + (u-\tau)^2\gamma''(u)/2 \\ &- (u-\tau)^3\gamma'''(u)/6 + (u-\tau)^4\int_0^1 \Theta^3\gamma^{(4)}(\tau + (u-\tau)\Theta)d\Theta/6,\\ e^a &= 1 + a + a^2/2 + a^3/6 + a^4\int_0^1 \Theta^3 e^{(1-\Theta)a}d\Theta/6.\end{aligned}$$

Further, for $s < u$,

$$\begin{aligned}&\gamma^{(4)}(s)/\gamma^{(4)}(u) \le [1-(u-s)/u]^{r-1}\exp\{(u^2-s^2)/2\}\\ &< \exp\{(u-s)[-(r-1)/u + (u+s)/2]\} < \exp\{-(u-s)q''/q'\}\\ &= \exp\{(1-\Theta)a_u\},\ \ \Theta = (s-\tau)/(u-\tau),\ \ a_u = 3qq''/q'^2.\end{aligned}$$

Thus,

$$\begin{aligned}\gamma(\tau) &< \gamma(u) - (u-\tau)\gamma'(u) + (u-\tau)^2\gamma''(u)/2\\ &- (u-\tau)^3\gamma'''(u)/6 + (u-\tau)^4\gamma^{(4)}(u)\mathcal{J}(a_u)/6,\end{aligned}$$

$$\begin{aligned}F(\tau,u)/\mathbf{P}(\chi_r \ge u) &< \gamma\cdot(u-\tau)^{-3}/q - \gamma'\cdot(u-\tau)^{-2}/q\\ &+ \gamma''\cdot(u-\tau)^{-1}/(2q) - \gamma'''/(6q) + \gamma^{(4)}\cdot(u-\tau)\mathcal{J}(a_u)/(6q). \qquad (7.13)\end{aligned}$$

Further,

$$\gamma^{(4)}\cdot(u-\tau)/(6q) = 3, \qquad (7.14)$$

$$-\gamma'''/(6q) = 1. \qquad (7.15)$$

Put $f_2 = \gamma'' + 6q^2/q'$. Then

$$f_2' = \gamma''' + 12q - 6q^2q''/q'^2 = 6q(1 - qq''/q'^2) > 0$$

because of (7.7); hence, $f_2 < 0$, i.e.,

$$\gamma''\cdot(u-\tau)^{-1}/(2q) < 1. \qquad (7.16)$$

Put $f_1 = -\gamma' - 6q^3/q'^2$. Then, by (7.7),

$$f_1' = -\gamma'' - 18q^2/q' + 12q^3q''/q'^3 > -\gamma'' - 6q^2/q' = -f_2 > 0;$$

hence, $f_1 < 0$, i.e.,

$$-(u-\tau)^{-2}\gamma'/q < 2/3. \tag{7.17}$$

Put $f_0 = \gamma + 6q^4/q'^3$. Then, by (7.7),

$$f_0' = \gamma' + 24q^3/q'^3 - 18q^4q/q'^4 > \gamma' + 6q^3/q'^2 = -f_1 > 0;$$

hence, $f_0 < 0$, i.e.,

$$(u-\tau)^{-3}\gamma/q < 2/9. \tag{7.18}$$

Getting now (7.12) - (7.18), (4.15), (4.16) together, we obtain

$$\begin{aligned}\Lambda_r(u) &< 2/9 + 2/3 + 1 + 1 + 3\mathcal{J}(a_u) = \\ &= 2e^3/9 + 3[\mathcal{J}(a_u) - \mathcal{J}(3)] < 2e^3/9, \;\; u \geq \mu_r.\end{aligned}$$

Thus (4.12) and (4.13) for $u \geq \mu_r$ are proved.

Now consider the case $r^{\frac{1}{2}} \leq u \leq \mu_r$. Then (see (4.7))

$$\Lambda_r(u) = rC_r^{-1}/(u^2q(u)), \;\; (u^2q(u))' = ug(u),$$

where we put $g(u) = 2q(u) - u^re^{-u^2/2}$. We have

$$g'(u) = [u^2 - (r+2)]u^{r-1}e^{-u^2/2} < 0$$

when $r^{\frac{1}{2}} \leq u \leq \mu_r (< (r+2)^{\frac{1}{2}}$; see (4.10)). Thus,

$$g(r^{\frac{1}{2}}) \leq 0 \Rightarrow \Lambda_r(u) \leq \Lambda_r(\mu_r) < \frac{2e^3}{9}, \;\; r^{\frac{1}{2}} \leq u \leq \mu_r,$$

in view of (4.12). If, on the contrary, $g(r^{\frac{1}{2}}) > 0$, then

$$q_r(r^{\frac{1}{2}}) > (r/e)^{r/2}/2. \tag{7.19}$$

If $u_*\epsilon(r^{\frac{1}{2}}, \mu_r)$ is a root of the equation $g(u) = 0$, then

$$u_*^2 q(u_*) = h(u_*)/2,$$

where we set $h(u) = u^{r+2}e^{-u^2/2}$; but

$$h'(u) = -[u^2 - (r+2)]h(u)/u > 0$$

when $0 \le u \le \mu_r (< (r+2)^{\frac{1}{2}})$; thus $u_*^2 q(u_*) > rq(r^{\frac{1}{2}})$; hence

$$\max\{\Lambda_r(u) : \;\; r^{\frac{1}{2}} \le u \le \mu_r\} = \max\{\Lambda_r(r^{\frac{1}{2}}), \Lambda_r(\mu_r)\}.$$

By (4.8), (4.12), $\Lambda_r(\mu_r) < 2e^3/9$. Let us prove that $\Lambda_r(r^{\frac{1}{2}}) < 2e^3/9$ or, equivalently, $q_r(0) < (2e^3/9)q_r(r^{\frac{1}{2}})$ under (7.19). It suffices to show that

$$q_r(0) < \frac{e^3}{9}(r/e)^{\frac{r}{2}}. \tag{7.20}$$

We shall do it by induction. This is easy when $r = 1, 2$. If (7.20) is true for some $r \ge 1$, then

$$\begin{aligned} &q_{r+2}(0) = rq_r(0) < r \cdot (e^3/9)(r/e)^{\frac{r}{2}} = \\ &= (e^3/9)[r/(r+2)]^{(r+2)/2} \cdot e \cdot [(r+2)/e]^{(r+2)/2} < (e^3/9)[(r+2)/e]^{(r+2)/2}. \end{aligned}$$

Thus, (4.13) is proved in the case $r^{\frac{1}{2}} \le u \le \mu_r$. Consider, finally, $0 \le u \le r^{\frac{1}{2}}$. Then

$$\Lambda_r(u) = 1/\mathbf{P}(\chi_r \ge u) \le \Lambda_r(r^{\frac{1}{2}}) < 2e^3/9$$

as we have just shown. So (4.13) and the whole proposition are completely proved.

Proof of proposition 4.10. Consider

$$\zeta(u) = Q_1'(u)/(C_1 q_1'(u)), \;\; u \ge \mu_1,$$

where $Q_1(u) = W_1(u) = F(\mu^{-1}(u), u)$ (see (4.8), (7.11)). Let t stand for $\mu^{-1}(u)$. Calculations show that

$$Q_1'(u) = -\gamma'^3/(27\gamma^2),$$

$$(d/dt)\log\zeta(u) = -3F^2/(\gamma\gamma'^3) \ge 0,$$

in view of (7.2), where $F = \gamma'^2 - \gamma\gamma''$, $\gamma = \gamma_1(t)$, $\gamma' = \gamma_1'(t)$, $\gamma'' = \gamma''(t)$; moreover,

$$((((F/\gamma'')'\gamma''^2/(\gamma'\gamma'''))'\gamma'''^2/(\gamma''\gamma^{(4)}))'\gamma^{(4)^2}/(\gamma^{(5)}\gamma'''))' = 6t^{-2}e^{-t^2/2} > 0$$

when $t > 0$, which, together with (7.2), (7.3), (7.5), leads to the inequality $F > 0$ and then to

$$(d/dt)\log\zeta(u) > 0, \;\; t > 0.$$

Now, in view of proposition 5.1 and remark 5.7, we obtain the statement of proposition 4.10.

Proof of theorem 6.3. Inequality (6.3) is a consequence of (6.1). By (5.5), $x_d(\delta) > (d-1)^{\frac{1}{2}}$ when $\delta \leq 0.5$. Put $f(u) = cq(u+h) - q(u)$, where

$$c = 2e^3/9, h = h(u) = (\log c)/[u - (d-1)/u], u > (d-1)^{\frac{1}{2}}.$$

Then

$$\begin{aligned} f'(u) &= -q'(u) + cq'(u+h)\cdot(1+h') > -q'(u) + cq'(u+h), \\ f'(u)u^{1-d}e^{u^2/2} &> 1 - c\cdot(1+h/u)^{d-1}\exp\left\{\frac{u^2}{2} - \frac{(u+h)^2}{2}\right\} \\ &> 1 - c\cdot\exp\{-[u-(d-1)/u]\cdot h\} = 0. \end{aligned}$$

Hence $f(u) < 0 \;\; \forall u > (d-1)^{\frac{1}{2}}$. In view of (6.1), this implies (6.3), (6.4).

Using again (5.5), we see that if $\delta \downarrow 0$, then $x_d(\delta) - (d-1)^{\frac{1}{2}} \to \infty$; hence we obtain (6.5), (6.6). Finally, (6.7) is trivial.

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