Localisation of Geometric Anisotropy

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Abstract

The class of 2-D nonseparable geometrically anisotropic localisation operators is defined, containing highly anisotropic nearly unidirectional localisation operators, as well as isotropic localisation operators. A continuum of anisotropic operators between the extremes of near unidirectionality and isotropy are treated in a single class. The eigensystem of any given operator in this family is determined, thus specifying geometrically anisotropic optimally concentrated functions, and their degree of localisation.

Index Terms

Localisation operator, wavelets, anisotropic and directional variation.

I. Introduction

THis correspondence introduces a new class of spatially anisotropic and nonseparable 2-D localisation operators, namely the class of *geometrically anisotropic* localisation operators. The operation of 'localisation' in this context refers to limiting variation in a 2-D square integrable function to variation associated with a given set of spatial locations and spatial frequencies, see also Daubechies [1][ch. 2]. By formally using a localisation operator, the energy concentration of a given function to an anisotropic and nonseparable 4-D region of 2-D space and 2-D spatial frequency, may be *exactly* quantified. Optimally localized functions can then be derived.

The important set of tools which motivates the need for deriving optimally localised functions is that based on the local representation of functions. A Fourier Transform (FT) represents a signal globally, *i.e.* it decomposes a signal in terms of modes present over the full length of a signal. Given many signals exhibit transient features, it is necessary to represent the local properties of the signal. Examples of local decompositions include the Wavelet Transform (WT) and the windowed FT [2].

To be able to form a local decomposition of a given function, a set of well-localised decomposition functions must be used, or the utility of the decomposition vanishes. The locality of a given decomposition function must be adjusted to the class of signal that will be analysed. By the construction procedure used for the WT [2], the locality of any member of the family of functions used for analysis, is in some sense 'equivalent' to the locality of the mother wavelet. Thus it is sufficient to determine appropriate mother wavelets for a given problem. The eigensystem of a localisation operator corresponds to a whole set of mutually orthogonal and optimally localized functions, where the eigenvalues of the functions, measure their degree of localisation to a given localisation region [3], [1]. Thus eigenfunctions of localisation operators are suitable mother wavelets.

By determining the eigensystem of the operators proposed in this correspondence, we obtain classes of mother wavelets suitable for analysis of non-stationary geometrically anisotropic fields. These facts justify the construction and study of the proposed operators. 1-D localisation operators have already been the focus of considerable study, see for example the references in [1], [4], [5]. Consecutive truncations form a possible method of constructing localisation operators: such procedures treat the space and spatial frequency variables inhomogeneously. This results in an unequal compromise between concentration in space and in spatial frequency, a clearly undesirable feature.

Also it is desirable to define operators localising functions to regions that do not contain zero frequency, if we wish to build for example families of wavelets. In this case the relocation in frequency is not implemented by frequency shifts. Daubechies and Paul [4] defined a set of homogeneous localisation operators, that treated time and frequency variation on an equal footing. These operators were used to derive optimal 1-D mother wavelet functions, and the 1-D Morse wavelets were thus obtained. The Morse wavelets have been used to analyse geophysical, astrophysical, and medical time series [6], [7], and have been extended to radial wavelets in 2-D [3]. The construction of localisation

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operators in both cases started from forming a local decomposition of the observed function, discussed in section II.

We seek to develop operators for the study of geometric anisotropy, formally defined in section III-A. Our motivation for studying this class is that it contains isotropic functions, and anisotropic functions that are nearly unidirectional, as well as the full continuum of structures in between isotropy and unidirectionality. Geometric anisotropy is used for modelling covariance structures in geostatistics (see for example Christakos [8][p. 61]). Thus the optimally localised geometrically anisotropic mother wavelets will have a natural application area, in the study of non-stationary geometrically anisotropic random fields.

In general it is very hard to define operators so that their eigensystem may be determined [1][p. 41], and without a known eigensystem the operators lose most of their utility. We shall therefore define the anisotropic operator with great care, in a series of steps, in section III. We start by defining the appropriate 'fiducial vector' from which to build the operator. Geometric anisotropy is characterised by a transformation matrix, and thus a transformed distance metric is defined. The fiducial vector is picked to be a function marginally well localized in space in terms of the transformed distance metric, and also well-concentrated in transformed frequency. From the fiducial vector a family of coherent states must subsequently be defined. We construct the family of coherent states in section III-B, using the transformation matrix of the geometric anisotropy. In general the family of coherent states will not be constructed by the usual 2-D operations of translating, scaling and rotating the fiducial vector [3], but rather by a set of *transformed* operations.

We subsequently need to determine the localisation of an arbitrary element of the family of coherent states, as this in conjunction with the choice of region of integration, will determine the localisation of the operator, see section III-C. This enables the definition of operators with given localisation regions, see section III-D. We subsequently determine the eigensystem of the proposed localisation operators, see section IV. We derive explicit forms for both eigenfunctions and eigenvalues. We give specific examples, for given transformation matrices, of functions in this class.

These developments combine to define new classes of 'optimally' localised nonseparable anisotropic functions. The developments span previously derived optimally localised functions, and form a cohesive framework for treating geometrically anisotropic 2-D localisation.

II. Local Decompositions

We shall construct the localisation operator starting from a local decomposition, similar to the 2-D Continuous WT (CWT), see for example Antoine *et al.* [2]. We define a family of 'coherent states' $v_{\boldsymbol{\xi}}(\mathbf{x})$ from the 'fiducial vector' [3], a single 2-D function $v(\mathbf{x})$. $v(\mathbf{x})$ is assumed to be marginally well-localised in space and spatial frequency. The index $\boldsymbol{\xi} = [a, \theta, \mathbf{b}]^T \in \mathcal{A} \subset \mathbb{R}^4$ regulates the localisation of the elements of the family of coherent states. For example, the usual WT corresponds to taking: $v_{\boldsymbol{\xi}}(\mathbf{x}) = \mathcal{D}_a \mathcal{R}_\theta \mathcal{T}_{\mathbf{b}} \{v\}(\mathbf{x})$, where for $a > 0$, $\mathcal{D}_a v(\mathbf{x}) = a^{-1} v(\mathbf{x}/a)$ is the dilation operator, for $\theta \in [0, 2\pi)$, $\mathcal{R}_\theta v(\mathbf{x}) = v(\mathbf{r}_{-\theta}\mathbf{x})$ is the rotation operator, with $\mathbf{r}_\theta$ as the rotation matrix [2], and for $\mathbf{b} \in \mathbb{R}^2$, $\mathcal{T}_{\mathbf{b}} v(\mathbf{x}) = v(\mathbf{x} - \mathbf{b})$ is the translation operator.

Let $v(\mathbf{x})$ have a FT given by $V(\boldsymbol{\omega})$, in angular frequency $\boldsymbol{\omega}$. We do not refer to $v(\mathbf{x})$ as a 'mother wavelet', as we intend to introduce a different set of operations to construct $v_{\boldsymbol{\xi}}(\mathbf{x})$. We define the 'local coefficients' of $v, g \in L^2(\mathbb{R}^2)$ by:

$$w_v(\boldsymbol{\xi}; g) \;=\; \int\int_{\mathbb{R}^2} v^*_{\boldsymbol{\xi}}(\mathbf{x}) g(\mathbf{x})\, d^2\mathbf{x} = \langle v_{\boldsymbol{\xi}}(\mathbf{x}), g(\mathbf{x}) \rangle = \frac{1}{(2\pi)^2} \int\int_{\mathbb{R}^2} V^*_{\boldsymbol{\xi}}(\boldsymbol{\omega}) G(\boldsymbol{\omega})\, d^2\boldsymbol{\omega}. \tag{1}$$

Eqn. (1) represents a projection of $g(\mathbf{x})$ into 'local contributions' $w_v(\boldsymbol{\xi}; g)$. The function $g(\mathbf{x})$ can be reconstructed from the local coefficients, if $\boldsymbol{\xi}$ and $v(\mathbf{x})$ satisfy a suitable set of constraints. For example if $v(\mathbf{x})$ is an admissible mother wavelet, with admissibility constant C_v and $\boldsymbol{\xi}$ corresponds to the indexing denoting the translation, scaling and rotation operation, then we may reconstruct $g(\mathbf{x})$ by:

$$g(\mathbf{x}) = \frac{1}{C_v} \int_{\mathcal{A}} v_{\boldsymbol{\xi}}(\mathbf{x}) w_v(\boldsymbol{\xi}; g)\; dA_{\boldsymbol{\xi}}, \quad dA_{\boldsymbol{\xi}} = a^{-3} da\, d^2\mathbf{b}\, d\theta. \tag{2}$$

The interpretability of eqns. (1) and (2) depends on the locality of $v_{\boldsymbol{\xi}}(\mathbf{x})$, in turn determined from the locality of $v(\mathbf{x})$ and the choice of operations denoted by $\boldsymbol{\xi}$. Depending on the type of function $g(\mathbf{x})$ that we are decomposing, different families of coherent states are suitable to use for the analysis of this function. To focus our interest on a special class of anisotropy, we shall now introduce the class of geometric anisotropy.

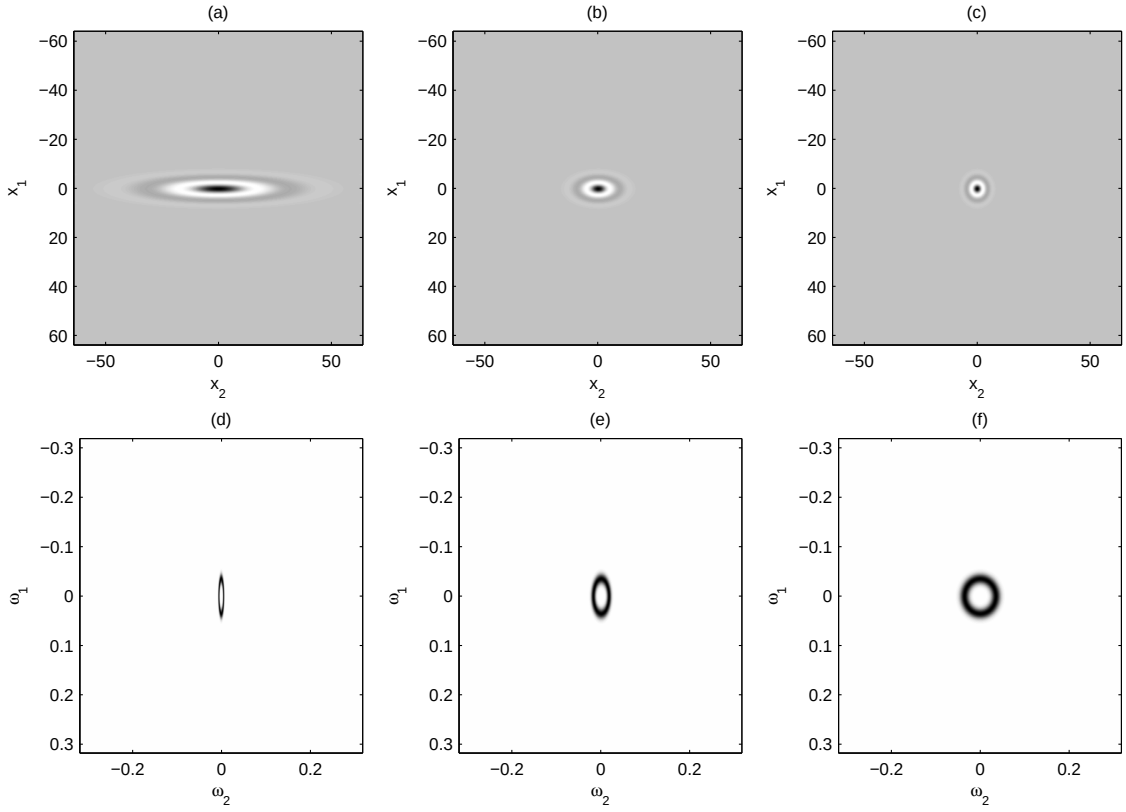

Fig. 1. The $n = 0$ eigenfunctions in space when a) $\mathbf{P_H} = ([1 \quad 0]\,;[0 \quad 0.15])$, b) $\mathbf{P_H} = ([1 \quad 0]\,;[0 \quad 0.5])$ and c) $\mathbf{P_H} = ([1 \quad 0]\,;[0 \quad 1])$, with $(\beta, \gamma) = (8, 3)$. The same functions are also plotted in frequency, in d), e) and f).

III. Constructing the Geometrically Anisotropic Operator

A. Geometrically Anisotropic Functions

Definition 3.1: A Geometrically Anisotropic Function

A function $g_A(\mathbf{x})$ is said to exhibit *geometric anisotropy* if for a fixed non-negative 2×2 symmetric matrix $\mathbf{H}$ and $g_r(\cdot)$ a 1-D function, it takes the form: $g_A(\mathbf{x}) = g_r\left(\mathbf{x}^T\mathbf{H}\mathbf{x}\right)$.

An application of such functions is to model auto-covariance of random fields having geometric anisotropy, see Christakos [8][p. 61]. For such random fields the auto-covariance of the field $h(\mathbf{x})$ at spatial locations $\mathbf{x} \in \mathbb{R}^2$ and at $\mathbf{y} \in \mathbb{R}^2$ takes the form $b_h(\mathbf{x}, \mathbf{y}) = \operatorname{cov}\{h(\mathbf{x}), h(\mathbf{y})\} = b_r\left((\mathbf{x}-\mathbf{y})^T\mathbf{H}(\mathbf{x}-\mathbf{y})\right)$. An example of such functions is an isotropic covariance, given by $\mathbf{H} = \mathbf{I}$, where $b_{\mathbf{H}}(\mathbf{x}_1, \mathbf{x}_2)$ is specified as a function of the Cartesian distance between $\mathbf{x}_1$ and $\mathbf{x}_2$, but also for certain observed phenomena it is natural to model the auto-covariance between the field at two points as depending on a local affine transformation of the two variables [8], [9]. Note that the anisotropic extreme of this form corresponds to having $\mathbf{H} = \mathbf{I}_1(\varepsilon) = \left([1 \quad 0]\,;\left[0 \quad \varepsilon^2\right]\right)$. The function becomes approximately unidirectional, as $\varepsilon \to 0$.

We form the decomposition of $\mathbf{H}$ as $\mathbf{H} = \mathbf{P}_{\mathbf{H}}^T\mathbf{P}_{\mathbf{H}}$, and refer to $\mathbf{P_H}$ as the 'transformation matrix' of $\mathbf{H}$. Unfortunately this specification is not sufficiently constrained to uniquely determine $\mathbf{P_H}$, however as we in this correspondence only seek to determine the optimal decomposition for a set $\mathbf{H}$, any of the $\mathbf{P_H}$ will work. For $\mathbf{H} = \mathbf{I}$, we have $\mathbf{P_H} = \mathbf{I}$ whilst for $\mathbf{H} = \mathbf{I}_1(\varepsilon)$, $\mathbf{P_H}(\varepsilon) = ([1 \quad 0]\,;[0 \quad \varepsilon])$. Let $\mathbf{Q_H} = \mathbf{P}_{\mathbf{H}}^{-1}$. Define the 'transformed spatial variables' $\mathbf{y}$ and the 'transformed frequency variables' $\boldsymbol{\nu}$ by: $\mathbf{y} = \mathbf{P_H}\mathbf{x}$ and $\boldsymbol{\nu} = \mathbf{Q}_{\mathbf{H}}^T\boldsymbol{\omega}$. Then $\mathbf{x} = \mathbf{Q_H}\mathbf{y}$ and $\boldsymbol{\omega} = \mathbf{P}_{\mathbf{H}}^T\boldsymbol{\nu}$. For convenience we also define $y = \|\mathbf{y}\|$, and $\nu = \|\boldsymbol{\nu}\|$, to complement $x = \|\mathbf{x}\|$ (the modulus of the spatial coordinate) and $\omega = \|\boldsymbol{\omega}\|$.

B. Defining the Geometrically Anisotropic Coherent States

We shall now construct a localisation operator that measures the localisation to regions that spatially decay in $y^2(\mathbf{x})$, and decay in frequency $\boldsymbol{\nu}(\boldsymbol{\omega})$, away from some set of frequencies. We first define the fiducial vector, very

much in analogue with choosing a mother wavelet, and from the fiducial vector define a family of coherent states. Define as in Metikas and Olhede [3][eqn. 4], $V^{(1-D,\beta,\gamma)}(\omega) = \frac{2^{(r+1)/2}\sqrt{\pi\gamma}}{\sqrt{\Gamma(r)}}\omega^{\beta}e^{-\omega^{\gamma}}$ for $\omega > 0$, and 0 for $\omega < 0$. We use the radial coherent states used in [3][eqn. 8]:

$$V_r^{(\beta,\gamma)}(\boldsymbol{\omega}) = \omega^{-1/2+\gamma/2}V_{1D}^{(1-D,\beta,\gamma)}(\omega), \quad v_r^{(\beta,\gamma)}(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\omega^{-1/2+\gamma/2}}{(2\pi)^2}V^{(1-D,\beta,\gamma)}(\omega)e^{\boldsymbol{\jmath}\boldsymbol{\omega}^T\mathbf{x}}\,d^2\boldsymbol{\omega}. \tag{3}$$

$v_r^{(\beta,\gamma)}(\mathbf{x})$ is marginally well localised, see [3]. Define the geometrically anisotropic fiducial vector by:

$$v_{\mathbf{H}}(\mathbf{x}) = \sqrt{|\mathbf{P_H}|}v_r(\mathbf{y}), \quad V_{\mathbf{H}}(\boldsymbol{\omega}) = |\mathbf{P_H}|^{-1/2}\,V_r(\mathbf{Q}_{\mathbf{H}}^T\boldsymbol{\omega}) = |\mathbf{P_H}|^{-1/2}\,V_r(\boldsymbol{\nu}). \tag{4}$$

Let $\omega_0 = \arg_{\omega>0}\max\omega^{-1+\gamma}|V_{1D}^{(1-D,\beta,\gamma)}(\omega)|^2$. $v_r^{(\beta,\gamma)}(\mathbf{x})$ is decaying radially in space from $\mathbf{x} = \mathbf{0}$, and is supported at angular frequencies such that $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, where $\omega \approx \omega_0$. Thus $v_{\mathbf{H}}(\mathbf{x})$ is local in terms of $\mathbf{x}^T\mathbf{H}\mathbf{x}$ and is associated with frequencies $\boldsymbol{\omega}$ such that $\left|\mathbf{Q}_{\mathbf{H}}^T\boldsymbol{\omega}\right| = \omega_0$.

We now need to define the coherent states from the fiducial vector defined in eqn. (4). We define the transformed rotation operator $\tilde{\mathcal{R}}_{\theta,\mathbf{H}}$ by: $\tilde{\mathcal{R}}_{\theta,\mathbf{H}}\left\{g\right\}(\mathbf{x}) = g\left(\tilde{\mathbf{r}}_{-\theta,\mathbf{H}}\mathbf{x}\right),\ \tilde{\mathbf{r}}_{-\theta,\mathbf{H}} = \left[\left(\mathbf{P}_{\mathbf{H}}^T\mathbf{r}_{-\theta}\mathbf{Q}_{\mathbf{H}}^T\right)^T\right]^{-1}$.Thus if $\mathbf{P_H}$ is itself a rotation, then $\tilde{\mathcal{R}}_{\theta,\mathbf{H}}$ corresponds to a normal rotation, otherwise $\tilde{\mathcal{R}}_{\theta,\mathbf{H}}$ defines the act of rotation in the transformed space, i.e. in the $\mathbf{y}$ coordinates.

We define the generalized spatial shift, also in the transformed space, for any function $g(\mathbf{x})$ with FT $G(\boldsymbol{\omega})$ by:

$$\tilde{\mathcal{T}}_{\mathbf{b},\mathbf{H}}^{(\gamma)}G(\boldsymbol{\omega}) = G(\boldsymbol{\omega})e^{-\boldsymbol{\jmath}\mathbf{b}^T\boldsymbol{\nu}\nu^{\gamma-1}}, \quad \tilde{\mathcal{T}}_{\mathbf{b},\mathbf{H}}^{(\gamma)}g(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\boldsymbol{\omega})e^{\boldsymbol{\jmath}\boldsymbol{\omega}^T(\mathbf{x}-\nu^{\gamma-1}\mathbf{Q_H}\mathbf{b})}\,d^2\boldsymbol{\omega}. \tag{5}$$

Thus if $\gamma = 1$ this simplifies to the usual spatial shift, if by $\mathbf{Q_H}\mathbf{b}$ rather than by $\mathbf{b}$. The generalized spatial shift is thus adapted to the transformed geometry. If $\gamma \neq 1$ then the local effect on $g(\mathbf{x})$, local to any given wavenumber $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, is a shift in space by $|\mathbf{P_H}\boldsymbol{\omega}_0|^{\gamma-1}\mathbf{Q_H}\mathbf{b}$. The magnitude of the shift depends on $|\mathbf{P_H}\boldsymbol{\omega}_0|^{\gamma-1}$. The act of a implementing a generalized spatial shift corresponds to shifting the fiducial vector in space by different amounts at different frequencies, in precise analogue with the radial [3] and 1-D [4] generalized spatial shifts. Introducing the generalized spatial shift enables us to treat a larger class of localisation regions. A family of coherent states is then defined by:

$$\begin{aligned} v_{\mathbf{H};\boldsymbol{\xi}}(\mathbf{x}) &= \mathcal{D}_{a^{1/\gamma}}\tilde{\mathcal{R}}_{\theta,\mathbf{H}}\tilde{\mathcal{T}}_{\mathbf{b},\mathbf{H}}^{(\gamma)}v_{\mathbf{H}}(\mathbf{x}),\ V_{\mathbf{H};\boldsymbol{\xi}}(\boldsymbol{\omega}) = a^{1/\gamma}V_{\mathbf{H}}(a^{1/\gamma}\left([\tilde{\mathbf{r}}_{-\theta,\mathbf{H}}]^{-1}\right)^T\boldsymbol{\omega})e^{-\boldsymbol{\jmath}\boldsymbol{\omega}^T\mathbf{Q_H}\mathbf{b}\|\mathbf{Q_H}\boldsymbol{\omega}\|^{\gamma-1}} \\ &= e^{-\boldsymbol{\jmath}\boldsymbol{\nu}^T\mathbf{b}\nu^{\gamma-1}}a^{1/\gamma}\,|\mathbf{P_H}|^{-1/2}\,V_r(a^{1/\gamma}\mathbf{r}_{-\theta}\boldsymbol{\nu}), \end{aligned} \tag{6}$$

and $V_{\mathbf{H};\boldsymbol{\xi}}^{(\beta,\gamma)}(\boldsymbol{\omega}) = e^{-(a\nu+\boldsymbol{\jmath}\boldsymbol{\nu}^T\mathbf{b})\nu^{\gamma-1}}a^{1/\gamma}2^{\frac{r+1}{2}}\sqrt{\pi\gamma}\left(a^{1/\gamma}\nu\right)^{\frac{2\beta+\gamma-1}{2}}/(\sqrt{|\mathbf{P_H}|\,\Gamma(r)})$.

C. The Localisation of the Coherent States

Before expressing any function in terms of the family of coherent states, and proving that a resolution of identity may be achieved, we shall determine the localisation of the coherent states. We need to determine the spatial and spatial frequency locality of $v_{\mathbf{H};\boldsymbol{\xi}}^{(\beta,\gamma)}(\mathbf{x})$ as a function of $\boldsymbol{\xi}$, to be able to determine the localisation of the operator. We firstly determine the energy of the coherent states (see eqn. (22) in the appendix), and noting that $d^2\boldsymbol{\omega} = |\mathbf{P_H}|\,d^2\boldsymbol{\nu}$, this then yields:

$$\langle 1\rangle_{\mathbf{H}}(\boldsymbol{\xi}) = \langle v_{\mathbf{H};\boldsymbol{\xi}}^{(\beta,\gamma)}, v_{\mathbf{H};\boldsymbol{\xi}}^{(\beta,\gamma)}\rangle = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|V_{\mathbf{H};\boldsymbol{\xi}}^{(\beta,\gamma)}(\boldsymbol{\omega})\right|^2\,d^2\boldsymbol{\omega} = \frac{1}{2\pi}\int_0^{\infty}\omega^{\gamma}\left|V_{1D}^{(\beta,\gamma)2}(\omega)\right|^2\,d\omega.$$

We then calculate the spatial and spatial frequency average of the function, just like Daubechies and Paul [4][eqn. 2.10] . It is easiest to calculate the transformed coordinate averages, rather than the averages of $\mathbf{x}$ and $\boldsymbol{\omega}$. We determine that (see eqn. (23) in the appendix):

$$\begin{aligned} \langle\nu\rangle_{\mathbf{H}}(\boldsymbol{\xi}) &= \frac{4}{\langle 1\rangle_{\mathbf{H}}(\boldsymbol{\xi})}\frac{1}{(2\pi)^2}\int_0^{\infty}\int_0^{\infty}\nu\,|V_{\mathbf{H};\boldsymbol{\xi}}(\boldsymbol{\omega})|^2\,d^2\boldsymbol{\omega} = \frac{|\mathbf{P_H}|}{\langle 1\rangle_{\mathbf{H}}(\boldsymbol{\xi})}\frac{a^{2/\gamma}}{\pi^2}\int_0^{\infty}\int_0^{\infty}\nu\left|V_{\mathbf{H}}(a^{1/\gamma}\boldsymbol{r}_{-\theta}\boldsymbol{\nu})\right|^2\,d^2\boldsymbol{\nu} \\ &= \frac{\Gamma\left(r+\frac{1}{\gamma}+1\right)}{2^{1/\gamma}\Gamma(r+1)}\frac{1}{a^{1/\gamma}} = \frac{C_3^{(\beta,\gamma)}}{a^{1/\gamma}}. \end{aligned} \tag{7}$$

Similarly we can determine that the average spatial position of any member of the family of coherent states is given (see eqn. (24) in the appendix) by:

$$\langle y_l \rangle_{\mathbf{H}}(\boldsymbol{\xi}) \;=\; \frac{1}{\langle 1 \rangle_{\mathbf{H}}(\boldsymbol{\xi})} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_l \left| v_{\mathbf{H};\boldsymbol{\xi}}(\mathbf{x}) \right|^2 \, d^2\mathbf{x} = \frac{2^{1/\gamma-2}(\gamma+1)\Gamma(r-1/\gamma+2)}{\Gamma(r+1)} a^{-1+1/\gamma} b_l.$$

We define $C_4^{(\beta,\gamma)} = 2^{1/\gamma-2}(\gamma+1)\Gamma(r-1/\gamma+2)/\Gamma(r+1)$, to be the term multiplying $a^{-1+1/\gamma}b_l$. The average spatial position $\langle \mathbf{x} \rangle_{\mathbf{H}}(\boldsymbol{\xi})$ has components $\langle x_l \rangle_{\mathbf{H}}(\boldsymbol{\xi}) = \left[\mathbf{Q}_{\mathbf{H}} \left(\langle y_1 \rangle_{\mathbf{H}}(\boldsymbol{\xi}) \quad \langle y_2 \rangle_{\mathbf{H}}(\boldsymbol{\xi}) \right) \right]_l$. With this set of relationships we may note the locality of $v_{\mathbf{H}}^{(\beta,\gamma)}(\mathbf{x})$ and use this to construct a suitable localisation region for a function exhibiting geometric anisotropy.

D. Defining the Localisation Operator & Resolution of Identity

1) General Operator: For any $g(\mathbf{x}) \in L^2(\mathbb{R})$ we define for any region $\mathcal{A} \subset \mathbb{R}^+ \times [0, 2\pi) \times \mathbb{R}^2$ the localisation operator. Define firstly the region of space and spatial frequency given by:

$$\mathcal{D}_{\mathbf{H}} = \left\{ (\mathbf{x}, \boldsymbol{\omega}) : \;\; \mathbf{x} = \langle \mathbf{x} \rangle_{\mathbf{H}}(\boldsymbol{\xi}), \;\; \left| \mathbf{Q}_{\mathbf{H}}^T \boldsymbol{\omega} \right| = \langle \nu \rangle_{\mathbf{H}}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathcal{A} \right\}. \tag{8}$$

Then subsequently define the projection operator by:

$$\begin{aligned}
\mathcal{P}_{\mathcal{D}_{\mathbf{H}}} g(\mathbf{x}) &= C_{\mathbf{H}} \int_{\mathcal{A}} \langle v_{\mathbf{H};\boldsymbol{\xi}}, g \rangle v_{\mathbf{H};\boldsymbol{\xi}}(\mathbf{x}) \, dA_{\boldsymbol{\xi}} \\
\mathcal{P}_{\mathcal{D}_{\mathbf{H}}} G(\boldsymbol{\omega}_1) &= \frac{C_{\mathbf{H}}}{(2\pi)^2} \int_{\mathcal{A}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_{\mathbf{H};\boldsymbol{\xi}}^*(\boldsymbol{\omega}_2) G(\boldsymbol{\omega}_2) V_{\mathbf{H};\boldsymbol{\xi}}(\boldsymbol{\omega}_1) \, d^2\boldsymbol{\omega}_2 \, dA_{\boldsymbol{\xi}} \\
\mathcal{P}_{\mathcal{D}_{\mathbf{H}}}^{(\beta,\gamma)} G(\boldsymbol{\omega}_1) &= \frac{C_{\mathbf{H}}}{(2\pi)^2} \int_{\mathcal{A}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a^{2/\gamma} V_r^{(\beta,\gamma)}(a^{1/\gamma} \mathbf{r}_{-\theta} \boldsymbol{\nu}_1) e^{-\boldsymbol{\jmath} \mathbf{b}^T \boldsymbol{\nu}_1 \nu_1^{\gamma-1}} \\
&\qquad V_r^{(\beta,\gamma)*}(a^{1/\gamma} \mathbf{r}_{-\theta} \boldsymbol{\nu}_2) e^{\boldsymbol{\jmath} \mathbf{b}^T \boldsymbol{\nu}_2 \nu_2^{\gamma-1}} G\left(\mathbf{P}_{\mathbf{H}}^T \boldsymbol{\nu}_2 \right) \, d^2\boldsymbol{\nu}_2 \, dA_{\boldsymbol{\xi}}.
\end{aligned} \tag{9}$$

We now let $\mathcal{A} \to \mathbb{R}^+ \times [0, 2\pi) \times \mathbb{R}^2 = \mathcal{A}_{\text{all}}$ and note from Metikas and Olhede [3][section III.D] that assuming $C_{\mathbf{H}}^{-1} = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} \nu^{-2\gamma} \left| V_r^{(\beta,\gamma)}(\boldsymbol{\nu}) \right|^2 \, d^2\boldsymbol{\nu} < \infty$, we have that $\mathcal{P}_{\mathcal{D}_{\text{all}}}^{(\beta,\gamma)} G(\boldsymbol{\omega}_1) = G\left(\mathbf{P}_{\mathbf{H}}^T \boldsymbol{\nu}_1 \right) = G\left(\boldsymbol{\omega}_1 \right)$. We thus achieve a 'resolution of identity'. This demonstrates that eqn. (9) is a suitable localisation operation. If we include all space when integrating then we retrieve the full function $g(\mathbf{x})$. Of course, to appropriately define a localised function, the operator needs to be specified very carefully.

We start by defining the appropriate restriction of the local index $\boldsymbol{\xi}$ for a fixed $C > 1$ by:

$$\mathcal{A}(C) = \left\{ (a, \theta, \mathbf{b}) : \quad a^2 + b^2 + 1 \leq 2aC, \quad \theta \in [0, 2\pi), \quad b = \sqrt{b_1^2 + b_2^2} \right\}. \tag{10}$$

Note that $\mathcal{A}(C)$ does not depend on $\mathbf{H}$, the transformation of the distance metric, or the shape parameters (β, γ). We shall now use the coherent states defined by eqn. (6) so that $\mathcal{A}(C)$ of equation (10) gets mapped onto a region of space and spatial frequency that is given by:

$$\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C) = \left\{ (\mathbf{x}, \boldsymbol{\omega}) : \quad \left(y \nu E_3^{(\beta,\gamma)} \right)^2 + \left(\nu^\gamma - C E_4^{(\beta,\gamma)} \right)^2 \leq E_4^{(\beta,\gamma)2} \left(C^2 - 1 \right) \right\}, \tag{11}$$

with $C_3^{(\beta,\gamma)}$, and $C_4^{(\beta,\gamma)}$, constants whose values are given by eqns. (7) and (8). For simplicity we have also defined:

$$E_3^{(\beta,\gamma)} = \left(\frac{C_3^{(\beta,\gamma)(\gamma-1)}}{C_4^{(\beta,\gamma)}} \right), \quad E_4^{(\beta,\gamma)} = C_3^{(\beta,\gamma)\gamma}. \tag{12}$$

Thus the region $\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)$ is specified by four different constituent parts: the transformation $\mathbf{P}_{\mathbf{H}}$, the shape parameters (β, γ) and C, the hypervolume parameter.

We note from eqn. (10) that $C - \sqrt{C^2-1} \leq a \leq C + \sqrt{C^2-1}$ and so the maximum and minimum wave number in $\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)$ are therefore given by: $\nu_{\max} = C_3^{(\beta,\gamma)} / \left[\sqrt[\gamma]{\left(C - \sqrt{C^2-1} \right)} \right]$, $\nu_{\min} = C_3^{(\beta,\gamma)} / \left[\sqrt[\gamma]{\left(C + \sqrt{C^2-1} \right)} \right]$. The local transformation $\mathbf{P}_{\mathbf{H}}$ specifies the local region $\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)$ in terms of the anisotropy between x_1 and x_2, as

well as ω_1 and ω_2, as the space is parameterised in $\mathbf{y}$ and $\boldsymbol{\nu}$. The shape parameters (β,γ) determine the compromise between $\mathbf{y}$ and $\boldsymbol{\nu}$, and C determines the hypervolume of the region. We reparameterise the region $\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)$ in $\mathbf{y}$ and $\boldsymbol{\nu}$, as $\mathcal{D}_{\mathbf{H},y}^{(\beta,\gamma)}(C)$. For a fixed value of C, the hypervolume of $\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)$, is given by:

$$\left|\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)\right| = \int_{\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)} d^2\mathbf{x}\, d^2\boldsymbol{\omega} = \int_{\mathcal{D}_{\mathbf{H},y}^{(\beta,\gamma)}(C)} d^2\mathbf{y}\, d^2\boldsymbol{\nu} = 4\pi^2 \int_{\mathcal{D}_{\mathbf{H},y}^{(\beta,\gamma)}(C)} y\, dy\, \nu\, d\nu.$$

We change variables to $a = \frac{C_3^{(\beta,\gamma)\gamma}}{\nu^\gamma}$, $b_l = \frac{C_3^{(\beta,\gamma)(\gamma-1)} y_l}{C_4^{(\beta,\gamma)}\nu^{\gamma-1}}$, and $b = \sqrt{b_1^2+b_2^2} = \frac{C_3^{(\beta,\gamma)(\gamma-1)} y}{C_4^{(\beta,\gamma)}\nu^{\gamma-1}}$. Then:

$$\begin{aligned} A(\beta,\gamma)\mathrm{Area}(C) &= \left|\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)\right| = 4\pi^2 \int_{C-\sqrt{C^2-1}}^{C+\sqrt{C^2-1}} \int_0^{2aC-1-a^2} \frac{C_4^{(\beta,\gamma)} b}{a^{1-1/\gamma}} \frac{C_3^{(\beta,\gamma)}}{a^{1/\gamma}} \frac{C_3^{(\beta,\gamma)} C_4^{(\beta,\gamma)}}{\gamma}\, \frac{da\, db}{a^2} \\ &= \frac{(\gamma+1)^2\Gamma^2(r-\frac{1}{\gamma}+2)\Gamma^2(r+\frac{1}{\gamma}+1)}{2^5\Gamma^4(r+1)\gamma} \left[2C\sqrt{C^2-1} + \log\left(\frac{C-\sqrt{C^2-1}}{C+\sqrt{C^2-1}}\right)\right]. \end{aligned}$$

We obtain the form starting from eqn. (24) in the appendix. This defines $A(\beta,\gamma) = \frac{(\gamma+1)^2\Gamma^2(r-\frac{1}{\gamma}+2)\Gamma^2(r+\frac{1}{\gamma}+1)}{2^5\Gamma^4(r+1)\gamma}$ and $\mathrm{Area}(C) = \left[2C\sqrt{C^2-1} + \log\left(\frac{C-\sqrt{C^2-1}}{C+\sqrt{C^2-1}}\right)\right]$. Note that $A(\beta,\gamma)$ is only a function of (β,γ) via r and so does not depend on C, whilst $\mathrm{Area}(C)$ is only a function of C and does not change with (β,γ). Also $\frac{d}{dC}\mathrm{Area}(C) > 0$ and so there is a 1-1 map between C and the area.

Metikas and Olhede [3][eqn. 22] defined a genuinely 2-D localisation operator, but were only able to determine its approximate eigenfunctions. To be able to exactly derive the eigenfunctions, they defined an operator only valid for radial functions, in [3][eqn. 12]. Similarly, we intend to define an operator only valid for geometrically anisotropic functions. We assume that $g(\mathbf{x})$ is a geometrically anisotropic function, and write it as $g_A(\mathbf{x})$. Then in analogue with [3] we define the geometrically anisotropic coherent state in terms of the simpler indexing of $\boldsymbol{\xi}_I = (a,b)^T$ by:

$$V_{\mathbf{H},\boldsymbol{\xi}_I}^{(1,\beta,\gamma)}(\boldsymbol{\omega}) = \frac{a^{1/\gamma}\left(a^{1/\gamma}\nu\right)^{-1/2+\gamma/2} V^{(1-D,\beta,\gamma)}(a^{1/\gamma}\nu)\cos\left(\nu^\gamma b - \pi/4\right)}{\sqrt{2^{-1}\pi\nu^\gamma b\left|\mathbf{P}_{\mathbf{H}}\right|}}, \tag{13}$$

$$v_{\mathbf{H},\boldsymbol{\xi}_I}^{(1,\beta,\gamma)}(\mathbf{x}) = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} V_{\mathbf{H},\boldsymbol{\xi}_I}^{(1,\beta,\gamma)}(\boldsymbol{\omega})\, e^{i\boldsymbol{\omega}^T\mathbf{x}}\, d^2\boldsymbol{\omega}. \tag{14}$$

As we may consider $v_{\mathbf{H},\boldsymbol{\xi}_H}^{(1,\beta,\gamma)}(\mathbf{x})$ as the coherent state $v_{\mathbf{H},\boldsymbol{\xi}}^{(\beta,\gamma)}(\mathbf{x})$ that has been averaged across directions in space over b constant (see [3]), we may consider $v_{\mathbf{H},\boldsymbol{\xi}_H}^{(1,\beta,\gamma)}(\mathbf{x})$ local to the set of spatial positions

$$\mathbf{x} = \mathbf{Q}_{\mathbf{H}}\mathbf{y}: \quad y^2 = \left|\langle\mathbf{y}\rangle_{\mathbf{H}}(\boldsymbol{\xi})\right|^2, \quad \boldsymbol{\omega} = \mathbf{Q}_{\mathbf{H}}\boldsymbol{\nu}: \quad \nu = \langle\nu\rangle_{\mathbf{H}}(\boldsymbol{\xi}), \tag{15}$$

noting that $|\langle\mathbf{y}\rangle_{\mathbf{H}}(\boldsymbol{\xi})|^2$ only depends on a and b (see eqn. (8)), whilst $\langle\nu\rangle_{\mathbf{H}}(\boldsymbol{\xi})$ only depends on a (see eqn. (7)). By only using $v_{\mathbf{H},\boldsymbol{\xi}_I}^{(1,\beta,\gamma)}(\mathbf{x})$ for $\boldsymbol{\xi}_H \in \mathcal{A}_r(C)$ with $\mathcal{A}_r(C) = \left\{(a,\theta,\mathbf{b}):\ a^2+b^2+1 \le 2aC\right\}$ we construct a function local in space and spatial frequency to

$$\mathcal{D}_{r,\mathbf{H}}(C) = \left\{(\mathbf{x},\boldsymbol{\omega}):\ \left|\mathbf{P}\mathbf{x}\right| = \left|\langle\mathbf{y}\rangle_{\mathbf{H}}(\boldsymbol{\xi})\right|,\ \left|\mathbf{Q}_{\mathbf{H}}^T\boldsymbol{\omega}\right| = \langle\nu\rangle_{\mathbf{H}}(\boldsymbol{\xi}),\quad b_1^2+b_2^2 = b^2,\quad (a,b)\in\mathcal{A}_r(C)\right\}.$$

In analogue with Metikas and Olhede [3] we then may define the localisation operator with $\langle g_1, g_2\rangle_{\mathbf{H}} = (2\pi)^{-1}\int G_1^*(\omega)G_2(\omega)\ \nu\ \left|\mathbf{P}_{\mathbf{H}}\right|\ d\nu$, by:

$$\mathcal{P}_{\mathcal{D}_{r,\mathbf{H}}(C)}^{(\beta,\gamma)}\left\{G_A\right\}(\boldsymbol{\omega}) = C_{\mathbf{H},A}\int\int_{\mathcal{A}_r(C)} V_{\mathbf{H},\boldsymbol{\xi}_H}^{(1,\beta,\gamma)}(\boldsymbol{\omega})\langle v_{\mathbf{H},\boldsymbol{\xi}_H}^{(1,\beta,\gamma)}, g_A\rangle_{\mathbf{H}}\frac{da}{a^3}b\ db. \tag{16}$$

Clearly by appropriate change of variables, we can note directly from Metikas and Olhede [3][eqn. 14] that a resolution of identity may be achieved as long as $C_{\mathbf{H},A} = (r-1)/2$.

IV. Determining the Eigensystem

We now intend to demonstrate that the eigenfunctions of the proposed operators can in the instance of eqn. (16) be determined exactly, or in the instance of eqn. (9) be determined approximately.

Firstly we note that by eqn. (9) that determining the eigenfunctions of the operator defined from the Morse coherent states for an arbitrary function corresponds to for a given $\mathbf{P_H}$, (β,γ) and C solving the set of eqns. $\mathcal{P}_{\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)}\Psi(\boldsymbol{\omega}_1)=\lambda\Psi(\boldsymbol{\omega}_1)$, or:

$$\begin{aligned}\mathcal{P}^{(\beta,\gamma)}_{\mathcal{D}_{\mathbf{H}}(C)}\Psi(\boldsymbol{\omega}_1) &= \frac{C_{\mathbf{H}}}{(2\pi)^2}\int_{\mathcal{A}(C)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a^{2/\gamma}V_r^{(\beta,\gamma)}(a^{1/\gamma}\mathbf{r}_{-\theta}\boldsymbol{\nu}_1)e^{-\boldsymbol{\jmath}\mathbf{b}^T\boldsymbol{\nu}_1\nu_1^{\gamma-1}} \\ &\quad V_r^{(\beta,\gamma)*}(a^{1/\gamma}\mathbf{r}_{-\theta}\boldsymbol{\nu}_2)e^{\boldsymbol{\jmath}\mathbf{b}^T\boldsymbol{\nu}_2\nu_2^{\gamma-1}}\Psi(\boldsymbol{\omega}_2)\,\frac{d^2\boldsymbol{\omega}_2}{|\mathbf{P_H}|}\,dA_{\boldsymbol{\xi}} \\ &= \frac{C_{\mathbf{H}}}{(2\pi)^2}\int_{\mathcal{A}(C)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a^{2/\gamma}V_r^{(\beta,\gamma)}(a^{1/\gamma}\mathbf{r}_{-\theta}\mathbf{Q}_{\mathbf{H}}^T\boldsymbol{\omega}_1)e^{-\boldsymbol{\jmath}\mathbf{b}^T\mathbf{Q}_{\mathbf{H}}^T\boldsymbol{\omega}_1|\mathbf{Q}_{\mathbf{H}}^T\boldsymbol{\omega}_1|^{\gamma-1}} \\ &\quad V_r^{(\beta,\gamma)*}(a^{1/\gamma}\mathbf{r}_{-\theta}\boldsymbol{\nu}_2)e^{\boldsymbol{\jmath}\mathbf{b}^T\boldsymbol{\nu}_2\nu_2^{\gamma-1}}\Psi(\mathbf{P}_{\mathbf{H}}^T\boldsymbol{\nu}_2)\,d^2\boldsymbol{\nu}_2\,dA_{\boldsymbol{\xi}}.\end{aligned} \tag{17}$$

Of course we note that $\boldsymbol{\omega}_1=\mathbf{P}_{\mathbf{H}}^T\boldsymbol{\nu}_1$, and so we find that:

$$\begin{aligned}\mathcal{P}^{(\beta,\gamma)}_{\mathcal{D}_{\mathbf{H}}(C)}\Psi(\mathbf{P}_{\mathbf{H}}^T\boldsymbol{\nu}_1) &= \frac{C_{\mathbf{H}}}{(2\pi)^2}\int_{\mathcal{A}(C)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a^{2/\gamma}V_r^{(\beta,\gamma)}(a^{1/\gamma}\mathbf{r}_{-\theta}\boldsymbol{\nu}_1)e^{-\boldsymbol{\jmath}\mathbf{b}^T\boldsymbol{\nu}_1|\boldsymbol{\nu}_1|^{\gamma-1}} \\ &\quad V_r^{(\beta,\gamma)*}(a^{1/\gamma}\mathbf{r}_{-\theta}\boldsymbol{\nu}_2)e^{\boldsymbol{\jmath}\mathbf{b}^T\boldsymbol{\nu}_2\nu_2^{\gamma-1}}\Psi(\mathbf{P}_{\mathbf{H}}^T\boldsymbol{\nu}_2)\,d^2\boldsymbol{\nu}_2\,dA_{\boldsymbol{\xi}}.\end{aligned} \tag{18}$$

Let $l=\beta-\frac{1}{2}$ and $m=\gamma$. Comparing eqn. (18) with [3][eqns. 7, 21], and denoting the eigenfunctions of the operator defined by [3][eqns. 7, 21] as $\Psi^{(e)}_{n;l,m}(\boldsymbol{\omega})$, we obtain that the eigenfunctions of equation (18), denoted by $\psi^{(A)}_{n;l,m,\mathbf{H}}(\boldsymbol{\omega})$, are given by the same functional form, but where the argument has been adjusted to the geometric anisotropy. $\psi^{(A)}_{n;l,m,\mathbf{H}}(\boldsymbol{\omega})$ have the same eigenvalues as the isotropic Morse wavelets, and we denote them $\{\lambda_{n;r,\mathbf{H}}(C)\}$. We may then note that the geometrically anisotropic eigensystem is given by:

$$\Psi^{(A)}_{n;l,m,\mathbf{H}}(\boldsymbol{\omega})=\Psi^{(e)}_{n;l,m}(\mathbf{Q}_{\mathbf{H}}^T\boldsymbol{\omega}),\quad \lambda_{n;r,\mathbf{H}}(C)=\lambda_{n,r}(C). \tag{19}$$

Thus, whenever a set of eigenfunctions are determined for the operator of eqns. (7) and (21) in [3], these automatically correspond to eigenfunctions of the geometrically anisotropic localisation operator, once the argument has been adjusted. Determining the exact eigenfunctions of eqn. (18) is in general not an analytically tractable problem. In analogue with Metikas and Olhede [3] we instead determine the eigenfunctions of the geometrically anisotropic localisation operator:

$$\mathcal{P}^{(\beta,\gamma)}_{\mathcal{D}_{r,\mathbf{H}}(C)}\left\{\psi_A\right\}(\mathbf{x}) = \lambda\psi_A(\mathbf{x}) \tag{20}$$

$$\begin{aligned}\mathcal{P}^{(\beta,\gamma)}_{\mathcal{D}_{r,\mathbf{H}}(C)}\left\{\Psi_A\right\}(\boldsymbol{\omega}_1) &= C_{\mathbf{H},A}\int\int_{\mathcal{A}_r(C)} V^{(1,\beta,\gamma)}_{\mathbf{H},\boldsymbol{\xi}_I}(\boldsymbol{\omega})\langle v^{(1,\beta,\gamma)}_{\mathbf{H},\boldsymbol{\xi}_I},g_A\rangle_{\mathbf{H}}\frac{da}{a^3}b\,db \\ &= C_{\mathbf{H},A}\int_0^{\infty}\int\int_{\mathcal{A}_r(C)} a^{1/\gamma}(\nu_1\nu_2)^{-1/2+\gamma/2}V^{(1-D,\beta,\gamma)}(a^{1/\gamma}\nu_1)\frac{\cos\left(\nu_1^{\gamma}b-\pi/4\right)}{\sqrt{2^{-1}\pi\nu_1^{\gamma}b}} \\ &\quad \Psi_A\left(\mathbf{P}_{\mathbf{H}}^T\boldsymbol{\nu}_2\right)V^{(1-D,\beta,\gamma)}(a^{1/\gamma}\nu_2)\frac{\cos\left(\nu_2^{\gamma}b-\pi/4\right)}{\sqrt{2^{-1}\pi\nu_2^{\gamma}b}}\,\nu_2\,d\nu_2\frac{da\;b\;db}{a^2}.\end{aligned} \tag{21}$$

Clearly comparing this with eqn. (13) of [3], we determine that the eigensystem is given by: $\Psi^{(A)}_{n;l,m,\mathbf{H}}(\boldsymbol{\omega})=\Psi^{(e)}_{n;l,m}(\mathbf{Q}_{\mathbf{H}}^T\boldsymbol{\omega})$ and $\lambda_{n;r,\mathbf{H}}(C)=\lambda_{n,r}(C)$, where we note from Metikas and Olhede [3] that the eigenfunctions and eigenvalues take the form:

$$\Psi^{(A)}_{n;l,m}(\boldsymbol{\omega})=\sqrt{2}A_{n;l+1/2,m}\nu^l e^{-\nu^m}L_n^c(2\nu^m),\quad \lambda_{n,r}(C)=\frac{\Gamma(r+n)}{\Gamma(N+1)\Gamma(r-1)}\int_0^{\frac{C-1}{C+1}}x^n\left(1-x\right)^{r-2}\,dx,$$

where $c=(2l+2)/m-1$ and $L_n^c(\cdot)$ denotes the generalized Laguerre polynomial [10][p. 783].

As an example of geometrical anisotropy for $0<\varepsilon\ll 1$ we take: $\mathbf{P_H}(\varepsilon)=\left[1\quad 0;\ 0\quad \varepsilon\right]$, $\mathbf{Q_H}(\varepsilon)=\left[1\quad 0;\ 0\quad \varepsilon^{-1}\right]$, $\mathbf{H}(\varepsilon)=\left[1\quad 0;\ 0\quad \varepsilon^2\right]$. We note that $\omega=\sqrt{\omega_1^2+\varepsilon^{-2}\omega_2^2}$. We obtain increasing anisotropy as ε approaches zero.

In the isotropic limit we obtain the results of Metikas and Olhede [3] and with $\mathbf{P_H} = \mathbf{I}$, the isotropic Morse wavelets are retrieved [3]. Compare the range of possible localisation regions that may be found in this class: see Figure 1. As we do not change the values of β and γ these functions have the same localisation value for a fixed value of C. As we are for the highly anisotropic case extending the extent of the function in one direction, trivially, to conserve the area size, it is compressed in the corresponding canonical variable, as is apparent from the frequency domain plot. As the spatial extent extends in x_2, the function squashes towards $f_2 = 0$. The proposed framework allows the quantification of the stretching explicitly, and the discussion of both $\varepsilon \rightarrow 0$ and $\varepsilon \rightarrow \infty$.

V. Conclusions

This correspondence has defined geometric anisotropic localisation, and derived the optimally local functions for geometrically anisotropic and nonseparable regions of space and spatial frequency. A full set of eigenfunctions with associated eigenvalues have been determined for this class, useful for the study of non-stationary fields in geostatistics.

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Appendix

Renormalising the functions to unit energy using eqn. (6) we have:

$$\begin{aligned}
\langle 1\rangle_{\mathbf{H}}\left(\boldsymbol{\xi}\right) &= \langle v^{(\beta,\gamma)}_{\mathbf{H};\boldsymbol{\xi}}, v^{(\beta,\gamma)}_{\mathbf{H};\boldsymbol{\xi}}\rangle = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V^{(\beta,\gamma)*}_{\mathbf{H},\boldsymbol{\xi}}\left(\boldsymbol{\omega}\right) V^{(\beta,\gamma)}_{\mathbf{H},\boldsymbol{\xi}}\left(\boldsymbol{\omega}\right)\, d^2\boldsymbol{\omega} \qquad (22)\\
&= (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left|a^{1/\gamma}V^{(\beta,\gamma)}_r(a^{1/\gamma}\mathbf{r}_{-\theta}\boldsymbol{\nu})\right|^2\, d^2\boldsymbol{\omega}}{|\mathbf{P_H}|} = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega^{-1+\gamma}\left|V^{(\beta,\gamma)2}_{1D}(\omega)\right|^2\, d^2\boldsymbol{\omega}.
\end{aligned}$$

$$\begin{aligned}
\langle\nu\rangle_{\mathbf{H}}(\boldsymbol{\xi}) &= \frac{1}{\langle 1\rangle_{\mathbf{H}}(\boldsymbol{\xi})}\frac{1}{(2\pi)^2 a^{1/\gamma}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nu\left|V^{(\beta,\gamma)}_{\mathbf{H}}(\boldsymbol{\nu})\right|^2\, d^2\boldsymbol{\nu} = \frac{2}{r}\frac{2^{r+1}\pi\gamma}{\Gamma(r)}\frac{1}{2\pi a^{1/\gamma}}\int_0^{\infty}\omega^{\gamma+1}\omega^{2\beta}e^{-2\omega^{\gamma}}\, d\omega\\
&= \frac{1}{\Gamma(r+1)}\frac{2^{r+1}}{a^{1/\gamma}}\int_0^{\infty} s^{(2\beta+2)/\gamma}e^{-2s}\, ds = \frac{\Gamma\left(r+\frac{1}{\gamma}+1\right)}{2^{r+\frac{1}{\gamma}+1}\Gamma(r+1)}\frac{2^{r+1}}{a^{1/\gamma}} = \frac{C_3^{(\beta,\gamma)}}{a^{1/\gamma}}. \qquad (23)
\end{aligned}$$

$$\begin{aligned}
\langle y_l\rangle_{\mathbf{H}}(\boldsymbol{\xi}) &= \frac{1}{\langle 1\rangle_{\mathbf{H}}(\boldsymbol{\xi})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_l\left|v^{(\beta,\gamma)}_{\mathbf{H};\boldsymbol{\xi}}(\mathbf{x})\right|^2\, d^2\mathbf{x} = \boldsymbol{j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\nu_l}V^{(\beta,\gamma)}_{\mathbf{H};\boldsymbol{\xi}}(\boldsymbol{\omega})\right]\frac{V^{(\beta,\gamma)*}_{\mathbf{H};\boldsymbol{\xi}}(\boldsymbol{\omega})}{(2\pi)^2\langle 1\rangle_{\mathbf{H}}(\boldsymbol{\xi})}\, d^2\boldsymbol{\omega}\\
&= \frac{2^r\gamma a^{r+1}}{\Gamma(r+1)\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[b_l\omega^{\gamma-1} + (\gamma-1)b_l\omega^{\gamma-2}\frac{\omega_l^2}{\omega}\right]\omega^{2\beta+(\gamma-1)}e^{-2a\omega^{\gamma}}\, d^2\boldsymbol{\omega}\\
&= \frac{2^r a^{r+1}b_l(\gamma+1)}{\Gamma(r+1)}\int_0^{\infty} u^{2\beta/\gamma+2-1/\gamma}e^{-2au}\, du u^{1/\gamma-1} = \frac{2^{1/\gamma-2}(\gamma+1)\Gamma(r-1/\gamma+2)a^{-1+1/\gamma}b_l}{\Gamma(r+1)}.
\end{aligned}$$

We define $C_4^{(\beta,\gamma)}$ by $C_4^{(\beta,\gamma)} = \langle y_l \rangle_{\mathbf{H}}(\boldsymbol{\xi})/(b_l a^{1/\gamma - 1})$ Furthermore the localisation region has size:

$$\begin{aligned}
\left|\mathcal{D}_{\mathbf{H}}^{(\beta,\gamma)}(C)\right| &= 4\pi^2 \int_{C-\sqrt{C^2-1}}^{C+\sqrt{C^2-1}} \int_0^{2aC-1-a^2} \frac{C_4^{(\beta,\gamma)} b}{a^{1-1/\gamma}} \frac{C_3^{(\beta,\gamma)}}{a^{1/\gamma}} \frac{C_3^{(\beta,\gamma)} C_4^{(\beta,\gamma)}}{\gamma} \; \frac{da \; db}{a^2} \\
&= \frac{2\pi^2 \left(C_3^{(\beta,\gamma)} C_4^{(\beta,\gamma)}\right)^2}{\gamma} \int_{C-\sqrt{C^2-1}}^{C+\sqrt{C^2-1}} \left(2aC - 1 - a^2\right) \frac{da}{a^3} \\
&= \frac{(\gamma+1)^2 \Gamma^2(r - \frac{1}{\gamma} + 2)\Gamma^2(r + \frac{1}{\gamma} + 1)}{2^5 \Gamma^4(r+1)\gamma} \left[2C\sqrt{C^2-1} + \log\left(\frac{C - \sqrt{C^2-1}}{C + \sqrt{C^2-1}}\right)\right] \\
&= A(\beta,\gamma)\mathrm{Area}(C). \qquad (24)
\end{aligned}$$