Bahadur representation of sample quantiles for functional of Gaussian dependent sequences under a minimal assumption

By Jean-François Coeurjolly[1]
SAGAG, Department of Statistics, Grenoble, FRANCE

September 30, 2008

We obtain a Bahadur representation for sample quantiles of nonlinear functional of Gaussian sequences with correlation function decreasing as $k^{-\alpha}$ for some $\alpha > 0$. This representation is derived under a mimimal assumption.

1 Introduction

We consider the problem of obtaining a Bahadur representation of sample quantiles in a certain dependence context. Before stating in what a Bahadur representation consists, let us specify some general notation. Given some random variable Y, $F(\cdot) = F_Y(\cdot)$ is referred as the cumulative distribution function of Y, $\xi(p) = \xi_Y(p)$ for some $0 < p < 1$ as the quantile of order p. If $F(\cdot)$ is absolutely continuous with respect to Lebesgue measure, the probability density function is denoted by $f(\cdot) = f_Y(\cdot)$. Based on the observation of a vector $\boldsymbol{Y} = (Y(1), \ldots, Y(n))$ of n random variables distributed as Y, the sample cumulative distribution function and the sample quantile of order p are respectively denoted by $\widehat{F}_Y(\cdot; \boldsymbol{Y})$ and $\widehat{\xi}_Y(p; \boldsymbol{Y})$ or simply by $\widehat{F}(\cdot; \boldsymbol{Y})$ and $\widehat{\xi}(p; \boldsymbol{Y})$.

Let $\boldsymbol{Y} = (Y(1), \ldots, Y(n))$ a vector of n i.i.d. random variables such that $F''(\xi(p))$ exists and is bounded in a neighborhood of $\xi(p)$ and such that $F'(\xi(p)) > 0$, Bahadur proved that as $n \to +\infty$,

$$\widehat{\xi}(p) - \xi(p) = \frac{p - \widehat{F}(p)}{f(\xi(p))} + r_n,$$

[1] *Key words and phrases.* Gaussian processes, Bahadur representation of sample quantiles, Hermite expansions.

with $r_n = \mathcal{O}_{a.s.}\left(n^{-3/4}\log(n)^{3/4}\right)$ where a sequence of random variables U_n is said to be $\mathcal{O}_{a.s.}(v_n)$ if U_n/v_n is almost surely bounded. Kiefer obtained the exact rate $n^{-3/4}\log\log(n)^{3/4}$. Under an Assumption on $F(\cdot)$ which is quite similar to the one done by Bahadur, extensions of above results to dependent random variables have been pursued in Sen and Ghosh (1972) for $\phi-$mixing variables, in Yoshihara (1995) for strongly mixing variables, and recently in Wu (2005) for short-range and long-range dependent linear processes, following works of Hesse (1990) and Ho and Hsing (1996). Finally, such a representation has been obtained by Coeurjolly (2007) for nonlinear functional of Gaussian sequences with correlation function decreasing as $k^{-\alpha}$ for some $\alpha > 0$.

Ghosh (1971) proposed in the i.i.d. case a much simpler proof of Bahadur's result which suffices for many statistical applications. He established under a weaker assumption on $F(\cdot)$ ($F'(\cdot)$ exists and is bounded in a neighborhood of $\xi(p)$ and $f(\xi(p)) > 0$) that the remainder term satisfies $r_n = o_{\mathbb{P}}(n^{-1/2})$, which means that $n^{1/2}r_n$ tends to 0 in probability. This result is sufficient for example to establish a central limit theorem for the sample quantile. Our goal is to extend Ghosh's result to nonlinear functional of Gaussian sequences with correlation function decreasing as $k^{-\alpha}$. The Bahadur representation is presented in Section 2 and is applied to a central limit theorem for the sample quantile. Proofs are deferred in Section 3.

2 Main result

Let $\{Y(i)\}_{i=1}^{+\infty}$ be a stationary (centered) gaussian process with variance 1, and correlation function $\rho(\cdot)$ such that, as $i \to +\infty$

$$|\rho(i)| \sim i^{-\alpha} \tag{1}$$

for some $\alpha > 0$.

Let us recall some background on Hermite polynomials: the Hermite polynomials form an orthogonal system for the Gaussian measure and are in particular such that $\mathbf{E}\left(H_j(Y)H_k(Y)\right) = j!\ \delta_{j,k}$, where Y is referred to a standard Gaussian variable. For some measurable function $g(\cdot)$ defined on $\mathbb{R}$ such that $\mathbf{E}(g(Y)^2) < +\infty$, the following expansion holds

$$g(t) = \sum_{j\geq\tau} \frac{c_j}{j!} H_j(t) \quad \text{with} \quad c_j = \mathbf{E}\left(g(Y)H_j(Y)\right),$$

where the integer τ defined by $\tau = \inf\{j \geq 0, c_j \neq 0\}$, is called the Hermite rank of the function g. Note that this integer plays an important role. For example, it is related to the correlation of $g(Y_1)$ and $g(Y_2)$, for Y_1 and Y_2 two standard gaussian variables with correlation ρ, since $\mathbf{E}(g(Y_1)g(Y_2) = \sum_{k\geq\tau} \frac{(c_k)^2}{k!}\rho^k = \mathcal{O}(\rho^\tau)$.

Our result is based on the assumption that $F'_{g(Y)}(\cdot)$ exists and is bounded in a neighborhood of $\xi(p)$. This is achieved if the function $g(\cdot)$ satisfies the following assumption (see *e.g.* Dacunha-Castelle and Duflo (1982), p.33).

Assumption $\boldsymbol{A(\xi(p))}$ **:** there exist U_i, $i = 1, \ldots, L$, disjoint open sets such that U_i contains a unique solution to the equation $g(t) = \xi_{g(Y)}(p)$, such that $F'_{g(Y)}(\xi(p)) > 0$ and such that g is a $\mathcal{C}^1-$diffeomorphism on $\cup_{i=1}^L U_i$.
Note that this assumption allows us to obtain

$$F'_{g(Y)}(\xi_{g(Y)}(p)) = f_{g(Y)}(\xi_{g(Y)}(p)) \; = \; \sum_{i=1}^{L} \frac{\phi(g_i^{-1}(t))}{g'(g_i^{-1}(t))},$$

where $g_i(\cdot)$ is the restriction of $g(\cdot)$ on U_i and where $\phi(\cdot)$ is referred to the probability density function of a standard Gaussian variable.

Now, define, for some real u, the function $h_u(\cdot)$ by:

$$h_u(t) = \mathbf{1}_{\{g(t)\leq u\}}(t) - F_{g(Y)}(u). \tag{2}$$

We denote by $\tau(u)$ the Hermite rank of $h_u(\cdot)$. For the sake of simplicity, we set $\tau_p = \tau(\xi_{g(Y)}(p))$. For some function $g(\cdot)$ satisfying Assumption $\boldsymbol{A(\xi(p))}$, we denote by

$$\overline{\tau}_p = \inf_{\gamma\in\cup_{i=1}^L g(U_i)} \tau(\gamma), \tag{3}$$

that is the minimal Hermite rank of $h_u(\cdot)$ for u in a neighborhood of $\xi_{g(Y)}(p)$. Denote also by $c_j(u)$ the j-th Hermite coefficient of the function $h_u(\cdot)$.

Theorem 1 *Under Assumption* $\boldsymbol{A(\xi(p))}$*, the following result holds as* $n \to +\infty$

$$\widehat{\xi}\left(p; \boldsymbol{g(Y)}\right) - \xi_{g(Y)}(p) = \frac{p - \widehat{F}\left(\xi_{g(Y)}(p); \boldsymbol{g(Y)}\right)}{f_{g(Y)}(\xi_{g(Y)}(p))} \; + \; o_{\mathbb{P}}\left(r_n(\alpha, \overline{\tau}_p)\right), \tag{4}$$

where $\boldsymbol{g(Y)} = (g(Y(1), \ldots, g(Y(n)))$*, for* $i = 1, \ldots, n$ *and where the sequence* $(r_n(\alpha, \overline{\tau}_p))_{n\geq 1}$ *is defined by*

$$r_n(\alpha, \overline{\tau}_p) = \begin{cases} n^{-1/2} & \text{if } \alpha\overline{\tau}_p > 1, \\ n^{-1/2}\log(n)^{1/2} & \text{if } \alpha\overline{\tau}_p = 1, \\ n^{-\alpha\overline{\tau}_p/2} & \text{if } \alpha\overline{\tau}_p < 1. \end{cases} \tag{5}$$

Remark 1 *The sequence $r_n(\alpha,\overline{\tau}_p)$ is related to the behaviour short-range or long-range dependent behaviour of the sequence $h_u(Y(1)),\ldots,h_u(Y(n))$ for u in a neighborhood of $\xi(p)$. More precisely, it corresponds to the asymptotic behaviour of the sequence*

$$\left(\frac{1}{n}\sum_{|i|<n}\rho(i)^{\overline{\tau}_p}\right)^{1/2}.$$

Corollary 2 *Under Assumption $\boldsymbol{A(\xi(p))}$, then the following convergence in distribution hold as $n\to+\infty$*

(i) if $\alpha\overline{\tau}_p>1$

$$\sqrt{n}\left(\widehat{\xi}\,(p;\boldsymbol{g}(\boldsymbol{Y}))-\xi_{g(Y)}(p)\right)\xrightarrow{d}\mathcal{N}(0,\sigma_p^2), \tag{6}$$

where

$$\sigma_p^2=\frac{1}{f(p)^2}\sum_{i\in\mathbb{Z}}\sum_{j\geq\overline{\tau}_p}\frac{c_j(p)^2}{j!}\rho(i)^j \text{ with } f(p)=f_{g(Y)}(\xi_{g(Y)}(p)) \text{ and } c_j(p)=c_j(\xi_{g(Y)}(p)).$$

(ii) if $\alpha\overline{\tau}_p<1$

$$n^{\alpha\overline{\tau}_p/2}\left(\widehat{\xi}\,(p;\boldsymbol{g}(\boldsymbol{Y}))-\xi_{g(Y)}(p)\right)\xrightarrow{d}\frac{c_{\overline{\tau}_p}(p)}{\overline{\tau}_p!f(p)}Z_{\overline{\tau}_p}, \tag{7}$$

where

$$Z_{\overline{\tau}_p}=K(\overline{\tau}_p,\alpha)\int'_{\mathbb{R}^{\overline{\tau}_p}}\frac{\exp(i(\lambda_1+\cdots+\lambda_{\overline{\tau}_p}))-1}{i(\lambda_1+\cdots+\lambda_{\overline{\tau}_p})}\prod_{j=1}^{\overline{\tau}_p}|\lambda_j|^{(\alpha-1)/2}\widetilde{B}(d\lambda_j)$$

and

$$K(\overline{\tau}_p,\alpha)=\left(\frac{(1-\alpha\overline{\tau}_p/2)(1-\alpha\overline{\tau}_p)}{\overline{\tau}_p!\,(2\Gamma(\alpha)\sin(\pi(1-\alpha)/2))^{\overline{\tau}_p}}\right)^{1/2}.$$

The measure $\widetilde{B}$ is a Gaussian complex measure and the symbol $\int'$ means that the domain of integration excludes the hyperdiagonals $\{\lambda_i=\pm\lambda_j, i\neq j\}$.

The proof of this result is omitted since it is a direct application of Theorem 1 and general limit theorems adapted to nonlinear functional of Gaussian sequences, *e.g.* Breuer and Major (1983) and Dehling and Taqqu (1989).

3 Proofs

3.1 Auxiliary Lemma

Lemma 3 *For every $j \geq 1$ and for all positive sequence $(u_n)_{n\geq 1}$ such that $u_n \to 0$, as $n \to +\infty$, we have, under Assumption $\mathbf{A}(\xi(\mathbf{p}))$*

$$I = \int_{\mathbb{R}} H_j(t)\phi(t)\mathbf{1}_{\{|g(t)-\xi_{g(Y)}(p))|\leq u_n\}} dt \sim u_n\ \kappa_j, \tag{8}$$

where κ_j is defined, for every $j \geq 1$,by

$$\kappa_j = \begin{cases} -2\sum_{i=1}^{L} \frac{\phi'\left(g_i^{-1}(\xi(p)\right)}{g'\left(g_i^{-1}(\xi(p))\right)} & \text{if } j=1, \\ 2(-1)^j \sum_{i=1}^{L} \frac{\phi^{(j-2)}\left(g_i^{-1}(\xi(p)\right)}{g'\left(g_i^{-1}(\xi(p))\right)} & \text{if } j>1. \end{cases} \tag{9}$$

Proof. Under Assumption $\mathbf{A}(\xi(\mathbf{p}))$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$I = \sum_{i=1}^{L} I_i \quad \text{with} \quad I_i = \int_{U_i} H_j(t)\phi(t)\mathbf{1}_{\{\xi(p)-u_n\ \leq\ g(t)\ \leq\ \xi(p)+u_n\}} dt. \tag{10}$$

Assume without loss of generality that the restriction of $g(\cdot)$ on U_i (denoted by $g_i(\cdot)$) is an increasing function, we have

$$\begin{aligned} I_i &= \int_{U_i} H_j(t)\phi(t)\mathbf{1}_{\{\xi(p)-u_n\ \leq\ g(t)\ \leq\ \xi(p)+u_n\}} dt \\ &= \int_{g_i^{-1}(\xi(p)-u_n)}^{g_i^{-1}(\xi(p)+u_n)} H_j(t)\phi(t)dt \\ &= \begin{cases} \phi(m_{i,n}) - \phi(M_{i,n}) = (m_{i,n} - M_{i,n}) & \text{if } j=1 \\ (-1)^j \left(\phi^{(j-1)}(M_{i,n}) - \phi^{(j-1)}(m_{i,n})\right) & \text{if } j>1, \end{cases} \end{aligned}$$

where $M_{i,n} = g_i^{-1}(\xi(p)+u_n)$ and $m_{i,n} = g_i^{-1}(\xi(p)-u_n)$. Then, there exists $\omega_{n,i,j} \in [m_{i,n}, M_{i,n}]$ for every $j \geq 1$ such that

$$I_i = \begin{cases} (m_{i,n} - M_{i,n})\ \phi^{(1)}(\omega_{n,i,1}) & \text{if } j=1 \\ (-1)^j\ (M_{i,n} - m_{i,n})\,\phi^{(j-2)}(\omega_{n,i,j}) & \text{if } j>1. \end{cases},$$

Under Assumption $\mathbf{A}(\xi(\mathbf{p}))$, we have, as $n \to +\infty$

$$\omega_{n,i,j} \sim g_i^{-1}(\xi(p)) \quad \text{and} \quad M_{i,n} - m_{i,n} \sim 2u_n\ \frac{1}{g'(g_i^{-1}(\xi(p)))},$$

which ends the proof. ■

3.2 Proof of Theorem 1

For the sake of simplicity, we set $\widehat{\xi}(p) = \widehat{\xi}(p; \boldsymbol{g}(\boldsymbol{Y}))$, $\xi(p) = \xi_{g(Y)}(p)$, $\widehat{F}(\cdot) = \widehat{F}(\cdot; \boldsymbol{g}(\boldsymbol{Y}))$, $F(\cdot) = F_{g(Y)}(\cdot)$ et $f(\cdot) = f_{g(Y)}(\cdot)$ and $r_n = r_n(\alpha, \overline{\tau}_p)$. Define,

$$V_n = r_n^{-1}\left(\widehat{\xi}(p) - \xi(p)\right) \quad \text{and} \quad W_n = r_n^{-1}\left(\frac{p - F(p)}{f(p)}\right).$$

The result is established if $V_n - W_n \xrightarrow{\mathbb{P}} 0$ as $n \to +\infty$. It suffices to prove that V_n and W_n satisfy the conditions of Lemma 1 of Ghosh (1971):

- **condition (a)** : for all $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta)$ such that $\mathbb{P}(|W_n| > \varepsilon) < \delta$.
- **condition (b)** : for all $y \in \mathbb{R}$ and for all $\varepsilon > 0$

$$\lim_{n\to+\infty} \mathbb{P}(V_n \leq y, W_n \geq k + \varepsilon) \quad \text{and} \quad \lim_{n\to+\infty} \mathbb{P}(V_n \geq y + \varepsilon, W_n \geq k)$$

condition (a) : from Bienaymé-Tchebyshev's inequality it is sufficient to prove that $\mathbf{E}W_n^2 = \mathcal{O}(1)$. Rewrite $W_n = \frac{r_n^{-1}}{n}\sum_{i=\ell+1}^{n} h_{\xi(p)}(Y(i))$. Let c_j (for some $j \geq 0$) denote the j-th Hermite coefficient of $h_{\xi(p)}(\cdot)$. Since $h_{\xi(p)}(\cdot)$ has at least Hermite rank $\overline{\tau}_p$, then

$$\begin{aligned}
\mathbf{E}W_n^2 &= \frac{r_n^{-2}}{n^2}\sum_{i_1,i_2=1}^{n} \mathbf{E}\left(h_{\xi(p)}(Y(i_1))\, h_{\xi(p)}(Y(i_2))\right) \\
&= \frac{r_n^{-2}}{n^2}\sum_{i_1,i_2=1}^{n}\sum_{j_1,j_2\geq\tau_p} c_{j_1}c_{j_2}\mathbf{E}\left(H_{j_1}(Y(i_1))\, H_{j_2}(Y(i_2))\right) \\
&= \frac{r_n^{-2}}{n^2}\sum_{i_1,i_2=1}^{n}\sum_{j\geq\overline{\tau}_p} \frac{(c_j)^2}{(j)!}\,\rho(i_2-i_1)^j \\
&= \mathcal{O}\left(r_n^{-2}\times\frac{1}{n}\sum_{|i|<n}\rho(i)^{\overline{\tau}_p}\right) = \mathcal{O}(1),
\end{aligned}$$

from Remark 1.

condition (b) : let $y \in \mathbb{R}$, we have

$$\begin{aligned}
\{V_n \leq y\} &= \left\{\widehat{\xi}(p) \leq y \times r_n + \xi(p)\right\} \\
&= \left\{p \leq \widehat{F}(y \times r_n + \xi(p))\right\} = \{Z_n \leq y_n\},
\end{aligned} \tag{11}$$

with

$$Z_n = \frac{r_n^{-1}}{f(\xi(p))}\left(F\Big(y\times r_n + \xi(p)\Big) - \widehat{F}\left(\frac{y}{\sqrt{r_n}} + \xi(p)\right)\right)$$

and

$$y_n = \frac{r_n^{-1}}{f(\xi(p))}\Big(F\Big(y \times r_n + \xi(p)\Big) - p\Big)$$

Under Assumption $\mathbf{A}(\xi(\mathbf{p}))$, we have $y_n \to y$, as $n \to +\infty$. Now, prove that $Z_n - W_n \xrightarrow{\mathbb{P}} 0$. Without loss of generality, assume $y > 0$. Then, we have

$$\begin{aligned} W_n - Z_n &= \frac{r_n^{-1}}{f(p)}\Big(\widehat{F}\Big(y \times r_n + \xi(p)\Big) - F\Big(y \times r_n + \xi(p)\Big) - \widehat{F}\left(\xi(p)\right) + F(\xi(p))\Big) \\ &= \frac{r_n^{-1}}{n}\frac{1}{f(\xi(p))}\sum_{i=1}^{n} h_{\xi(p),n}\left(Y(i)\right) \end{aligned}$$

where $h_{\xi(p),n}(\cdot)$ is the function defined for $t \in \mathbb{R}$ by :

$$h_{\xi(p),n}(t) = \mathbf{1}_{\Big\{\xi(p) \le g(t) \le \xi(p) + y \times r_n\Big\}}(t) - \mathbb{P}\Big(\xi(p) \le g(Y) \le \xi(p) + y \times r_n\Big).$$

For n sufficiently large, the function $h_{\xi(p),n}(\cdot)$ has Hermite rank $\overline{\tau}_p$. Denote by $c_{j,n}$ the j-th Hermite coefficient of $h_{\xi(p),n}(\cdot)$. From Lemma 3, there exists a sequence $(\kappa_j)_{j \ge \overline{\tau}_p}$ such that, as $n \to +\infty$

$$c_{j,n} \sim \kappa_j \times r_n.$$

Since, for all $n \ge 1$ $\mathbf{E}(h_n(Y)^2) = \sum_{j \ge \overline{\tau}_p}(c_{j,n})^2/j! < +\infty$, it is clear that the sequence $(\kappa_j)_{j \ge \overline{\tau}_p}$ is such that $\sum_{j \ge \overline{\tau}_p}(\kappa_j)^2/j! < +\infty$. By denoting λ a positive constant, we get, as $n \to +\infty$

$$\begin{aligned} \mathbf{E}(W_n - Z_n)^2 &= \frac{r_n^{-2}}{n^2}\frac{1}{f(\xi(p))^2}\sum_{i_1,i_2=1}^{n}\mathbf{E}\left(h_{\xi(p),n}\left(Y(i_1)\right)h_{\xi(p),n}\left(Y(i_2)\right)\right) \\ &= \frac{r_n^{-2}}{n^2}\frac{1}{f(\xi(p))^2}\sum_{i_1,i_2=1}^{n}\sum_{j_1,j_2 \ge \overline{\tau}_p} c_{j_1,n}c_{j_2,n}\mathbf{E}\left(H_{j_1}\left(Y(i_1)\right)H_{j_2}\left(Y(i_2)\right)\right) \\ &= \frac{r_n^{-2}}{n^2}\frac{1}{f(\xi(p))^2}\sum_{i_1,i_2=1}^{n}\sum_{j \ge \overline{\tau}_p}\frac{c_{j,n}^2}{j!}\rho(i_2 - i_1)^j \\ &\le \lambda\,\frac{r_n^{-2}}{n}\sum_{j \ge \overline{\tau}_p}\frac{(\kappa_j)^2}{j!}\,r_n^2\sum_{|i|<n}\rho(i)^j = \mathcal{O}\left(\frac{1}{n}\sum_{|i|<n}\rho(i)^{\overline{\tau}_p}\right) = \mathcal{O}(r_n^2), \end{aligned}$$

from Remark 1. Therefore, $W_n - Z_n$ converges to 0 in probability, as $n \to +\infty$. Thus, for all $\varepsilon > 0$, we have, as $n \to +\infty$,

$$\mathbb{P}\left(V_n \le y, W_n \ge y + \varepsilon\right) = \mathbb{P}\left(Z_n \le y_n, W_n \ge y + \varepsilon\right) \to 0.$$

Following the sketch of this proof, we also have $\mathbb{P}(V_n \geq y+\varepsilon, W_n \leq y) \to 0$, ensuring condition (b). Therefore, $W_n - Z_n$ converges to 0 in probability, as $n \to +\infty$. Thus, for all $\varepsilon > 0$, we have, as $n \to +\infty$,

$$\mathbb{P}(V_n \leq y, W_n \geq y+\varepsilon) = \mathbb{P}(Z_n \leq y_n, W_n \geq y+\varepsilon) \to 0.$$

Following the sketch of this proof, we also have $\mathbb{P}(V_n \geq y+\varepsilon, W_n \leq y) \to 0$, ensuring condition (b).

Acknowledgement The author would like to thank the referee for his coments improving the statement of Corollary 2.

References

Bahadur R.R. (1966). A note on quantiles in large samples. *Ann. Math. Statist.* **37** 577-580.

Breuer P. and Major P. (1983). Central limit theorems for non-linear functionals of Gaussian fields. *J. Mult. Ann.* **13** 425-441.

Coeurjolly J.-F. (2007). Hurst exponent estimation of locally self-similar Gaussian processes using sample quantiles. *E-print arxiv:* `http://fr.arxiv.org/abs/math.ST/0506290`, 1-44.

Dacunha-Castelle D. and Duflo M. (1982) *Exercices de probabilités et statistiques. Tome 1 Problèmes à temps fixe.* Collection Mathémathiques Appliquées pour la maîtrise, Masson, Paris.

Dehling H. and Taqqu M.S. (1989) The empirical process of some long-range dependent sequences with an application to U-statistics. *Ann. Statist.* **17** 1767-1783.

Ghosh J.K. (1971). A new proof of the Bahadur Representation of Quantiles and an application. *Ann. Math. Statist.* **42** 1957-1961.

Hesse C.H. (1990). A Bahadur-type representation for empirical quantiles of a large class of stationary, possibly infinite-variance, linear processes. *Ann. Statist.* **18** 1188-1202.

Ho H.C. and Hsing T. (1996) On the asymptotic expansion of the empirical process of long-memory moving averages. *Ann. Statist.* **24** 992-1024.

Kiefer J. (1967). On Bahadur's representation of sample quantiles. *Ann. Math. Statist.* **38** 1323-1342.

Sen P.K. (1972). On the Bahadur representation of sample quantiles for sequences of ϕ-mixing random variables. *J. Multivariate Anal.* **2** 77-95.

Yoshihara K.-I. (1995). The Bahadur representation of sample quantiles for sequences of strongly mixing random variables. *Stat. Probab. Lett.* **24** 299-304.

Wu W.-B. (2005). On the Bahadur representation of sample quantiles for dependent sequences. *Ann. Statist.* **33** 1934-1963.

J.-F. Coeurjolly
SAGAG, Department of Statistics, Grenoble
1251 Av. Centrale BP 47
38040 GRENOBLE Cedex 09
France
E-mail: Jean-Francois.Coeurjolly@upmf-grenoble.fr