# EXPECTED NUMBER OF SLOPE CROSSINGS OF CERTAIN GAUSSIAN RANDOM POLYNOMIALS

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**Abstract**

Let $Q_n(x) = \sum_{i=0}^{n} A_i x^i$ be a random polynomial where the coefficients $A_0, A_1, \cdots$ form a sequence of centered Gaussian random variables. Moreover, assume that the increments $\Delta_j = A_j - A_{j-1}$, $j = 0, 1, 2, \cdots$ are independent, assuming $A_{-1} = 0$. The coefficients can be considered as $n$ consecutive observations of a Brownian motion. We study the number of times that such a random polynomial crosses a line which is not necessarily parallel to the $x$-axis. More precisely we obtain the asymptotic behavior of the expected number of real roots of the equation $Q_n(x) = Kx$, for the cases that $K$ is any non-zero real constant $K = o(n^{1/4})$, and $K = o(n^{1/2})$ separately.

Key words and Phrases: random algebraic polynomial, number of real zeros,slope crossing, expected density, Brownian motion.
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## 1 Preliminaries

The theory of the expected number of real zeros of random algebraic polynomials was addressed in the fundamental work of M. Kac[6] (1943). The works of Logan and Shepp [7, 8], Ibragimov and Maslova [5], Wilkins [14], Farahmand [3], and Sambandham [12, 13] are other fundamental contributions

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to the subject. For various aspects on random polynomials see Bharucha-Reid and Sambandham [1], and Farahmand[4].

There has been recent interest in cases where the coefficients form certain random processes, Rezakhah and Soltani [10, 11], Rezakhah and Shemehsavar [9].

Let $A_0, A_1, \cdots$ be a mean zero Gaussian random sequence for which the increments $\Delta_i = A_i - A_{i-1},\ i = 1, 2, \cdots$ are independent, assuming $A_{-1} = 0$. The sequence $A_0, A_1 \cdots$ may be considered as successive Brownian points, i.e., $A_j = W(t_j),\ j = 0, 1, \cdots,$ where $t_0 < t_1 < \cdots$ and $\{W(t),\ t \geq 0\}$ is the standard Brownian motion. In this physical interpretation, $\mathrm{Var}(\Delta_j)$ is the distance between successive times $t_{j-1},\ t_j$. Let

$$Q_n(x) = \sum_{i=0}^{n} A_i x^i,\ \ -\infty < x < \infty. \tag{1.1}$$

We study the number of times that $Q_n(x)$ crosses a line which is not necessarily parallel to the $x$-axis. Let $N_K(a, b)$ denote the number of real roots of the equation $Q_n(x) = Kx$, in the interval $(a, b)$, where $K$ is any real constant independent of $x$, and multiple roots are counted only once.

We note that $A_j = \Delta_0 + \Delta_1 + \cdots + \Delta_j,\quad j = 0, 1, \cdots,$ where $\Delta_i \sim N(0, \sigma_i^2)$ and $\Delta_i$ are independent, $i = 0, 1, \cdots$. Thus $Q_n(x) = \sum_{k=0}^{n} (\sum_{j=k}^{n} x^j) \Delta_k = \sum_{k=0}^{n} a_k(x)\Delta_k$, and $Q'_n(x) = \sum_{k=0}^{n} (\sum_{j=k}^{n} j x^{j-1}) \Delta_k = \sum_{k=0}^{n} b_k(x)\Delta_k$, where

$$a_k(x) = \sum_{j=k}^{n} x^j, \qquad b_k(x) = \sum_{j=k}^{n} j x^{j-1}, \qquad\quad k = 0, \cdots, n. \tag{1.2}$$

Using the function $\mathrm{erf}(t) = 2\Phi(t\sqrt{2}) - 1,$ and the fact that $t(\mathrm{erf}(t)) = -t(\mathrm{erf}(-t))$, the classical result of Cramer and Leadbetter [1967 p 285] can be applied to obtain ;

$$EN_K(a, b) = \int_a^b f_n(x)dx \tag{1.3}$$

where

$$f_n(x) = g_1(x)\exp(g_2(x)) + g_3(x)\exp(g_4(x))\,\mathrm{erf}(g_5(x)), \tag{1.4}$$

in which

$$g_1(x) = \frac{1}{\pi} E A^{-2},\ \ g_2(x) = -\frac{K^2(A^2 - 2Cx + B^2x^2)}{2E^2},$$

$$g_3(x) = \frac{1}{\sqrt{2\pi}} K(A^2 - Cx)A^{-3}\ \ g_4(x) = \frac{-K^2x^2}{2A^2},\ \ g_5(x) = \frac{K(A^2 - Cx)}{AE\sqrt{2}},$$

and

$$A^2 = \mathrm{Var}(Q_n(x)) \;=\; \sum_{k=0}^{n} a_k^2(x)\sigma_k^2, \quad B^2 = \mathrm{Var}(Q_n'(x)) = \sum_{k=0}^{n} b_k^2(x)\sigma_k^2,$$

$$C = \mathrm{Cov}(Q_n(x), Q_n'(x)) = \sum_{k=0}^{n} a_k(x) b_k(x)\sigma_k^2, \quad \text{and} \quad E^2 = A^2B^2 - C^2,$$

where $a_k(x)$, $b_k(x)$ is defined by (1.2).

## 2 Asymptotic behaviour of $EN_K$

In this section we obtain the asymptotic behaviour of the expected number of real zeros of the equation $Q_n(x) = Kx$, where $Q_n(x)$ is defined by (1.1). We prove the following theorem for the case that the increments $\Delta_1 \cdots \Delta_n$ have the same distributions, and $\sigma_k^2 = 1$, $k = 1 \cdots n$.

**Theorem 2.1.** Let $Q_n(x)$ be the random algebraic polynomial given by (1.1) for which $A_j = \Delta_1 + \cdots \Delta_j$, where $\Delta_i$, $i = 1, \cdots n$, are independent and $\Delta_j \sim N(0,1)$, for $j = 1, 2, \cdots, n$. The expected number of real roots of $Q_n(x) = Kx$ satisfies the following equations:

(i) - for $K = o(n^{1/4})$, and for large $n$

$$\begin{aligned} EN_K(-\infty,\infty) \;=\; & \frac{1}{\pi}(\log(2n+1)) + \frac{1}{\pi}(1.920134478) \\ & - \frac{1}{\pi\sqrt{2n}}\left(\pi - 2\arctan(\frac{1}{2\sqrt{2n}})\right) + \frac{K^2}{n\pi}(3.126508929) \\ & + \frac{1}{n\pi} C_1 + o(n^{-1}) \end{aligned} \qquad (2.1)$$

where for $n$ odd $C_1 = 1.715215531$, and for $n$ even $C_1 = -0.7200279388$.

(ii) - for $K = o(n^{1/2})$, and for large $n$

$$EN_K(-\infty,\infty) = \frac{1}{\pi}(\log(2n+1)) + \frac{1}{\pi}(1.920134478) + o(1) \qquad (2.2)$$

**Proof.** Due to the behaviour of $Q_n(x)$, the asymptotic behaviour is treated separately on the intervals $1 < x < \infty$, $-\infty < x < -1$, $0 < x < 1$ and $-1 < x < 0$. For $1 < x < \infty$, we use the change of variable $x = 1 + \frac{t}{n}$ and the equality $\left(1 + \frac{t}{n}\right)^n = e^t\left(1 - \frac{t^2}{n}\right) + O\left(\frac{1}{n^2}\right)$. Using (1.3), we find that

$$EN_K(1,\infty) = \frac{1}{n}\int_0^\infty f_n(1 + \frac{t}{n})dt,$$

where by (1.4) and by tedious manipulation we have that

$$g_2\left(1+\frac{t}{n}\right)=o(n^{-2}),\quad g_3\left(1+\frac{t}{n}\right)=o(1),$$

$$g_4\left(1+\frac{t}{n}\right)=o(n^{-2}),\quad g_5\left(1+\frac{t}{n}\right)=o(n^{-1}) \tag{2.3}$$

and

$$n^{-1}g_1\left(1+\frac{t}{n}\right)=\frac{1}{\pi}\left(R_1(t)+\frac{S_1(t)}{n}+O\left(\frac{1}{n^2}\right)\right),\quad n\to\infty, \tag{2.4}$$

where

$$R_1(t)=\frac{\sqrt{(4t-15)e^{4\,t}+(24t+32)e^{3\,t}-(8t^3+12t^2+36t+18)e^{2\,t}+8e^t t+1}}{2\,t\,(-1-3\,e^{2\,t}+4\,e^t+2\,te^{2\,t})}$$

and $S_1(t)=S_{11}(t)/S_{12}(t)$ in which

$$S_{11}(t)=-0.25\Big((4t^2-6t-27)e^{6t}+(156-84t+116t^2-24t^3)e^{5t}$$
$$+(16t^5-72t^4+96t^3-212t^2+220t-331)e^{4t}+(328-168t+128t^2-104t^3)e^{3t}$$
$$+(8t^4+8t^3-32t^2+42t-153)e^{2t}+(28-4t-4t^2)e^t-1\Big)$$

$$S_{12}(t)=\Big((2t-3)e^{2t}+4e^t-1\Big)^2\Big((4t-15)e^{4t}+(32+24t)e^{3t}$$
$$-(8t^3+12t^2+36t+18)e^{2t}+8te^t+1\Big)^{1/2}.$$

One can easily verify that as $t\to\infty$,

$$R_1(t)=\frac{1}{2t^{3/2}}+O(t^{-2}),\qquad S_1(t)=-\frac{1}{8t^{1/2}}+O(t^{-3/2}).$$

As (2.4) can not be integrated term by term, by noting that

$$\frac{I_{[t>1]}}{8n\sqrt{t}}=\frac{I_{[t>1]}}{8n\sqrt{t}+t\sqrt{t}}+O\left(\frac{1}{n^2}\right), \tag{2.5}$$

where

$$I_{[t>1]}=\begin{cases}1 & \text{if}\quad t\ge 1\\ 0 & \text{if}\quad t<1\end{cases},$$

Also by (2.3), $\operatorname{erf}(\,g_5(t)\,)=o(n^{-1})$. Therefore

$$n^{-1}f_n\left(1+\frac{t}{n}\right)=\frac{1}{\pi}\left(R_1(t)+\frac{S_1(t)}{n}\right)+O\left(\frac{1}{n^2}\right).$$

Thus by (2.5) we have that

$$n^{-1} f_n\Big(1+\frac{t}{n}\Big) + \frac{I_{[t>1]}}{\pi(8n\sqrt{t}+t\sqrt{t})} = \frac{R_1(t)}{\pi} + \frac{1}{\pi}\left(\frac{S_1(t)}{n} + \frac{I_{[t>1]}}{8n\sqrt{t}}\right) + O\left(\frac{1}{n^2}\right).$$

This expression is term by term integrable, and provides that

$$\begin{aligned} EN_K(1,\infty) &= \frac{1}{n}\int_0^\infty f_n\Big(1+\frac{t}{n}\Big)dt = \frac{1}{2\pi\sqrt{2n}}\left(-\pi + 2\arctan(\frac{1}{2\sqrt{2n}})\right) \\ &\quad + \frac{1}{\pi}\int_0^\infty R_1(t)dt + \frac{1}{\pi n}\int_0^\infty \left(S_1(t) + \frac{I_{[t>1]}}{8\sqrt{t}}\right)dt + O\left(\frac{1}{n^2}\right), \end{aligned}$$

where $\int_0^\infty R_1(t)dt \simeq .734874192$ , and $\int_0^\infty \left(S_1(t) + \frac{I_{[t>1]}}{8\sqrt{t}}\right)dt = -.25460172372.$

For $-\infty < x < -1$, we use the change of variable $x = -1 - \frac{t}{n}$. So by (1.3) we have that $EN_K(-\infty,-1) = \frac{1}{n}\int_0^\infty f_n\left(-1-\frac{t}{n}\right)dt$. Using (1.4) we find

$$n^{-1} g_1\left(-1-\frac{t}{n}\right) = \frac{1}{\pi}\left(R_2(t) + \frac{S_2(t)}{n} + O\left(\frac{1}{n^2}\right)\right) \tag{2.6}$$

where

$$R_2(t) = 1/2\sqrt{\frac{(1+4t)e^{4\,t} - (2+4t+12t^2+8t^3)e^{2\,t}+1}{t^2\left((2t+1)e^{2\,t}-1\right)^2}}.$$

Also we find that for $n$ even $S_2(t) = (S_{21}(t)+S_{22}(t))/(4S_{23}(t))$ , and for $n$ odd $S_2(t) = (S_{21}(t)-S_{22}(t))/(4S_{23}(t))$, in which

$$\begin{aligned} S_{21}(t) &= 1 + \left(-8\,t^4 + 30\,t - 8\,t^3 + 48\,t^2 - 3\right)e^{2\,t} \\ &\quad + \left(3 - 12t + 52t^2 + 96t^3 + 40t^4 - 16\,t^5\right)e^{4\,t} - \left(18t + 4t^2 + 1\right)e^{6\,t}, \\ S_{22}(t) &= 4\,e^t t + \left(8\,t + 32\,t^3 + 40\,t^2\right)e^{3\,t} + \left(-8\,t^2 - 12\,t\right)e^{5\,t} \\ S_{23}(t) &= \left(e^{4\,t}(4t+1) - 2e^{2t}(1+2t+6t^2+4t^3)+1\right)^{1/2}\left(e^{2\,t}(2t+1)-1\right)^2. \end{aligned}$$

Also

$$g_2\left(-1-\frac{t}{n}\right) = \left(\frac{K^2}{n}\right) g_{2,1}(t) + o(n^{-1}) \tag{2.7}$$

where

$$g_{2,1}(t) = \frac{(-32\,t^4 - 16\,t^3 + 16\,t^2 - 8\,t)\,e^{2\,t} + 8\,t}{(4t+1)e^{4\,t} - (8\,t^3 + 12\,t^2 + 4\,t + 2)\,e^{2\,t} + 1}.$$

Also we find that

$$n^{-1} g_3\left(-1-\frac{t}{n}\right) = \left(\frac{K}{\sqrt{n\pi}}\right) g_{3,1}(t) + o(n^{-1}) \tag{2.8}$$

where $g_{3,1}(t) = \frac{-\left(1+\left(4\,t^2+2\,t-1\right)e^{2\,t}\right)}{\sqrt{t}((2t+1)\,e^{2\,t}-1)^{3/2}}$. Also

$$g_4\left(-1-\frac{t}{n}\right) = o\left(n^{-1/2}\right). \tag{2.9}$$

Finally

$$g_5\left(-1-\frac{t}{n}\right) = \left(\frac{K}{\sqrt{n}}\right) g_{5,1}(t) + o(n^{-1}), \tag{2.10}$$

where

$$\begin{aligned} g_{5,1}(t) = -\ & 2\sqrt{t}\left(1+(4t^2+2t-1)e^{2\,t}\right)\left\{\left(6\,t+1+8\,t^2\right)e^{6\,t}\right. \\ & +\left(-12\,t-3-32\,t^3-20\,t^2-16\,t^4\right)e^{4\,t} \\ & \left.+\left(3+6\,t+12\,t^2+8\,t^3\right)e^{2\,t}-1\right\}^{-1/2}. \end{aligned}$$

It can be seen that as $n \to \infty$,

$$R_2(t) = \frac{1}{2t^{3/2}} + O(t^{-2}), \qquad S_2(t) = \frac{-1}{8t^{1/2}} + O(t^{-3/2}), \qquad g_{2,1}(t) = o(e^{-t}),$$

$$g_{3,1}(t) = O(e^{-t}), \qquad\qquad g_{5,1}(t) = O(t^{3/2}e^{-t}).$$

Now by using (1.4) we find that $f_n(-1-t/n)/n := I_{21} + I_{22}$ where

$$I_{21} = \frac{1}{\pi}\left\{R_2(t) + \frac{S_2(t)}{n} + \frac{K^2}{n}R_2(t)g_{2,1}(t)\right\} + o(n^{-1})$$

and

$$I_{22} = \frac{1}{\pi}\left\{\frac{2K^2}{n}g_{3,1}(t)g_{5,1}(t)\right\} + o(n^{-1})$$

then by using (2.5) we have that

$$\begin{aligned} EN_K(-\infty,-1) &= \frac{1}{n}\int_0^\infty f_n\Big(-1-\frac{t}{n}\Big)dt \\ &= \frac{1}{2\pi\sqrt{2n}}\left(-\pi + 2\arctan(\frac{1}{2\sqrt{2n}})\right) + \frac{1}{\pi}\int_0^\infty R_2(t)dt \\ &\quad + \frac{1}{\pi n}\int_0^\infty \left(S_2(t) + \frac{I_{[t>1]}}{8\sqrt{t}}\right)dt \\ &\quad + \frac{K^2}{n\pi}\int_0^\infty \Big(R_2(t)g_{2,1}(t) + 2g_{3,1}(t)g_{5,1}(t)\Big)dt + o(n^{-1}), \end{aligned}$$

where $\int_0^\infty \Big(S_2(t)+\frac{I_{[t>1]}}{8\sqrt{t}}\Big)dt = -0.0322863$, for $n$ odd, and $\int_0^\infty \Big(S_2(t)+\frac{I_{[t>1]}}{8\sqrt{t}}\Big)dt = -0.4677136958$ for $n$ even. Also $\int_0^\infty (R_2(t))\,dt = 1.09564006$, and

$$\int_0^\infty \Big(R_2(t)g_{2,1}(t) + 2g_{3,1}(t)g_{5,1}(t)\Big)dt = 1.593359902.$$

For $0 < x < 1$, let $x = 1 - \frac{t}{n+t}$, then $EN_K(0,1) = \left(\frac{n}{(n+t)^2}\right)\int_0^\infty f_n\left(1 - \frac{t}{n+t}\right)dt$, in which

$$g_2\left(1 - \frac{t}{n+t}\right) = o(n^{-2}), \qquad \left(\frac{n}{(n+t)^2}\right) g_3\left(1 - \frac{t}{n+t}\right) = o(n^{-1})$$

$$g_4\left(1 - \frac{t}{n+t}\right) = o(n^{-2}), \qquad g_5\left(1 - \frac{t}{n+t}\right) = o(n^{-1}) \tag{2.11}$$

and

$$\begin{aligned}\left(\frac{n}{(n+t)^2}\right) g_1\left(1 - \frac{t}{n+t}\right) &= \left(1 - \frac{2t}{n} + O\left(\frac{1}{n^2}\right)\right)\frac{1}{\pi}\left(R_3(t) + \frac{S_3(t)}{n} + O\left(\frac{1}{n^2}\right)\right)\\ &= \frac{1}{\pi}\left(R_3(t) + \frac{S_3(t) - 2tR_3(t)}{n}\right) + O\left(\frac{1}{n^2}\right),\end{aligned} \tag{2.12}$$

where we observe that $R_3(t) \equiv R_1(-t)$ and $S_3(t) = S_{31}(t)/S_{32}(t)$, in which

$$\begin{aligned}S_{31}(t) \;=\; &\Bigg(\left(-7t^2 - \frac{69}{2}t - \frac{61}{4}\right)e^{-6t} + \left(6t^3 + 35t - 55t^2 + 39\right)e^{-5t} +\\ &\left(49t - 4t^5 + 22t^4 + 91t^2 - \frac{63}{4} - 12t^3\right)e^{-4t} - \left(6t^3 + 30 + 44t^2 + 66t\right)e^{-3t}\\ &+\left(\frac{35}{2}t + 2t^4 - 6t^3 + 16t^2 + \frac{123}{4}\right)e^{-2t} + \left(-9 - t - t^2\right)e^{-t} + 3/4\Bigg),\end{aligned}$$

and $S_{32}(t) \equiv S_{12}(-t)$. We see that, as $t \to \infty$, $R_3(t) = (2t)^{-1} + O(t^{-1/2}e^{-t/2})$, and $S_3(t) = \frac{3}{4} + O(t^2 e^{-t})$. Now using the equality

$$\frac{I_{[t>1]}}{2t} - \frac{I_{[t>1]}}{4n + 2t} = \frac{I_{[t>1]}}{2t} - \frac{I_{[t>1]}}{4n} + O\left(\frac{1}{n^2}\right), \tag{2.13}$$

we have

$$\begin{aligned}\frac{n}{(n+t)^2} f_n\left(1 - \frac{t}{n+t}\right) \;=\; &\frac{1}{\pi}\left(R_3(t) - \frac{I_{[t>1]}}{2t}\right) + \frac{1}{\pi}\left(\frac{I_{[t>1]}}{2t} - \frac{I_{[t>1]}}{4n + 2t}\right)\\ &+\frac{1}{n\pi}\left(S_3(t) - 2tR_3(t) + \frac{I_{[t>1]}}{4}\right) + O\left(\frac{1}{n^2}\right),\end{aligned}$$

Thus

$$\begin{aligned}\frac{n}{(n+t)^2}\int_0^\infty f_n\left(1 - \frac{t}{n+t}\right)dt \;=\; &\frac{1}{\pi}\int_0^\infty \left(R_3(t) - \frac{I_{[t>1]}}{2t}\right)dt + \frac{\log(2n+1)}{2\pi}\\ &+\frac{1}{n\pi}\int_0^\infty \left(S_3(t) - 2tR_3(t) + \frac{I_{[t>1]}}{4}\right)dt + O\left(\frac{1}{n^2}\right).\end{aligned} \tag{2.14}$$

where $\int_0^\infty (R_3(t) - I_{[t>1]}/2t)dt = -0.28977126$ and $\int_0^\infty (S_3(t) - 2tR_3(t) + I_{[t>1]}/4)dt = 0.497593957$.

For $-1 < x < 0$, using the change of variable let $x = -1 + t(n+t)^{-1}$, we have that $EN_K(-1,0) = n(n+t)^{-2} \int_0^\infty f_n\left(-1 + \frac{t}{n+t}\right) dt$, we have

$$\left(\frac{n}{(n+t)^2}\right) g_1(t) = \left(\frac{n^2}{(n+t)^2}\right) \frac{1}{\pi} \left(R_4(t) + \frac{S_4(t)}{n} + O\left(\frac{1}{n^2}\right)\right), \tag{2.15}$$

in which $R_4(t) \equiv R_2(-t)$. For $n$ even $S_4(t) = (S_{41}(t) + S_{42}(t))/(4S_{43}(t))$, and for $n$ odd $S_4(t) = (S_{41}(t) - S_{42}(t))/(4S_{43}(t))$, where

$$\begin{aligned} S_{41}(t) \;=\; 8\Bigg\{ & \left(\frac{15}{4}t - 7/2t^2 - 3/8\right) e^{-6t} + \left(15t^4 - 3/2t - 22t^3 + \frac{9}{8} + 19/2t^2 - 2t^5\right) e^{-4t} \\ & + \left(-9/4t + 6t^2 - 3t^3 - \frac{9}{8} + t^4\right) e^{-2t} + 3/8 \Bigg\} \end{aligned}$$

$S_{42}(t) \equiv S_{22}(-t)$, and $S_{43}(t) \equiv S_{23}(-t)$. Also we have that

$$g_2\left(-1 + \frac{t}{n+t}\right) = \left(\frac{K^2}{n}\right) g_{2,1}(t) + o(n^{-1}) \tag{2.16}$$

where

$$g_{2,1}(t) = \frac{8t\,(1 + (4t^3 - 2t^2 - 2t - 1)e^{-2t})}{(4t-1)e^{-4t} + (2 - 4t + 12t^2 - 8t^3)e^{-2t} - 1}.$$

Also

$$\left(\frac{n}{(n+t)^2}\right) g_3(t) = \left(\frac{K}{\sqrt{n\pi}}\right) g_{3,1}(t) + o(n^{-1}) \tag{2.17}$$

where $g_{3,1}(t) = \frac{-((4t^2 - 2t + 1)e^{-2t} - 1)}{\sqrt{t}((2t-1)e^{-2t} + 1)^{3/2}}$. Also we have that

$$g_4(t) = o(n^{-1/2}), \tag{2.18}$$

Finally we find that

$$g_5(t) = \frac{K}{\sqrt{n}}\, g_{5,1}(t) + o(n^{-1}) \tag{2.19}$$

where

$$g_{5,1}(t) \;=\; \frac{\sqrt{t}\,((-8t^2 + 4t + 2)\,e^{-2t} - 2)}{\sqrt{(2t-1)e^{-2t} + 1}\sqrt{(1-4t)e^{-4t} + (8t^3 - 12t^2 + 4t - 2)e^{-2t} + 1}}.$$

As $t \to \infty$ we have

$$R_4(t) = \frac{1}{2t} + O\left(t^{1/2}e^{-t}\right), \qquad S_4(t) = \frac{3}{4} + O\left(te^{-t}\right), \qquad g_{2,1}(t) = -8t + O(t^4 e^{-2t})$$

$$g_{3,1}(t) = t^{-1/2} + O(t^{3/2}e^{-2t}), \qquad g_{5,1}(t) = -2\sqrt{t} + O(t^{5/2}e^{-2t}).$$

then we have by (1.4) that

$$\frac{n}{(n+t)^2} f_n\Big(-1 + \frac{t}{n+t}\Big) := I_{41} + I_{42}$$

where

$$I_{41} = \frac{1}{\pi}\frac{n^2}{(n+t)^2}\Big(R_4(t) + \frac{S_4(t)}{n} + \frac{K^2}{n}R_4(t)g_{2,1}(t)\Big) + o(n^{-1})$$

and

$$I_{42} = \frac{1}{\pi}\frac{n^2}{(n+t)^2}\Big(\frac{2K^2}{n}g_{3,1}(t)g_{5,1}(t)\Big) + o(n^{-1})$$

Now using (2.13) we have that

$$\begin{aligned} EN_K(-1,0) &= \frac{n}{(n+t)^2}\int_0^\infty f_n\Big(-1+\frac{t}{n+t}\Big) \qquad (2.20)\\ &= \frac{1}{\pi}\int_0^\infty \Big(R_4(t) - \frac{I_{[t>1]}}{2t}\Big)\,dt \\ &+ \frac{1}{\pi}\int_0^\infty \Big(\frac{I_{[t>1]}}{2t} - \frac{I_{[t>1]}}{4n+2t}\Big)\,dt + \frac{1}{n\pi}\int_0^\infty \Big(S_4(t) - 2tR_4(t) + \frac{I_{[t>1]}}{4}\Big)\,dt \\ &+ \frac{K^2}{n\pi}\int_0^\infty (R_4(t)g_{2,1}(t) + 2g_{3,1}(t)g_{5,1}(t))\,dt + o(n^{-1}). \end{aligned}$$

where for $n$ odd $\int_0^\infty (S_4(t) - 2tR_4(t) + I_{[t>1]}/4)dt = 1.499908194$, and for $n$ even $\int_0^\infty (S_4(t) - 2tR_4(t) + I_{[t>1]}/4)dt = -0.4999082034$. Also $\int_0^\infty \Big(R_4(t) - I_{[t>1]}(2t)^{-1}\Big)\,dt = 0.3793914850$, and $\int_0^\infty (R_4(t)\,g_{2,1}(t) + 2\,g_{3,1}(t)g_{5,1}(t))\,dt = 1.533149028$. Thus we arrive at (2.1), and the first part of the theorem is proved.

Now for the proof of the second part of the theorem, that is for the case $K = o(\sqrt{n})$ we study the asymptotic behavior of $EN_K(a,b)$ for different intervals $(-\infty,-1)$, $(-1,0)$, $(0,1)$ and $(1,\infty)$ separately.

Let $1 < x < \infty$ by using the change of variable $x = 1 + \frac{t}{n}$ and (1.4) and by the result (2.4) we find that

$$\frac{1}{n}\int_0^\infty f_n(1+\frac{t}{n})dt = \frac{1}{\pi}\int_0^\infty R_1(t)dt + o(n^{-1})$$

where $\int_0^\infty R_1(t)dt = 0.734874192$ .

For $-\infty < x < -1$ we use the change of variable $x = -1 - \frac{t}{n}$. By the fact that $K^2/n = o(n)$, we have $\lim_{n\to\infty} K^2/n = 0$. So

$$exp\{K^2 g_{2,1}(-1-t/n)/n\} = 1 + o(1), \qquad exp\{K^2 g_{4,1}(-1-t/n)/n\} = 1 + o(1).$$

Therefore the relations (2.7) and (2.9) implies that $exp\{g_2(-1-t/n)\} = 1+o(1)$ and $exp\{g_4(-1-t/n)\} = 1+o(1)$. These and the relations (1.4), (2.6), (2.8), and (2.10) implies that

$$\frac{1}{n}\int_0^\infty f_n(-1-\frac{t}{n})dt = \frac{1}{\pi}\int_0^\infty R_2(t)dt + o(1)$$

where$\int_0^\infty R_2(t)dt = 1.095640061$.
For $0 < x < 1$ by using the change of variable $x = 1 - \frac{t}{n+t}$ and relations (1.4), (2.11), (2.12), (2.13) and by a similar method as in (2.14) we find that

$$\frac{n}{(n+t)^2}\int_0^\infty f_n(-1-\frac{t}{n+t})dt = \frac{1}{\pi}\int_0^\infty (R_3(t) - \frac{I[t>1]}{2t})dt + \frac{1}{2\pi}\log(2n+1)$$

where $\int_0^\infty (R_3(t) - \frac{I[t>1]}{2t})dt = -0.28977126$.
For $-1 < x < 0$ we use the change of variable $x = -1 + \frac{t}{n+t}$. Using (2.16) and (2.18), by the same reasoning as in the case $-\infty < x < -1$, we have that $exp\{g_2(-1+\frac{t}{n+t})\} = 1+o(1)$ and $exp\{g_4(-1+\frac{t}{n+t})\} = 1+o(1)$. Thus the relations (2.15),(2.16),(2.17),(2.18),(2.19), and by a similar method as in (2.20) we find that

$$\frac{n}{(n+t)^2}\int_0^\infty f_n(-1+\frac{t}{n+t})dt = \frac{1}{\pi}\int_0^\infty (R_4(t) - \frac{I[t>1]}{2t})dt + \frac{1}{2\pi}\log(2n+1)$$

where$\int_0^\infty (R_4(t) - \frac{I[t>1]}{2t})dt = 0.3793914850$. Thus we have (2.2) and the theorem is proved.

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