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Loss distributions conditional on defaults

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Abstract

The impact of default events on the loss distribution of a credit portfolio can be assessed by determining the loss distribution conditional on these events. While it is conceptually easy to estimate loss distributions conditional on default events by means of Monte Carlo simulation, it becomes impractical for two or more simultaneous defaults as the conditioning event is extremely rare. We provide an analytical approach to the calculation of the conditional loss distribution for the CreditRisk$^+$ portfolio model with independent random loss given default distributions. The analytical solution for this case can be used to study the properties of the conditional loss distributions and to discuss how they relate to the identification of risk concentrations.

1 Introduction

Loss distributions conditional on default of one or more obligors are promising means to identify vulnerabilities of banks. Default of a large obligor not only has a direct impact on the profit and loss of a bank and potentially also on its capital basis. Due to mutual dependence of default events the default of one or more obligors can have a significant impact on the loss distribution of the remaining portfolio, too. Determination of loss distributions conditional on defaults, therefore, can be considered a special stress testing technique. Such analysis, in particular, can help to decide whether a large exposure to a certain obligor is just a risk concentration for its size or, even worse, also significant part of a sector or industry risk concentration.

It is conceptually easy to determine the impact of the default of one or more obligors via a Monte Carlo simulation approach: Just eliminate all simulation iterations from the sample in which the obligor(s) on whose default(s) conditioning is to be conducted have not defaulted. This, however, is feasible in practice for one default, but becoming impracticable for two or more defaults.

Tasche (2004, equation (3.31)) showed how the loss distribution conditional on one default can be calculated analytically in the CreditRisk$^+$ model with random loss severities. In this note the related formulas for the case of two defaults are provided. Formulas for the cases of three or more defaults can be readily derived in the same way as the formula for the case of two default is derived. As a consequence of the likely lack of practical relevance of cases of three or more defaults, we do not provide the results for these cases here.

*The opinions expressed in this note are those of the author and do not necessarily reflect views of Lloyds Banking Group.

This note is focused on the theoretical derivation of the result and its interpretation. The question of practical numerical implementation is not considered. The reader is referred to Gundlach and Lehrbass (2004) or Schmock (2008) for detailed discussions of this topic.

The plan of this note is as follows:

- As background and for introducing the notation, Section 2 provides a description of the CreditRisk$^+$ model as presented in CSFB (1997) or Gundlach (2004). The CreditRisk$^+$ model described here is enhanced to allow for random loss severities[1].
- In Section 3 the results are presented and their application is discussed. To derive the results we revisit the approach used in Tasche (2004) to develop analytical representations of the Value-at-Risk and Expected Shortfall contributions of single obligors in CreditRisk$^+$.
- The note concludes with some comments in Section 4.

2 An analytical credit portfolio model with random loss severities

The approach to the CreditRisk$^+$ loss distribution in CSFB (1997) or Gundlach (2004) is driven by analytical considerations and – to some extent – hides the way in which the Poisson approximation is used to smooth the loss distribution. While preserving the notation of Gundlach (2004), therefore we review in this section the steps that lead to the formula for the generating function of the loss distribution in CSFB (1997) and Gundlach (2004). When doing so, we slightly generalize the methodology to the case of stochastic exposures – thus allowing for random loss severities – that are independent of the default events and the random factors expressing the dependence on sectors or industries. This generalization can be afforded at no extra cost as the result is again a generating function in the shape as presented in Gundlach (2004, equation (2.19)), the only difference being that the sector polynomials are composed another way.

Write $\mathbf{1}_A$ for the default indicator of obligor A, i.e. $\mathbf{1}_A = 0$ if A does not default in the observation period and $\mathbf{1}_A = 1$ if A defaults. In CSFB (1997) and Gundlach (2004), an approximation is derived for the distribution of the portfolio loss variable $X = \sum_A \mathbf{1}_A \, \nu_A$ with the ν_A denoting deterministic potential losses. A careful inspection of the beginning of Section 5 of Gundlach (2004) reveals that the main step in the approximation procedure is to replace the $\{0, 1\}$-valued indicators $\mathbf{1}_A$ by integer-valued random variables D_A with the same expected values. These variables D_A are conditionally Poisson distributed given the random factors $S_1, \ldots, S_N$.

Here, we want to study the distribution of the more general loss variable $X = \sum_A \mathbf{1}_A \, \mathcal{E}_A$, where $\mathcal{E}_A$ denotes the random outstanding exposure of obligor A. We assume that $\mathcal{E}_A$ takes on positive integer values. However, just replacing $\mathbf{1}_A$ by D_A as in the case of deterministic potential losses does not yield a nice generating function – "nice" in the sense that the CreditRisk$^+$ algorithms for extracting the loss distribution can be applied. We instead consider the approximate loss variable

$$X = \sum_A \sum_{i=1}^{D_A} \mathcal{E}_{A,i}, \tag{2.1a}$$

[1]Schmock (2008) describes a further generalisation of the model to include connected groups of obligors.

where $\mathcal{E}_{A,1}, \mathcal{E}_{A,2}, \ldots$ are independent copies of $\mathcal{E}_A$. Thus, we approximate the terms $\mathbf{1}_A \, \mathcal{E}_A$ by conditionally compound Poisson sums. For the sake of brevity, we write

$$Y_A = \sum_{i=1}^{D_A} \mathcal{E}_{A,i} \tag{2.1b}$$

for the loss suffered due to obligor A. A careful inspection of the arguments presented to derive Gundlach (2004, equation (2.19)) now yields the following result on the generating function of the distribution of X.

Theorem 2.1 *Define the "loss" variable X by (2.1a) and specify the distribution of X by the following assumptions:*

(i) The approximate default indicators D_A are conditionally independent given a set of "economic" factors $\mathbf{S} = (S_0, S_1, \ldots, S_N)$. The conditional distribution of D_A given $\mathbf{S}$ is Poisson with intensity $p_A^S = p_A \sum_{k=0}^N w_{Ak} \, S_k$ where $p_A \geq 0$ denotes the "probability of default" of obligor A and $0 \leq w_{Ak} \leq 1$ are "factor loadings" such that $\sum_{k=0}^N w_{Ak} = 1$ for each obligor A.

(ii) The idiosyncratic *factor S_0 is a constant and equals 1. The factors $S_1, \ldots, S_N$ are independent and Gamma-distributed with unit expectations $\mathbb{E}[S_k] = 1$ and parameters[2] $(\alpha_k, \beta_k) = (\alpha_k, 1/\alpha_k)$ for $k = 1, \ldots, N$.*

(iii) The random variables $\mathcal{E}_{A,1}, \mathcal{E}_{A,2}, \ldots$ are independent copies of a non-negative integer-valued random variable $\mathcal{E}_A$ and, additionally, are also independent of the D_A and $\mathbf{S}$. The distribution of $\mathcal{E}_A$ is given through its generating function

$$H_A(z) = \mathbb{E}\big[z^{\mathcal{E}_A}\big]. \tag{2.2a}$$

Define for $k = 0, 1, \ldots, N$ the sector polynomial *$\mathcal{Q}_k$ by*

$$\mathcal{Q}_k(z) = \frac{1}{\mu_k} \sum_A w_{Ak} \, p_A \, H_A(z), \tag{2.2b}$$

where the sector default intensities *μ_k are given by*

$$\mu_k = \sum_A w_{Ak} \, p_A. \tag{2.2c}$$

Then the generating function G_X of the loss variable X can be represented as

$$G_X(z) = e^{\mu_0 \, (\mathcal{Q}_0(z) - 1)} \prod_{k=1}^N \left(\frac{1 - \delta_k}{1 - \delta_k \, \mathcal{Q}_k(z)} \right)^{\alpha_k}, \tag{2.2d}$$

where the constants δ_k are defined as $\delta_k = \mu_k / (\mu_k + \alpha_k)$.

[2] $\beta_k = 1/\alpha_k$ is implied by the assumption that S_k has unit expectation.

Remark 2.2

1) *The case of deterministic severities can be regained from Theorem 2.1 by choosing the exposures constant, e.g.* $\mathcal{E}_A = \nu_A$. *Then the generating functions of the exposures turn out to be just monomials, namely* $H_A(z) = z^{\nu_A}$.

2) *Representation* (2.2d) *of the generating function of the portfolio loss distribution implies that the portfolio loss distribution can be interpreted as the distribution of a sum of* $N+1$ *independent* sector loss distributions *that correspond to the economic factors* $(S_0, S_1, \dots, S_N)$.
The term $e^{\mu_0\,(\mathcal{Q}_0(z)-1)}$ *is the generating function of a random variable with a compound*[3] *Poisson distribution that can be realised as* $\sum_{i=1}^{T_0} \eta_{0,i}$ *where* $T_0, \eta_{0,1}, \eta_{0,2}, \dots$ *are independent,* T_0 *is Poisson-distributed with intensity* μ_0, *and* $\eta_{0,1}, \eta_{0,2}, \dots$ *are i.i.d. with generating function* $\mathcal{Q}_0(s)$.
The terms $\left(\frac{1-\delta_k}{1-\delta_k\,\mathcal{Q}_k(z)}\right)^{\alpha_k}$, $k = 1, \dots, N$, *are the generating functions of random variables with compound negative binomial distributions that can be realised as* $\sum_{i=1}^{T_k} \eta_{k,i}$ *where* $T_k, \eta_{k,1}, \eta_{k,2}, \dots$ *are independent,* T_k *is negative binomially distributed*[4] *with* failure probability δ_k *and* success number parameter α_k, *and* $\eta_{k,1}, \eta_{k,2}, \dots$ *are i.i.d. with generating function* $\mathcal{Q}_k(s)$.
With this representation of the portfolio loss distribution as the convolution of compound Poisson and negative binomial distributions, the sector polynomials $\mathcal{Q}_k(s)$ *can be interpreted as the generating functions of typical loss severities in the respective sectors.*

By means of Theorem 2.1 the loss distribution of the generalized model (2.1a) can be calculated in principle with the same algorithms as in the case of the original CreditRisk$^+$ model. Once the probabilities $\mathbb{P}[X = x]$, x non-negative integer, are known, it is an easy task to calculate the loss quantiles $q_\theta(X)$ as defined by

$$q_\theta(X) \;=\; \min\{x \ge 0 : \mathrm{P}[X \le x] \ge \theta\}, \tag{2.3}$$

or related risk measures like Value-at-Risk or Expected Shortfall.

When working with Theorem 2.1, one has to decide whether random exposures shall be taken into account, and in case of a decision in favor of doing so, how the exposure distributions are to be modeled. Tasche (2004, Example 1) and Schmock (2008) present some possible choices of discrete exposure distributions. Gordy (2004) discusses an approximative but similar approach to random severities with continuous distributions.

3 Loss distributions conditional on defaults

The purpose of this section is to provide formulas for the portfolio loss distribution conditional on defaults that can be represented in similar terms as the unconditional loss distribution and hence be evaluated with the known CreditRisk$^+$ algorithms. The following theorem – a modification of Tasche (2004, Lemma 1) – yields the foundation of the results. Denote by $I(E)$ the indicator variable of the event E, i.e. $I(E; m) = 1$ if $m \in E$ and $I(E; m) = 0$ if $m \notin E$.

[3] See, e.g., Rolski et al. (1999) for background information on compound distributions and generating functions.

[4] If the success number parameter of a negative binomial distribution is a positive integer a then the distribution can be interpreted as the distribution of the number of failures in a series of independent identical experiments before the a-th success is observed.

Theorem 3.1 *Define the approximate default indicators D_A as in Theorem 2.1. Assume that $A(1), \ldots, A(r)$ are obligors such that $A(i) \neq A(j)$ for $i \neq j$. Under the assumptions and with the notation of Theorem 2.1 then we have*

$$\mathbb{E}\Big[I(X = x) \prod_{i=1}^{r} D_{A(i)}\Big] = \mathbb{E}\Big[I\Big(X = x - \sum_{j=1}^{r} \mathcal{E}_{A(j)}\Big) \prod_{i=1}^{r} p^S_{A(i)}\Big] \tag{3.1}$$

for any non-negative integer x. The random variables X and $\mathcal{E}_{A(1,)}, \ldots, \mathcal{E}_{A(r)}$ on the right-hand side of (3.1) are independent.

Proof. We provide the proof only for the case $r = 2$ as the proof for general r is not much different but the notation would be more cumbersome. Hence assume that two obligors $A(1) \neq A(2)$ have been selected. The assumptions on independence and conditional independence from Theorem 2.1 then imply

$$\begin{aligned}
&\mathbb{E}\big[D_{A(1)}\, D_{A(2)}\, I(X = x)\big] \\
&= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} k_1\, k_2\, \mathbb{P}\Big[D_{A(1)} = k_1,\, D_{A(2)} = k_2,\, \sum_{\substack{B \neq A(1),\\ B \neq A(2)}} Y_B + \sum_{i=1}^{k_1} \mathcal{E}_{A(1),i} + \sum_{j=1}^{k_2} \mathcal{E}_{A(2),j} = x\Big] \\
&= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} k_1\, k_2\, \mathbb{E}\Bigg[\frac{(p^S_{A(1)})^{k_1}}{k_1!}\, e^{-p^S_{A(1)}}\, \frac{(p^S_{A(2)})^{k_2}}{k_2!}\, e^{-p^S_{A(2)}} \\
&\qquad \times \mathbb{P}\Big[\sum_{\substack{B \neq A(1),\\ B \neq A(2)}} Y_B + \sum_{i=1}^{k_1} \mathcal{E}_{A(1),i} + \sum_{j=1}^{k_2} \mathcal{E}_{A(2),j} = x \,|\, \mathbf{S}\Big]\Bigg] \\
&= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \mathbb{E}\Bigg[p^S_{A(1)}\, p^S_{A(2)} \\
&\qquad \times \mathbb{P}\Big[D_{A(1)} = k_1,\, D_{A(2)} = k_2,\, \sum_{\substack{B \neq A(1),\\ B \neq A(2)}} Y_B + \sum_{i=1}^{k_1+1} \mathcal{E}_{A(1),i} + \sum_{j=1}^{k_2+1} \mathcal{E}_{A(2),j} = x \,|\, \mathbf{S}\Big]\Bigg] \\
&= \mathbb{E}\Big[I\Big(X = x - \mathcal{E}_{A(1)} - \mathcal{E}_{A(2)}\Big)\, p^S_{A(1)}\, p^S_{A(2)}\Big]
\end{aligned}$$

as stated in (3.1). q.e.d.

As the variable D_A approximates obligor A's default indicator the conditional expectation $\mathbb{E}[D_A \,|\, X = x]$ can be interpreted as an approximation of the conditional probability of obligor A's default given that the portfolio loss X assumes the value x. Tasche (2004, Corollary 1) observed the following result for $\mathbb{E}[D_A \,|\, X = x]$. It can be readily derived from Theorem 3.1.

Notation. *For any positive integers $i \leq n$ define the n-*dimensional *i-*th unit vector *$e_i^{(n)}$ by*

$$e_i^{(n)} = (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{n-i \text{ times}}). \tag{3.2}$$

Where the dimension is known from the context we write $e_i = e_i^{(n)}$ for short.

Corollary 3.2 (Probability of default conditional on portfolio loss)
Adopt the setting and the notation of Theorem 2.1 and Theorem 3.1. Write $\mathbb{P}_\alpha[X \in \cdot\,]$ for $\mathbb{P}[X \in \cdot\,]$ in order to express the dependence[5] of the portfolio loss distribution upon the exponents $\alpha = (\alpha_1, \ldots, \alpha_N)$ in (2.2d). Assume that x is an integer such that $\mathbb{P}_\alpha[X = x] > 0$. Then, in the CreditRisk$^+$ framework, the conditional probability of obligor A's default given that the portfolio loss X assumes the value x can be approximated by

$$\mathbb{E}[D_A \,|\, X = x] \;=\; p_A\, \mathbb{P}_\alpha[X = x]^{-1} \times \Big(w_{A0}\, \mathbb{P}_\alpha[X = x - \widetilde{\mathcal{E}}_A] + \sum_{j=1}^{N} w_{Aj}\, \mathbb{P}_{\alpha+e_j}[X = x - \widetilde{\mathcal{E}}_A] \Big), \quad (3.3)$$

where $\widetilde{\mathcal{E}}_A$ stands for a random variable that has the same distribution as $\mathcal{E}_A$ but is independent of X.

Intuitively, one might think that $\mathbb{P}[D_A > 0 \,|\, X = x]$ would be a better approximation of the conditional probability of default of obligor A than $\mathbb{E}[D_A \,|\, X = x]$. However, there is no such relatively simple representation of $\mathbb{P}[D_A > 0 \,|\, X = x]$ as (3.3) is for $\mathbb{E}[D_A \,|\, X = x]$. Moreover, by the assumption on the conditional Poisson distribution of D_A we have

$$\mathbb{E}\big[\mathbb{P}[D_A > 0 \,|\, X]\big] = \mathbb{P}[D_A > 0] < p_A = \mathbb{E}\big[\mathbb{E}[D_A \,|\, X]\big]. \quad (3.4)$$

Hence the bias of $\mathbb{P}[D_A > 0 \,|\, X = x]$ with respect to $\mathbb{P}[A \text{ defaults} \,|\, X = x]$ is likely to be greater than the bias of $\mathbb{E}[D_A \,|\, X = x]$.

The probabilities in the numerator of the right-hand side of (3.3) must be calculated by convolution if the loss severities $\mathcal{E}_A$ are non-deterministic. In any case, Corollary 3.2 can be used for constructing the portfolio loss distribution conditional on the default of an obligor. Observe that by the very definition of conditional probabilities we have

$$\mathbb{P}[X = x \,|\, A \text{ defaults}] \;=\; \mathbb{P}[A \text{ defaults} \,|\, X = x] \frac{\mathbb{P}[X = x]}{p_A}. \quad (3.5)$$

Since by Corollary 3.2 an approximation for $\mathbb{P}[A \text{ defaults} \,|\, X = x]$ is provided, the term-wise comparison of (3.3) and (3.5) yields

$$\mathbb{P}_\alpha[X = x \,|\, A \text{ defaults}] \;\approx\; w_{A0}\, \mathbb{P}_\alpha[X = x - \widetilde{\mathcal{E}}_A] + \sum_{j=1}^{N} w_{Aj}\, \mathbb{P}_{\alpha+e_j}[X = x - \widetilde{\mathcal{E}}_A]. \quad (3.6)$$

Note that according to (3.6), the conditional distribution $\mathbb{P}_\alpha[X = \cdot \,|\, A \text{ defaults}]$ of the portfolio loss X given that A defaults may be computed as a weighted mean of stressed portfolio loss distributions. The stresses are expressed by the exponents $\alpha_j + 1$ in the generating functions of $\mathbb{P}_{\alpha+e_j}[X = \cdot\,]$, $j = 1, \ldots, N$. In actuarial terms, incrementing the success number parameter of a negative binomial claim number distribution (cf. Remark 2.2) means to give the claim number distribution a heavier tail. Hence, this way the number of claims (sector-related defaults in CreditRisk+ terms) tends to be larger after the stress was applied. No change due to stress, however, occurs to the sector loss severity distributions as characterised by the sector polynomials $\mathcal{Q}_j$. This is no surprise as the loss severities in the setting of this note are assumed to be independent of the economic factors that define the sectors.

[5] Of course, the distribution also depends on μ_0, $\mathcal{Q}_0, \ldots, \mathcal{Q}_N$, and $\delta_1, \ldots, \delta_N$. However, these input parameters are considered constant in Corollary 3.2.

Remark 3.3

(i) *By (3.6) stressed portfolio loss distributions can be evaluated, conditional on the scenarios that single obligors have defaulted. If, for instance, the portfolio Value-at-Risk changes dramatically when obligor A's default is assumed, then one may find that the portfolio depends too strongly upon A's condition.*

(ii) *Equation (3.6) reflects a write-off or special provision due to obligor A's default. This is a consequence of the fact that on the right-hand side of the equation loss distributions of the shape $X + \widetilde{\mathcal{E}}_A$ appear, thus implying that losses X are added to a loss socket $\widetilde{\mathcal{E}}_A$ caused by obligor A's first default. However, usually in banks occurred losses are not taken into account for the determination of risk metrics (like quantiles as defined by* (2.3)*) but are deducted from the banks available capital buffer. In that sense (3.6) does not appropriately reflect banks' practice.*

(iii) *To deal with the issue observed in (ii), note that Theorem 2.1 and Corollary 3.2 also can be applied to the case $\mathcal{E}_A = 0$. In particular, dependencies within the portfolio are then still adequately reflected by obligor A's conditional default intensity p_A^S. With $\mathcal{E}_A = 0$, then (3.6) still holds, but the sector default intensities μ_k and the sector polynomials $\mathcal{Q}_k$ are slightly different to the case of obligor A not being in default.*

While Theorem 3.1 can be used to study the portfolio loss distributions conditional on any number of defaults, we confine ourselves in the following corollary and its consequences to considering only the case of two defaults as we already did in the proof of Theorem 3.1. The formulas for conditioning on three or more defaults can be derived in the same way as the formula for the case of two defaults. The cases of three or more defaults, however, are notationally and computationally much more inconvenient, presumably much less relevant for practice, and do not add much more theoretical insight compared to the case of two defaults.

Corollary 3.4 (Joint probability of default conditional on portfolio loss)
Adopt the setting and the notation of Corollary 3.2. Let $A(1) \neq A(2)$ denote two obligors who have been selected in advance. Assume that x is an integer such that $\mathbb{P}_\alpha[X = x] > 0$. Then, in the CreditRisk$^+$ framework, the conditional joint probability of obligor $A(1)$'s and obligor $A(2)$'s default given that the portfolio loss X assumes the value x may be approximated by

$$\begin{aligned}\mathbb{E}[D_{A(1)}\, D_{A(2)} \,|\, X = x] = {} & p_{A(1)}\, p_{A(2)}\, \mathbb{P}_\alpha[X = x]^{-1} \\ & \times \Big(w_{A(1)0}\, w_{A(2)0}\, \mathbb{P}_\alpha[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \\ & + \sum_{j=1}^{N} \Big(w_{A(1)0}\, w_{A(2)j} + w_{A(1)j}\, w_{A(2)0} \Big)\, \mathbb{P}_{\alpha+e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \\ & + \sum_{j=1}^{N} w_{A(1)j}\, w_{A(2)j}\, \frac{\alpha_j + 1}{\alpha_j}\, \mathbb{P}_{\alpha+2e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \\ & + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} w_{A(1)i}\, w_{A(2)j}\, \mathbb{P}_{\alpha+e_i+e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \Big)\end{aligned} \tag{3.7}$$

where $\widetilde{\mathcal{E}}_A$ stands for a random variable that has the same distribution as $\mathcal{E}_A$ but is independent of X.

While (3.7) in general looks like a straight-forward extension to (3.3), there is a subtle difference in the terms involving $\mathbb{P}_{\alpha+2e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}]$ which reflect double stress in the same sector. This double stress is enforced by the additional factors $\frac{\alpha_j+1}{\alpha_j} > 1$.

Proof of Corollary 3.4. We will derive (3.7) by comparing the coefficients of two power series. The first one will be $\mathbb{E}[D_{A(1)}\, D_{A(2)}\, z^X] = \sum_{k=0}^\infty \mathbb{E}[D_{A(1)}\, D_{A(2)}\, I(X=k)]\, z^k$, the second one will be an expression that is equivalent to $\mathbb{E}[D_{A(1)}\, D_{A(2)}\, z^X]$ but involves generating functions similar to (2.2d).

Recall that we denote the generating function of $\mathcal{E}_A$ by $H_A(z)$. By means of Theorem 3.1 and the independence of the random exposures, we can compute

$$\begin{aligned}
\mathbb{E}[D_{A(1)}\, D_{A(2)}\, z^X] &= \sum_{k=0}^\infty \mathbb{E}[p^S_{A(1)}\, p^S_{A(2)}\, I(X + \mathcal{E}_{A(1)} + \mathcal{E}_{A(2)} = k)]\, z^k \\
&= \mathbb{E}[p^S_{A(1)}\, p^S_{A(2)}\, z^{X+\mathcal{E}_{A(1)}+\mathcal{E}_{A(2)}}] \\
&= \mathbb{E}[p^S_{A(1)}\, p^S_{A(2)}\, z^X]\, \mathbb{E}[z^{\mathcal{E}_{A(1)}}]\, \mathbb{E}[z^{\mathcal{E}_{A(2)}}] \\
&= \mathbb{E}[p^S_{A(1)}\, p^S_{A(2)}\, z^X]\, H_{A(1)}(z)\, H_{A(2)}(z).
\end{aligned} \tag{3.8a}$$

Recall the definitions of the intensities p^S_A, the sector default intensities μ_k and the sector polynomials $\mathcal{Q}_k$ from Theorem 2.1. By making use of the fact that the scalar factors $(S_1, \ldots, S_N)$ are Gamma-distributed with parameters $(\alpha_k, 1/\alpha_k)$, $k = 1, \ldots, N$, and that $S_0 = 1$ we obtain for $\mathbb{E}[p^S_{A(1)}\, p^S_{A(2)}\, z^X]$ (cf. the proof of (3.25c) in Tasche (2004))

$$\begin{aligned}
\mathbb{E}[p^S_{A(1)}\, p^S_{A(2)}\, z^X] &= \mathbb{E}\big[p^S_{A(1)}\, p^S_{A(2)}\, \mathbb{E}[z^X \mid S]\big] && (3.8b) \\
&= p_{A(1)}\, p_{A(2)} \sum_{i=0}^N \sum_{j=0}^N w_{A(1)i}\, w_{A(2)j}\, \mathbb{E}\big[S_i\, S_j \prod_{k=0}^N \exp\big(S_k\, \mu_k\, (\mathcal{Q}_k(z) - 1)\big)\big].
\end{aligned}$$

Denote by

$$G_X^{(\alpha)}(z) = \sum_{k=0}^\infty \mathbb{P}_\alpha[X = k]\, z^k \tag{3.9}$$

the generating function of X according to (2.2d) as a function of the exponents $\alpha = (\alpha_1, \ldots, \alpha_N)$ on the right-hand side of the equation as has been explained in Corollary 3.2. Observe then that

$$\begin{aligned}
\mathbb{E}\big[S_0^2 \prod_{k=0}^N \exp\big(S_k\, \mu_k\, (\mathcal{Q}_k(z) - 1)\big)\big] &= G_X^{(\alpha)}(z) \\
\mathbb{E}\big[S_0\, S_j \prod_{k=0}^N \exp\big(S_k\, \mu_k\, (\mathcal{Q}_k(z) - 1)\big)\big] &= G_X^{(\alpha+e_j)}(z),\ j \ge 1 \\
\mathbb{E}\big[S_i\, S_j \prod_{k=0}^N \exp\big(S_k\, \mu_k\, (\mathcal{Q}_k(z) - 1)\big)\big] &= G_X^{(\alpha+e_i+e_j)}(z),\ i \ne j \\
\mathbb{E}\big[S_j^2 \prod_{k=0}^N \exp\big(S_k\, \mu_k\, (\mathcal{Q}_k(z) - 1)\big)\big] &= \frac{\alpha_j + 1}{\alpha_j}\, G_X^{(\alpha+2\,e_j)}(z),\ j \ge 1.
\end{aligned} \tag{3.10}$$

Note that $G_X^{(\alpha)}(z)\, H_{A(1)}(z)\, H_{A(2)}(z)$ is the generating function of the sequence $\mathbb{P}_\alpha[X + \widetilde{\mathcal{E}}_{A(1)} + \widetilde{\mathcal{E}}_{A(2)} = 0], \mathbb{P}_\alpha[X + \widetilde{\mathcal{E}}_{A(1)} + \widetilde{\mathcal{E}}_{A(2)} = 1], \ldots$ (i.e. of the distribution of $X + \widetilde{\mathcal{E}}_{A(1)} + \widetilde{\mathcal{E}}_{A(2)}$). Combining this observation with (3.8a), (3.8b), and (3.10) implies (3.7) by power series comparison. q.e.d.

As Corollary 3.2 can be used for constructing the portfolio loss distribution conditional on the default of an obligor, Corollary 3.4 can be used for the portfolio loss distribution conditional on the joint default of two obligors. Again by the definition of conditional probabilities we have

$$\mathbb{P}[X = x \,|\, A(1) \text{ and } A(2) \text{ default}] \;=\; \mathbb{P}[A(1) \text{ and } A(2) \text{ default} \,|\, X = x] \frac{\mathbb{P}[X = x]}{\mathbb{P}[A(1) \text{ and } A(2) \text{ default}]}. \quad (3.11)$$

Since by Corollary 3.4 an approximation for $\mathbb{P}[AA(1) \text{ and } A(2) \text{ default} \,|\, X = x]$ is provided, the term-wise comparison of (3.7) and (3.11) yields

$$\begin{aligned}
&\mathbb{P}_\alpha[X = x \,|\, A(1) \text{ and } A(2) \text{ default}] \\
&\approx \frac{p_{A(1)}\, p_{A(2)}}{\mathbb{P}[A(1) \text{ and } A(2) \text{ default}]} \qquad (3.12\text{a}) \\
&\quad \times \Big(w_{A(1)0}\, w_{A(2)0}\, \mathbb{P}_\alpha[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \\
&\quad + \sum_{j=1}^N \Big(w_{A(1)0}\, w_{A(2)j} + w_{A(1)j}\, w_{A(2)0} \Big)\, \mathbb{P}_{\alpha+e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \\
&\quad + \sum_{j=1}^N w_{A(1)j}\, w_{A(2)j}\, \frac{\alpha_j + 1}{\alpha_j}\, \mathbb{P}_{\alpha+2e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \\
&\quad + \sum_{i=1}^N \sum_{j=1, j\neq i}^N w_{A(1)i}\, w_{A(2)j}\, \mathbb{P}_{\alpha+e_i+e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \Big)
\end{aligned}$$

Making use of the well-known result (see Gundlach, 2004, Section 2.3)

$$\mathbb{E}[D_{A(1)}\, D_{A(2)}] \;=\; p_{A(1)}\, p_{A(2)} \left(1 + \sum_{k=1}^N \frac{w_{A(1)k}\, w_{A(2)k}}{\alpha_k} \right), \; A(1) \neq A(2), \qquad (3.12\text{b})$$

(3.12a) can be slightly simplified to

$$\begin{aligned}
&\mathbb{P}_\alpha[X = x \,|\, A(1) \text{ and } A(2) \text{ default}] \\
&\approx \frac{1}{1 + \sum_{k=1}^N \frac{w_{A(1)k}\, w_{A(2)k}}{\alpha_k}} \qquad (3.12\text{c}) \\
&\quad \times \Big(w_{A(1)0}\, w_{A(2)0}\, \mathbb{P}_\alpha[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \\
&\quad + \sum_{j=1}^N \Big(w_{A(1)0}\, w_{A(2)j} + w_{A(1)j}\, w_{A(2)0} \Big)\, \mathbb{P}_{\alpha+e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \\
&\quad + \sum_{j=1}^N w_{A(1)j}\, w_{A(2)j}\, \frac{\alpha_j + 1}{\alpha_j}\, \mathbb{P}_{\alpha+2e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \\
&\quad + \sum_{i=1}^N \sum_{j=1, j\neq i}^N w_{A(1)i}\, w_{A(2)j}\, \mathbb{P}_{\alpha+e_i+e_j}[X = x - \widetilde{\mathcal{E}}_{A(1)} - \widetilde{\mathcal{E}}_{A(2)}] \Big)
\end{aligned}$$

Comments similar to the comments on (3.6) also apply to (3.12c). The conditional distribution $\mathbb{P}_\alpha[X = x \,|\, A(1)$ and $A(2)$ default] of the portfolio loss X given that obligors $A(1)$ and $A(2)$ default can be computed as a weighted mean of stressed or double-stressed portfolio loss distributions. The stresses, however, are not only expressed by the exponents $\alpha_j + 1$ and $\alpha_j + 2$ in the generating functions of $\mathbb{P}_{\alpha+e_j}[X = \cdot]$ and $\mathbb{P}_{\alpha+e_i+e_j}[X = \cdot]$, $i, j = 1, \ldots, N$, but also by the factors $\frac{\alpha_j+1}{\alpha_j} > 1$ appearing on the right-hand side of (3.12c). Obviously, as a consequence of the $(N+1)^2$ terms on the right-hand side of (3.12c) instead of the only $N+1$ terms of the right-hand side of (3.6), it is much more expensive to calculate the loss distributions conditional on double defaults than to calculate the loss distributions conditional on simple defaults.

Observe that Remark 3.3 also applies to (3.12c). Hence it could make sense to do the calculations for (3.12c) with loss severities $\mathcal{E}_{A(1)} = 0$ and $\mathcal{E}_{A(2)} = 0$ to reflect the risk management attitude not to take account of occurred losses for the determination of living portfolio risk metrics.

4 Conclusions

We have studied the way in which defaults impact a credit portfolio loss distribution in an enhanced CreditRisk$^+$ model, by looking at the loss distribution conditional on a number – one or two in this note – of defaults. While the derived formulas are not necessarily easy to implement, they provide nonetheless insight in the details of how the default scenarios impact the conditional portfolio loss distribution.

The results of this note can be used for specific stress scenario analyses that are intended to identify whether large credit exposures besides having an obvious size impact additionally contribute to sector risk concentrations. Another more indirect application of the results would be to use them to check the accuracy of alternative approaches to such default scenario analyses. One potential alternative approach is Monte Carlo portfolio simulation which would suffer from rare event effects when deployed for estimating loss distributions conditional on two or more defaults.

Another potential alternative could be to calculate for each obligor the probability of default conditional on the joint default of a fixed set of obligors and then to use these conditional probabilities of default as input parameters to a portfolio model. This "stressed input parameters" approach ignores the exact dependence between the default events of the obligors considered defaulted under the scenario and the economic factors commonly used for modeling dependence in credit portfolio models. Therefore, the approach is principally biased. If the bias were not too big, the approach nonetheless would be useful for its conceptual simplicity.

A study on the numerical comparison of the three different approaches – analytical as described in this note, Monte Carlo Simulation, and stressed input parameters – within the CreditRisk$^+$ framework would help to establish their relative reliability.

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