

FORWARD-CONVEX CONVERGENCE OF SEQUENCES IN \mathbb{L}_+^0

CONSTANTINOS KARDARAS AND GORDAN ŽITKOVIĆ

ABSTRACT. For a sequence $(f_n)_{n \in \mathbb{N}}$ in \mathbb{L}_+^0 , we provide simple necessary and sufficient conditions to ensure that each sequence $(h_n)_{n \in \mathbb{N}}$ of its forward convex combinations, the latter meaning that $h_n \in \text{conv}(\{f_n, f_{n+1}, \dots\})$ for all $n \in \mathbb{N}$, \mathbb{L}_+^0 -converges to the same limit. These conditions correspond to a measure-free version of the notion of uniform integrability and are related to the numéraire problem of mathematical finance.

0. INTRODUCTION

A growing body of work in applied probability in general, and in the field of mathematical finance in particular, has singled out \mathbb{L}^0 , the Fréchet space of a.s.-equivalence classes of random variables topologized by the convergence in probability, as especially important (see, e.g., [3, 8, 13, 16, 17]). Reasons for this are multiple, but if a single commonality is to be found, it would have to be the fact that \mathbb{L}^0 is measure-free. More precisely, the \mathbb{L}^0 -spaces built over the same measure space with different probabilities will coincide as long as the probabilities are equivalent. The desirability and necessity of the measure-free property in mathematical finance stems from the central tenet of replication (popularized by the work of Black, Scholes, Merton and others) which finds its mathematical expression in the theory of stochastic integration. Since replication amounts to complete removal of risk, the probability measure under which a financial system is modeled should not matter, modulo its negligible sets. On the other hand, given that general stochastic integration does not admit a canonical pathwise definition, we are left with \mathbb{L}^0 as the only proper setting for the theory. The only other measure-free member of the $(\mathbb{L}^p)_{p \in [0, \infty]}$ family, namely \mathbb{L}^∞ , turns out to be inadequately small for a large number of modeling tasks.

It is important to note that the interplay between \mathbb{L}^0 , the measure-free property, and stochastic integration, reaches farther into the history than the relatively recent progress in mathematical finance. The seminal work of Stricker ([15]) on the semimartingale property under absolutely-continuous changes of measures and the celebrated result of Dellacherie and Bichteller ([1, 2, 7]) on the theory of \mathbb{L}^0 -integrators are but two early examples. Even before that, results related to the

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measure-free structure of \mathbb{L}^0 , but without relation to stochastic integration, have been published (see, e.g., [4, 14]).

While \mathbb{L}^0 seems to fit the modeling requirements perfectly, there is a steep price that needs to be paid for its use: a large number of classical functional-analytic tools which were developed for locally-convex (and, in particular, Banach) spaces must be renounced. Indeed, \mathbb{L}^0 fails the local-convexity property in a dramatic fashion: if $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic, the topological dual of \mathbb{L}^0 is trivial (see [9], Theorem 2.2, p. 18). Therefore, a new set of functional-analytic tools which do not rely on local convexity (and the related tools such as the Hahn-Banach theorem) are needed to treat even the most basic applied problems. Specifically, convexity has to be “supplied endogenously”, leading to various substitutes for indispensable notions such as compactness (see [6, 12, 16]). A central idea behind their introduction is that a passage to a sequence of convex combinations, instead of a more classical passage to a subsequence, yields practically the same analytic benefit, while working much better with the barren structure of \mathbb{L}^0 . The situation is not as streamlined as in the classical case where true subsequences are considered. Indeed, there are examples of sequences $(f_n)_{n \in \mathbb{N}}$ in \mathbb{L}_+^0 (the nonnegative orthant of \mathbb{L}^0) that converge to zero, whereas the set of all possible limits of the convergent sequences $(h_n)_{n \in \mathbb{N}}$ such that $h_n \in \text{conv}(\{f_n, f_{n+1}, \dots\})$ is the *entire* \mathbb{L}_+^0 (see Example 1.3 for details).

It is a goal of the present paper to give necessary and sufficient conditions on a sequence $(f_n)_{n \in \mathbb{N}}$ in \mathbb{L}_+^0 to be *forward-convexly convergent*, i.e., such that each sequence of its forward convex combinations (meaning a sequence $(h_n)_{n \in \mathbb{N}}$ with $h_n \in \text{conv}(\{f_n, f_{n+1}, \dots\})$ for all $n \in \mathbb{N}$) converges in \mathbb{L}_+^0 to the same limit. Arguably, forward-convex convergence plays as natural a role in \mathbb{L}^0 as the strong convergence does in \mathbb{L}^1 -spaces. It rules out certain pathological limits and, as will be shown, imposes a measure-free locally-convex structure on the sequence. Put simply, it brings the benefits of local convexity to a naturally non-locally-convex framework.

As far as *sufficient* conditions for forward-convex convergence are concerned, the reader will quickly think of an example: almost sure convergence of the original sequence will do, for instance. Other than the obvious ones, useful *necessary* conditions are much harder to come by, and it is therefore surprising that one of our main results has such a simple form. It says, *inter alia*, that a sequence $(f_n)_{n \in \mathbb{N}}$ is forward-convexly convergent *if and only if* there exists a probability measure \mathbb{Q} in the equivalence class that generates the topology of \mathbb{L}^0 such that $(f_n)_{n \in \mathbb{N}}$ is $\mathbb{L}^1(\mathbb{Q})$ -convergent. Effectively, it identifies forward-convex convergence as a measure-free version of the notion of uniform integrability.

Interestingly, there is an alternate route towards better understanding of the role played by forward-convex convergence; it is inspired by recent work in mathematical finance by one of the authors (see [10]). For a sequence $(f_n)_{n \in \mathbb{N}}$ in \mathbb{L}_+^0 , one can form a nested family $(\mathcal{C}_n)_{n \in \mathbb{N}}$, each \mathcal{C}_n being the \mathbb{L}^0 -closure of $\text{conv}(\{f_n, f_{n+1}, \dots\})$, and interpret forward-convex convergence as convergence of each sequence $(h_n)_{n \in \mathbb{N}}$ with $h_n \in \mathcal{C}_n$ to the same limit. If \mathcal{C} denotes $\bigcap_n \mathcal{C}_n$, it can be shown that $(f_n)_{n \in \mathbb{N}}$ is forward-convexly convergent if and only if its limit f is a numéraire in \mathcal{C} , i.e.,

if it essentially maximizes the expected logarithmic utility (again, under some probability in the equivalence class generating the topology of \mathbb{L}^0) among all elements of \mathcal{C} . This extremality property can be viewed as a measure-free no-loss-of-mass condition on the original sequence, giving further support to the interpretation of forward-convex convergence as a variant of uniform integrability.

After this introduction, we give a brief review of the notation and terminology (both about \mathbb{L}^0 and the notion of a numéraire) and state our main result in Section 2. The bulk of the paper (Section 3) is dedicated to its proof, which is further divided into 4 logically separate parts. Given that the majority of applications of our results are in applied probability, we choose to use probabilistic notation and talk about expectations, random variables and convergence in probability instead of integrals, measurable functions and convergence in measure.

1. THE RESULT

1.1. Preliminaries. Let $(\Omega, \mathcal{F}, \overline{\mathbb{P}})$ be a probability space, and let Π be the collection of all probabilities on (Ω, \mathcal{F}) that are equivalent to (the representative) $\overline{\mathbb{P}} \in \Pi$. All probabilities in Π have the same sets of zero measure, which we shall be calling Π -null. We write “ Π -a.s.” to mean \mathbb{P} -a.s. with respect to any, and then all, $\mathbb{P} \in \Pi$.

By \mathbb{L}_+ we shall be denoting the set of all (equivalence classes modulo Π of) *possibly infinite-valued* nonnegative random variables on (Ω, \mathcal{F}) . We follow the usual practice of not differentiating between a random variable and the equivalence class it generates in \mathbb{L}_+ . The expectation of $f \in \mathbb{L}_+$ under $\mathbb{P} \in \Pi$ is denoted by $\mathbb{E}_{\mathbb{P}}[f]$. For fixed $\mathbb{P} \in \Pi$, we define a metric $d_{\mathbb{P}}$ on \mathbb{L}_+ via $d_{\mathbb{P}}(f, g) = \mathbb{E}_{\mathbb{P}}[|\exp(-f) - \exp(-g)|]$ for $f \in \mathbb{L}_+$ and $g \in \mathbb{L}_+$. The topology on \mathbb{L}_+ that is induced by the previous metric does not depend on $\mathbb{P} \in \Pi$; convergence of sequences in this topology is simply (extended) convergence in probability under any $\mathbb{P} \in \Pi$.

A set $\mathcal{C} \subseteq \mathbb{L}_+$ is *convex* if $(\alpha f + (1 - \alpha)h) \in \mathcal{C}$ whenever $f \in \mathcal{C}$, $g \in \mathcal{C}$ and $\alpha \in [0, 1]$, where the multiplication convention $0 \times \infty = 0$ is used. For $\mathcal{A} \subseteq \mathbb{L}_+$, $\text{conv}(\mathcal{A})$ denotes the smallest convex set that contains \mathcal{A} ; $\text{conv}(\mathcal{A})$ is just the set of all possible finite convex combinations of elements in \mathcal{A} . Further, $\overline{\text{conv}}(\mathcal{A})$ will denote the \mathbb{L}_+ -closure of $\text{conv}(\mathcal{A})$.

If $(f_n)_{n \in \mathbb{N}}$ is a sequence in \mathbb{L}_+ , any sequence $(h_n)_{n \in \mathbb{N}}$ such that $h_n \in \text{conv}(\{f_n, f_{n+1}, \dots\})$ for all $n \in \mathbb{N}$ will be called a *sequence of forward convex combinations* of $(f_n)_{n \in \mathbb{N}}$.

The set of all $f \in \mathbb{L}_+$ such that $\{f = \infty\}$ is Π -null is denoted by \mathbb{L}_+^0 . We endow \mathbb{L}_+^0 with the restriction of the \mathbb{L}_+ -topology; convergence of sequences under this topology is simply convergence in probability under any $\mathbb{P} \in \Pi$. When we write \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} f_n = f$, we tacitly imply that both the sequence $(f_n)_{n \in \mathbb{N}}$ and the limit f are elements of \mathbb{L}_+^0 .

A set $\mathcal{B} \subseteq \mathbb{L}_+^0$ is called \mathbb{L}_+^0 -bounded if $\downarrow \lim_{\ell \rightarrow \infty} \sup_{f \in \mathcal{B}} \mathbb{P}[f > \ell] = 0$ holds for some (and then for all) $\mathbb{P} \in \Pi$. If $\mathcal{B} \subseteq \mathbb{L}_+^0$ is \mathbb{L}_+^0 -bounded, its \mathbb{L}_+ -closure is a subset of \mathbb{L}_+^0 , and coincides with its \mathbb{L}_+^0 -closure.

1.2. Numéraires. We start with a result about certain “optimal” (extremal) elements of subsets of \mathbb{L}_+^0 .

Proposition 1.1. For $\mathcal{K} \subseteq \mathbb{L}_+^0$, let $h \in \mathcal{K}$ be such that $\{h = 0\} \subseteq \{f = 0\}$ holds for all $f \in \mathcal{K}$. Then, the following statements are equivalent:

- (1) There exists a σ -finite measure μ on (Ω, \mathcal{F}) , equivalent to the probabilities in Π , such that $\int h d\mu = \sup_{f \in \mathcal{K}} \int f d\mu < \infty$.
- (2) There exists $\mathbb{P} \in \Pi$ such that $\mathbb{E}_{\mathbb{P}}[f/h \mid h > 0] \leq 1$ holds for all $f \in \mathcal{K}$.

Proof. We exclude from the discussion the trivial case $\mathcal{K} = \{0\}$ so that $\{h > 0\}$ is not Π -null.

First, assume (1). If $\mu[h = 0] = \infty$, we can easily redefine it so that $\mu[h = 0] < \infty$ without affecting the values of the integrals $\int f d\mu$, for $f \in \mathcal{K}$. Therefore, we can assume that $\mu[h = 0] < \infty$. Define $\mathbb{P} \in \Pi$ via

$$\mathbb{P}[A] = \frac{1}{2} \frac{\int_A h d\mu}{\int h d\mu} + \frac{1}{2} \frac{\mu[A \cap \{h = 0\}]}{\mu[\{h = 0\}]}, \text{ for } A \in \mathcal{F},$$

using the convention $0/0 = 1$. Then, $\mathbb{E}_{\mathbb{P}}[f/h \mid h > 0] = \int f d\mu / \int h d\mu \leq 1$ holds for all $f \in \mathcal{K}$.

Conversely, assume (2) and define $\mu : \mathcal{F} \mapsto \mathbb{R}_+ \cup \{\infty\}$ via

$$\mu[A] = \mathbb{E}_{\mathbb{P}} \left[\left(\frac{1}{h} \mathbb{I}_{\{h > 0\}} + \mathbb{I}_{\{h = 0\}} \right) \mathbb{I}_A \right], \text{ for } A \in \mathcal{F}.$$

It is apparent that μ is a σ -finite measure, equivalent to $\mathbb{P} \in \Pi$. Moreover, for any $f \in \mathcal{K}$, we have

$$\int f d\mu = \mathbb{E}_{\mathbb{P}}[(f/h) \mathbb{I}_{\{h > 0\}}] = \mathbb{E}_{\mathbb{P}}[f/h \mid h > 0] \mathbb{P}[h > 0] \leq \mathbb{P}[h > 0] = \int h d\mu,$$

which completes the proof. \square

Definition 1.2. An element $h \in \mathcal{K} \subseteq \mathbb{L}_+^0$ such that $\{h = 0\} \subseteq \{f = 0\}$ for all $f \in \mathcal{K}$ that has one of the equivalent properties of Proposition 1.1 will be called a *numéraire* in \mathcal{K} . The set of all possible numéraires in \mathcal{K} will be denoted by \mathcal{K}^{num} .

By condition (1) of Proposition 1.1, \mathcal{K}^{num} exactly consists of elements in \mathcal{K} that are supported by a “dual” σ -finite measure μ , equivalent to all probabilities in Π . Clearly, the linear mapping $\mathbb{L}_+^0 \ni f \mapsto \int f d\mu$ is in general extended-valued and only lower semicontinuous; therefore, μ does not define a dual element in the strict functional-analytic sense. However, “morally” speaking, \mathcal{K}^{num} coincides with the set of possible maximizers of strictly increasing and strictly concave functionals on \mathcal{K} . In fact, if $h \in \mathcal{K}^{\text{num}}$ and $\mathbb{P} \in \Pi$ are as described in condition (2) of Proposition 1.1, h is essentially the element in \mathcal{K} that maximizes the functional $\mathcal{K} \ni f \mapsto \mathbb{E}_{\mathbb{P}}[\log(f)]$. We use the quantifier “essentially” because the last problem might not be well-posed, and an approximating procedure has to be utilized in order to construct h , as can be seen from the proof of Theorem 1.1(4) in [10].

1.3. The main result. Having introduced all the ingredients, we are ready to state our main equivalence result. Before we do that, we pause to give an example of the kind of pathological behavior we are trying to outlaw.

Example 1.3. Take $\Omega = (0, 1]$ equipped with the Borel σ -field and Lebesgue measure \mathbb{P} , and define the sequence $(f_n)_{n \in \mathbb{N}}$ by

$$f_n = (m-1)2^{m-1} \mathbb{I}_{((k-1)/2^{m-1}, k/2^{m-1}]}, \text{ for } n = 2^{m-1} + k - 1 \text{ with } m \in \mathbb{N} \text{ and } 1 \leq k \leq 2^{m-1}.$$

It is straightforward to check that \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} f_n = 0$, but as we shall show below, this sequence behaves in a strange way: for any $f \in \mathbb{L}_+^0$, there exists a sequence $(h_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ such that \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} h_n = f$.

We start by noting that it suffices to establish the above claim only for $f \in \mathbb{L}_+^\infty$; and, consequently, pick $f \in \mathbb{L}_+^\infty$ with $f \leq M$ for some $M \in \mathbb{R}_+$. For each $m \in \mathbb{N}$, let \mathcal{F}_m be the σ -field on Ω generated by the intervals $((k-1)2^{-m}, k2^{-m}]$, $1 \leq k \leq 2^m$. For $m \in \mathbb{N}$, define $g_m := \mathbb{E}_{\mathbb{P}}[f \mid \mathcal{F}_m]$; by the martingale convergence theorem, \mathbb{L}_+^0 - $\lim_{m \rightarrow \infty} g_m = f$. Furthermore,

$$g_m = \sum_{k=1}^{2^m} 2^m \mathbb{E}_{\mathbb{P}} [h \mathbb{I}_{((k-1)/2^m, k/2^m]}] \mathbb{I}_{((k-1)/2^m, k/2^m]} = \sum_{k=1}^{2^m} \frac{\mathbb{E}_{\mathbb{P}} [h \mathbb{I}_{((k-1)/2^m, k/2^m]}]}{m} f_{2^m+k-1}.$$

Set $\alpha_{m,k} = m^{-1} \mathbb{E}_{\mathbb{P}} [h \mathbb{I}_{((k-1)/2^m, k/2^m]}] \in \mathbb{R}_+$ for $m \in \mathbb{N}$ and $1 \leq k \leq 2^m$, so that, for $m \geq M$, we have

$$\sum_{k=1}^{2^m} \alpha_{m,k} = \frac{\mathbb{E}_{\mathbb{P}}[h]}{m} \leq \frac{M}{m} \leq 1.$$

Define the sequence $(h_n)_{n \in \mathbb{N}}$ as follows: for $m \in \mathbb{N}$ with $m < M$, simply set $h_{2^{m-1}+k-1} = f_{2^{m-1}+k-1}$ for all $1 \leq k \leq 2^{m-1}$, while for $m \in \mathbb{N}$ with $m \geq M$ set

$$h_{2^{m-1}+k-1} = \left(1 - \sum_{\ell=1}^{2^m} \alpha_{m,\ell}\right) f_{2^m} + \sum_{k=1}^{2^m} \alpha_{m,\ell} f_{2^m+\ell-1} = \left(1 - \sum_{\ell=1}^{2^m} \alpha_{m,\ell}\right) f_{2^m} + g_m$$

for all $1 \leq k \leq 2^{m-1}$. Then, $(h_n)_{n \in \mathbb{N}}$ is a sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$, and \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} h_n = f$.

Theorem 1.4. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{L}_+^0 . Set $\mathcal{C} := \bigcap_{n \in \mathbb{N}} \overline{\text{conv}}(\{f_n, f_{n+1}, \dots\})$. If*

$$\text{(CONV)} \quad \mathbb{L}_+^0\text{-}\lim_{n \rightarrow \infty} f_n = f.$$

holds for some $f \in \mathbb{L}_+^0$, then the following statements are equivalent:

- (1) *Every sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ \mathbb{L}_+^0 -converges to f .*
- (2) *Whenever a sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ is \mathbb{L}_+ -convergent, its \mathbb{L}_+ -limit is f .*
- (3) $\mathcal{C} = \{f\}$.
- (4) $\mathcal{C} \subseteq \mathbb{L}_+^0$ and $f \in \mathcal{C}^{\text{num}}$.
- (5) *There exists $\mathbb{Q} \in \Pi$ such that $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[f_n] < \infty$ and $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[|f_n - f|] = 0$.*

With (CONV) holding, and under any of the above equivalent conditions, we have

$$(1.1) \quad \overline{\text{conv}}(\{f_1, f_2, \dots\}) = \left\{ \sum_{n \in \mathbb{N}} \alpha_n f_n + \left(1 - \sum_{n \in \mathbb{N}} \alpha_n\right) f \mid (\alpha_n)_{n \in \mathbb{N}} \in \Delta^{\mathbb{N}} \right\}.$$

where $\Delta^{\mathbb{N}}$ is the infinite-dimensional simplex:

$$\Delta^{\mathbb{N}} := \left\{ \alpha = (\alpha_n)_{n \in \mathbb{N}} \mid \alpha_n \in \mathbb{R}_+ \text{ for all } n \in \mathbb{N}, \text{ and } \sum_{n \in \mathbb{N}} \alpha_n \leq 1 \right\}.$$

Furthermore, with \mathbb{Q} being any probability in Π that satisfies statement (5) above, the \mathbb{L}_+^0 -topology on $\overline{\text{conv}}(\{f_1, f_2, \dots\})$ coincides with the $\mathbb{L}_+^1(\mathbb{Q})$ -topology, (in particular, $\overline{\text{conv}}(\{f_1, f_2, \dots\})$ with the \mathbb{L}_+^0 -topology is locally convex), and $\overline{\text{conv}}(\{f_1, f_2, \dots\})$ is \mathbb{L}_+^0 -compact; in fact, it is $\mathbb{L}_+^1(\mathbb{Q})$ -compact.

In the special case $f = 0$, the equivalences of the above five statements and the properties discussed after them hold even without assumption (CONV).

Implications (1) \Rightarrow (2), (2) \Rightarrow (3), (3) \Rightarrow (4) and (5) \Rightarrow (1) are all straightforward, and (CONV) is *not* required. Indeed, (1) \Rightarrow (2) and (3) \Rightarrow (4) are completely trivial. For implication (2) \Rightarrow (3), observe that \mathcal{C} coincides with the set of all possible \mathbb{L}_+ -limits of sequences $(h_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$. Finally, implication (5) \Rightarrow (1) is immediate since

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}} [|h_n - f|] \leq \limsup_{n \rightarrow \infty} \left(\sup_{\mathbb{N} \ni k \geq n} \mathbb{E}_{\mathbb{Q}} [|f_k - f|] \right) = 0$$

holds for any sequence $(h_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$.

The proof of implication (4) \Rightarrow (5) is significantly harder, and will be discussed in Section 2. We shall first deal with the case $f = 0$, where (CONV) will not be assumed. Then, we proceed with the proof of (4) \Rightarrow (5) in the general case. There is a simple argument that reduces the proof of implication (4) \Rightarrow (5) to the special case $f = 0$; however, in order to be able to carry out this argument one needs to first establish (4) \Rightarrow (1), which is quite technical.

Remark 1.5. Consider an \mathbb{L}_+^0 -convergent sequence $(f_n)_{n \in \mathbb{N}}$, and set $f := \mathbb{L}_+^0\text{-}\lim_{n \rightarrow \infty} f_n$. From a qualitative viewpoint, Theorem 1.4 helps to understand the cases where there exists a sequence $(h_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ that \mathbb{L}_+^0 -converges to some limit other than f . Indeed, by statement (4), this happens *if and only if* $f \notin \mathcal{C}^{\text{num}}$, in other words, if $(f_n)_{n \in \mathbb{N}}$ converges to a “suboptimal” limit of all the possible limits of its sequences of forward convex combinations.

Remark 1.6. In the special case $f = 0$, (CONV) is not needed in Theorem 1.4. However, when $f \neq 0$, (CONV) is crucial for (3) \Rightarrow (1) of Theorem 1.4 to hold. We present below an example to illustrate this fact.

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to accommodate a sequence $(f_n)_{n \in \mathbb{N}}$ of random variables that are independent under \mathbb{P} and have identical distributions given by $\mathbb{P}[f_n = 0] = \mathbb{P}[f_n = 2] = 1/2$. By Kolmogorov’s zero-one law, it follows that any possible \mathbb{L}_+ -limit of sequence of convex combinations of $(f_n)_{n \in \mathbb{N}}$ has to be constant. Now, $(f_n)_{n \in \mathbb{N}}$ is uniformly integrable (in fact, uniformly bounded) under \mathbb{P} , which means that $\mathcal{C} = \{1\}$. With $f = 1$ we have all (2), (3) and (4) of Theorem 1.4 holding. However, both (1) and (5) fail.

Remark 1.7. Our treatment only applies for sequences in \mathbb{L}_+^0 , since it uses the concept of the numéraire, only defined for subsets of \mathbb{L}_+^0 . It would be interesting to obtain a similar result for \mathbb{L}^0 .

2. PROOF OF THEOREM 1.4

We split the proof in four steps, indicating each time what is being proved. For the first two steps, and in particular until the case $f = 0$ has been treated, condition (CONV) is not assumed. We introduce some notation that will be used throughout the proof: for $n \in \mathbb{N}$, set $\mathcal{C}_n := \overline{\text{conv}}(\{f_n, f_{n+1}, \dots\}) \subseteq \mathbb{L}_+$ so that $\mathcal{C} = \bigcap_{n \in \mathbb{N}} \mathcal{C}_n$. Also, let $\mathcal{S}_n \subseteq \mathbb{L}_+$ be the *solid hull* of \mathcal{C}_n : $g \in \mathcal{S}_n$ if and only if $0 \leq g \leq h$ for some $h \in \mathcal{C}_n$. It is clear that \mathcal{S}_n is convex and solid, and that $\mathcal{C}_n \subseteq \mathcal{S}_n$.

2.1. $\mathcal{C} \subseteq \mathbb{L}_+^0$ implies that $\overline{\text{conv}}(\{f_1, f_2, \dots\})$ is \mathbb{L}_+^0 -bounded. We start by showing that \mathcal{S}_n is \mathbb{L}_+ -closed, for $n \in \mathbb{N}$. For that, we pick an \mathcal{S}_n -valued sequence $(g_k)_{k \in \mathbb{N}}$ that converges \mathbb{P} -a.s. to $g \in \mathbb{L}_+$. Let $(h_k)_{k \in \mathbb{N}}$ be a \mathcal{C}_n -valued sequence with $g_k \leq h_k$ for all $k \in \mathbb{N}$. By Lemma A.1 in [5], we can extract a sequence $(\tilde{h}_k)_{k \in \mathbb{N}}$ of forward convex combinations of $(h_k)_{k \in \mathbb{N}}$ such that $h := \lim_{k \rightarrow \infty} \tilde{h}_k \in \mathbb{L}_+$ Π -a.s. exists. Of course, $h \in \mathcal{C}_n$ and it is straightforward that $g \leq h$. We conclude that $g \in \mathcal{S}_n$, i.e., \mathcal{S}_n is \mathbb{L}_+ -closed.

Let $\mathcal{S} = \bigcap_{n \in \mathbb{N}} \mathcal{S}_n$; then, $\mathcal{C} \subseteq \mathcal{S}$ and \mathcal{S} is \mathbb{L}_+ -closed, convex and solid. We claim that \mathcal{S} actually is the solid hull of \mathcal{C} ; to show this, we only need to establish that for any $g \in \mathcal{S}$ there exists $h \in \mathcal{C}$ with $g \leq h$. For all $n \in \mathbb{N}$, since $g \in \mathcal{S} \subseteq \mathcal{S}_n$, there exists $h_n \in \mathcal{C}_n$ with $g \leq h_n$. By another application of Lemma A.1 from [5], we can extract a sequence $(\tilde{h}_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(h_n)_{n \in \mathbb{N}}$ such that $h := \mathbb{L}_+ \text{-} \lim_{k \rightarrow \infty} \tilde{h}_k$ exists. Then, $h \in \mathcal{C}$ and $g \leq h$.

Each \mathcal{S}_n is \mathbb{L}_+ -closed, convex and solid; therefore, a straightforward generalization of Lemma 2.3 in [3] gives, for each $n \in \mathbb{N}$, the existence of a partition $\Omega = \Phi_n \cup (\Omega \setminus \Phi_n)$, where $\Phi_n \in \mathcal{F}$, $\{f \mathbb{I}_{\Phi_n} \mid f \in \mathcal{S}_n\}$ is \mathbb{L}_+^0 -bounded, while $h \mathbb{I}_{\Omega \setminus \Phi_n} \in \mathcal{S}_n$ for all $h \in \mathbb{L}_+$. Clearly, $\mathcal{C}_n \supseteq \mathcal{C}_{n+1}$ implies $\Phi_n \subseteq \Phi_{n+1}$, for all $n \in \mathbb{N}$. However, since $f_n \in \mathbb{L}_+^0$, i.e., $\{f_n = \infty\}$ is Π -null for all $n \in \mathbb{N}$, it follows that $\Phi_{n+1} = \Phi_n$ for all $n \in \mathbb{N}$. In other words, $\Phi_n = \Phi_1$ for all $n \in \mathbb{N}$. Then, $h \mathbb{I}_{\Omega \setminus \Phi_1} \in \mathcal{S}$ for all $h \in \mathbb{L}_+$. Since $\mathcal{C} \subseteq \mathbb{L}_+^0$, and, therefore, $\mathcal{S} \subseteq \mathbb{L}_+^0$ as well, it follows that $\Omega \setminus \Phi_1$ is Π -null. Therefore, \mathcal{S}_1 is \mathbb{L}_+^0 -bounded, which completes this part of the proof. Observe that all \mathcal{S}_n , $n \in \mathbb{N}$, are convex, solid, \mathbb{L}_+^0 -bounded, and \mathbb{L}_+^0 -closed; we shall use this later.

2.2. Equivalence of (1), (2), (3), (4) and (5) in Theorem 1.4 when $f = 0$. As already discussed, the proofs of (1) \Rightarrow (2), (2) \Rightarrow (3), (3) \Rightarrow (4) and (5) \Rightarrow (1) are immediate, and (CONV) is not used. Here, we prove (4) \Rightarrow (5) when $f = 0$ without assuming (CONV).

Since \mathcal{S}_1 is \mathbb{L}_+^0 -bounded, there exists $\mathbb{P} \in \Pi$ such that $\sup_{h \in \mathcal{S}_1} \mathbb{E}_{\mathbb{P}}[h] < \infty$; in particular, $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}}[f_n] < \infty$. Given the existence of such $\mathbb{P} \in \Pi$, the following result will be useful in order to extract a probability $\mathbb{Q} \in \Pi$ that satisfies condition (5) of Theorem 1.4.

Lemma 2.1. *Fix $\mathbb{P} \in \Pi$ with $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}}[f_n] < \infty$. Then, the following statements are equivalent:*

- (1) *For some $\mathbb{Q} \in \Pi$, $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[f_n] < \infty$ and $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[f_n] = 0$.*
- (2) *For any $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{F}$ such that $\mathbb{P}[\Omega \setminus A_\epsilon] \leq \epsilon$ and $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{A_\epsilon}] = 0$.*

Proof. First assume (1) in the statement of Lemma 2.1. Define $Z := d\mathbb{Q}/d\mathbb{P}$; then, $\mathbb{P}[Z > 0] = 1$. For fixed $\epsilon > 0$, let $\delta = \delta(\epsilon) > 0$ be such that, with $A_\epsilon := \{Z > \delta\} \in \mathcal{F}$, $\mathbb{P}[\Omega \setminus A_\epsilon] \leq \epsilon$ holds. Then,

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{A_\epsilon}] = \limsup_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[(1/Z) f_n \mathbb{I}_{\{Z > \delta\}}] \leq (1/\delta) \limsup_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[f_n] = 0.$$

Now, assume (2) in the statement of Lemma 2.1. For each $k \in \mathbb{N}$, let $B_k \in \mathcal{F}$ be such that $\mathbb{P}[\Omega \setminus B_k] \leq 1/k$ and $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{B_k}] = 0$. By replacing B_k with $\bigcup_{m=1}^k B_m$ for each $k \in \mathbb{N}$ consecutively, we may assume without loss of generality that $(B_k)_{k \in \mathbb{N}}$ is a nondecreasing sequence of sets in \mathcal{F} with $\lim_{k \rightarrow \infty} \mathbb{P}[B_k] = 1$, as well as that $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{B_k}] = 0$ holds for each fixed $k \in \mathbb{N}$. Define $B_0 = \emptyset$, $n_0 = 0$, and a *strictly* increasing \mathbb{N} -valued sequence $(n_k)_{k \in \mathbb{N}}$ with the following property: for all $k \in \mathbb{N}$, $\mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{B_k}] \leq 1/k$ holds for all $n \geq n_{k-1}$. (Observe that this is trivially valid for $k = 1$.) Then, define a sequence $(E_n)_{n \in \mathbb{N}}$ of sets in \mathcal{F} by setting $E_n = B_k$ whenever $n_{k-1} \leq n < n_k$. It is clear that $(E_n)_{n \in \mathbb{N}}$ is a nondecreasing sequence, that $\lim_{n \rightarrow \infty} \mathbb{P}[E_n] = 1$, and that $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{E_n}] = 0$. With $E_0 := \emptyset$, define $Z := c \sum_{n \in \mathbb{N}} 2^{-n} \mathbb{I}_{E_n \setminus E_{n-1}}$, where $c > 0$ is a normalizing constant in order to ensure that $\mathbb{E}_{\mathbb{P}}[Z] = 1$. Define $\mathbb{Q} \in \Pi$ via $\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}[Z \mathbb{I}_A]$ for all $A \in \mathcal{F}$. With $K := \sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}}[f_n] < \infty$, $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[f_n] \leq c \sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}}[f_n] = cK < \infty$. Furthermore,

$$\mathbb{E}_{\mathbb{Q}}[f_n] = \mathbb{E}_{\mathbb{Q}}[f_n \mathbb{I}_{E_n}] + \mathbb{E}_{\mathbb{Q}}[f_n \mathbb{I}_{\Omega \setminus E_n}] \leq c \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{E_n}] + c 2^{-n} \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{\Omega \setminus E_n}] \leq c \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{E_n}] + cK 2^{-n}.$$

Since $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{E_n}] = 0$, we obtain $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[f_n] = 0$, which completes the argument. \square

We continue with the proof of the implication (4) \Rightarrow (5), *fixing* $\mathbb{P} \in \Pi$ *with* $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}}[f_n] < \infty$ *until the end of §2.2.*

For any $\mathcal{A} \subseteq \mathbb{L}_+^0$, define its *polar* $\mathcal{A}^\circ := \{g \in \mathbb{L}_+^0 \mid \mathbb{E}_{\mathbb{P}}[gh] \leq 1 \text{ for all } h \in \mathcal{A}\}$. It is straightforward that $(\bigcup_{n \in \mathbb{N}} \mathcal{A}_n)^\circ = \bigcap_{n \in \mathbb{N}} \mathcal{A}_n^\circ$, for all collections $\{\mathcal{A}_n \mid n \in \mathbb{N}\}$ of subsets of \mathbb{L}_+^0 . Also, consider the *bipolar* $\mathcal{A}^{\circ\circ} := (\mathcal{A}^\circ)^\circ$ of \mathcal{A} ; Theorem 1.3 of [3] states that if a set is convex and solid, $\mathcal{A}^{\circ\circ}$ coincides with the \mathbb{L}_+^0 -closure of \mathcal{A} .

For each $n \in \mathbb{N}$, $\mathcal{S}_n \subseteq \mathbb{L}_+^0$ is convex, solid and \mathbb{L}_+^0 -closed; therefore, $\mathcal{S}_n^{\circ\circ} = \mathcal{S}_n$. Since $\mathcal{S} = \bigcap_{n \in \mathbb{N}} \mathcal{S}_n$ is the solid hull of $\mathcal{C} = \{0\}$, i.e., $\mathcal{S} = \{0\}$, we have

$$\left(\bigcup_{n \in \mathbb{N}} \mathcal{S}_n^\circ \right)^{\circ\circ} = \left(\bigcap_{n \in \mathbb{N}} \mathcal{S}_n^{\circ\circ} \right)^\circ = \left(\bigcap_{n \in \mathbb{N}} \mathcal{S}_n \right)^\circ = \{0\}^\circ = \mathbb{L}_+^0.$$

Since $\bigcup_{n \in \mathbb{N}} \mathcal{S}_n^\circ$ is convex and solid, the above means that the \mathbb{L}_+^0 -closure of $\bigcup_{n \in \mathbb{N}} \mathcal{S}_n^\circ$ is \mathbb{L}_+^0 .

Fix $\epsilon > 0$. Define a \mathbb{N} -valued and strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ with the following property: for all $k \in \mathbb{N}$ there exists $g_k \in \mathcal{S}_{n_k}^\circ$ such that $\mathbb{P}[|g_k - 2k| \leq k] \leq \epsilon 2^{-(k+1)}$. (This can be done in view of the fact that the \mathbb{L}_+^0 -closure of $\bigcup_{n \in \mathbb{N}} \mathcal{S}_n^\circ$ is \mathbb{L}_+^0 .) In particular, $\mathbb{P}[g_k \leq k] \leq \epsilon 2^{-(k+1)}$ and $\mathbb{E}_{\mathbb{P}}[g_k f_n] \leq 1$ hold for all $k \in \mathbb{N}$ and $n \geq n_k$. Define $A_\epsilon := \bigcap_{k \in \mathbb{N}} \{g_k > k\}$; then, $\mathbb{P}[\Omega \setminus A_\epsilon] \leq \epsilon$. Furthermore, for all $k \in \mathbb{N}$ and $n \geq n_k$,

$$\mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{A_\epsilon}] \leq \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{\{g_k > k\}}] \leq \mathbb{E}_{\mathbb{P}}[(g_k/k) f_n \mathbb{I}_{\{g_k > k\}}] \leq (1/k) \mathbb{E}_{\mathbb{P}}[g_k f_n] \leq 1/k.$$

Then, $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[f_n \mathbb{I}_{A_\epsilon}] = 0$. Invoking Lemma 2.1, we obtain the existence of $\mathbb{Q} \in \Pi$ such that $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[f_n] < \infty$ and $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[f_n] = 0$.

2.3. Equivalence of (1), (2), (3), (4) and (5) in Theorem 1.4: general case. We shall now tackle the general case $f \in \mathbb{L}_+^0$. Of course (1) \Rightarrow (2), (2) \Rightarrow (3), (3) \Rightarrow (4) and (5) \Rightarrow (1) are still trivially valid. Here, we shall first show (4) \Rightarrow (1), and then use this to reduce the proof of (5) \Rightarrow (1) to the special case $f = 0$, which we have already established. *For the purposes of §2.3, we work under the assumption (CONV).*

2.3.1. Proof of (4) \Rightarrow (1). The first line of business is to reduce the proof of (4) \Rightarrow (1) to the case where $f = 1$. Consider a new \mathbb{L}_+^0 -valued sequence $(\tilde{f}_n)_{n \in \mathbb{N}}$ defined via $\tilde{f}_n = f_n \mathbb{I}_{\{f=0\}}$ for all $n \in \mathbb{N}$. With $\tilde{\mathcal{C}}$ being the set of all possible \mathbb{L}_+ -limits of $(\tilde{f}_n)_{n \in \mathbb{N}}$, i.e., the equivalent of the set \mathcal{C} for the sequence $(\tilde{f}_n)_{n \in \mathbb{N}}$ in place of $(f_n)_{n \in \mathbb{N}}$, we shall show that $\tilde{\mathcal{C}} = \{0\}$. Let $(\tilde{h}_n)_{n \in \mathbb{N}}$ be a sequence of forward convex combinations of $(\tilde{f}_n)_{n \in \mathbb{N}}$ such that $\tilde{h} := \mathbb{L}_+-\lim_{n \rightarrow \infty} \tilde{h}_n$ exists. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ such that $\tilde{h}_n = g_n \mathbb{I}_{\{f=0\}}$ holds for all $n \in \mathbb{N}$. Even though $(g_n)_{n \in \mathbb{N}}$ might not be \mathbb{L}_+ -convergent, Lemma A.1 in [5] gives the existence of a sequence $(h_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(g_n)_{n \in \mathbb{N}}$ that Π -a.s. converges to some $h \in \mathbb{L}_+$. Of course, $h \mathbb{I}_{\{f=0\}} = \mathbb{L}_+-\lim_{n \rightarrow \infty} (h_n \mathbb{I}_{\{f=0\}}) = \tilde{h}$, and $(h_n)_{n \in \mathbb{N}}$ is a sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$. It follows that $h \in \mathcal{C}$; since $f \in \mathcal{C}^{\text{num}}$, we have $h \mathbb{I}_{\{f=0\}} = 0$. This, in turn, implies that $\tilde{h} = 0$. Therefore, $\tilde{\mathcal{C}} = \{0\}$. From the already-established validity of (4) \Rightarrow (1) in the special case of zero limit, we obtain that $\mathbb{L}_+^0\text{-}\lim_{n \rightarrow \infty} (h_n \mathbb{I}_{\{f=0\}}) = 0$ holds for any sequence $(h_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$. It follows that in order to show that $\mathbb{L}_+^0\text{-}\lim_{n \rightarrow \infty} h_n = f$ holds for any sequence $(h_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ it suffices to show that $\mathbb{L}_+^0\text{-}\lim_{n \rightarrow \infty} (h_n \mathbb{I}_{\{f>0\}}) = f$ holds for any sequence $(h_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$. Redefine the sequence $(\tilde{f}_n)_{n \in \mathbb{N}}$ via $\tilde{f}_n = (f_n/f) \mathbb{I}_{\{f>0\}} + \mathbb{I}_{\{f=0\}}$, as well as the set $\tilde{\mathcal{C}}$. Clearly, $\tilde{\mathcal{C}} = \{(h/f) \mathbb{I}_{\{f>0\}} + \mathbb{I}_{\{f=0\}} \mid h \in \mathcal{C}\}$. By (CONV), $\mathbb{L}_+^0\text{-}\lim_{n \rightarrow \infty} \tilde{f}_n = 1$. Furthermore, $\mathcal{C} \subseteq \mathbb{L}_+^0$ and $f \in \mathcal{C}^{\text{num}}$ imply $\tilde{\mathcal{C}} \subseteq \mathbb{L}_+^0$ and $1 \in \tilde{\mathcal{C}}^{\text{num}}$. If we show that any sequence $(\tilde{h}_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(\tilde{f}_n)_{n \in \mathbb{N}}$ \mathbb{L}_+^0 -converges to $\tilde{f} = 1$, it will follow that $\mathbb{L}_+^0\text{-}\lim_{n \rightarrow \infty} (h_n \mathbb{I}_{\{f>0\}}) = f$ holds for any sequence $(h_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$.

To recapitulate: *we only have to show the implication (4) \Rightarrow (1) for the special case $f = 1$. Therefore, we assume that $f = 1$ until the end of the proof of implication (4) \Rightarrow (1).*

In order to proceed, we shall need a general result — see Theorem 1.1(4) of [10].

Theorem 2.2. *Let $\mathcal{K} \subseteq \mathbb{L}_+^0$ be convex, closed and \mathbb{L}_+^0 -bounded. Then, for all $\mathbb{P} \in \Pi$ there exists $\hat{f} = \hat{f}(\mathbb{P}) \in \mathcal{K}$ such that $\{\hat{f} = 0\} \subseteq \{f = 0\}$ and $\mathbb{E}_{\mathbb{P}}[f/\hat{f} \mid \hat{f} > 0] \leq 1$ holds for all $f \in \mathcal{K}$.*

For a convex, closed and \mathbb{L}_+^0 -bounded $\mathcal{K} \subseteq \mathbb{L}_+^0$ and $\mathbb{P} \in \Pi$, it is easy to see that an element $\hat{f} \in \mathcal{K}$ satisfying the condition of Theorem 2.2 is unique. We shall call it *the numéraire in \mathcal{K} under \mathbb{P}* .

The following two results will be used in the sequel. Although they appear also in [11], we include here their proofs for completeness.

Lemma 2.3. *Fix $\mathbb{P} \in \Pi$. Consider two \mathbb{L}_+^0 -valued sequences $(g_n)_{n \in \mathbb{N}}$, $(h_n)_{n \in \mathbb{N}}$ such that $\mathbb{E}_{\mathbb{P}}[g_n] \leq 1$ and $\mathbb{E}_{\mathbb{P}}[h_n] \leq 1$ for all $n \in \mathbb{N}$, as well as \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (g_n h_n) = 1$. Then, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} g_n = 1$ and \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} h_n = 1$.*

Proof. The fact that \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (g_n h_n) = 1$ implies that \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \sqrt{g_n h_n} = 1$; then

$$\limsup_{n \rightarrow \infty} \left(1 - \mathbb{E}_{\mathbb{P}} \left[\sqrt{g_n h_n} \right] \right) = 1 - \liminf_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[\sqrt{g_n h_n} \right] \leq 0,$$

as follows from Fatou's Lemma. Since

$$\mathbb{E}_{\mathbb{P}} \left[\left(\sqrt{g_n} - \sqrt{h_n} \right)^2 \right] = \mathbb{E}_{\mathbb{P}}[g_n] + \mathbb{E}_{\mathbb{P}}[h_n] - 2\mathbb{E}_{\mathbb{P}} \left[\sqrt{g_n h_n} \right] \leq 2 \left(1 - \mathbb{E}_{\mathbb{P}} \left[\sqrt{g_n h_n} \right] \right),$$

\mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (\sqrt{g_n} - \sqrt{h_n}) = 0$ holds. In view of $g_n - h_n = (\sqrt{g_n} - \sqrt{h_n})(\sqrt{g_n} + \sqrt{h_n})$ and the \mathbb{L}_+^0 -boundedness of both sequences $(g_n)_{n \in \mathbb{N}}$, $(h_n)_{n \in \mathbb{N}}$ (holding because $\mathbb{E}_{\mathbb{P}}[g_n] \leq 1$ and $\mathbb{E}_{\mathbb{P}}[h_n] \leq 1$ for all $n \in \mathbb{N}$), we also have \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (g_n - h_n) = 0$. Furthermore, the equality $g_n + h_n = (\sqrt{g_n} - \sqrt{h_n})^2 + 2\sqrt{g_n h_n}$ gives \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (g_n + h_n) = 2$. Finally, combining \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (g_n - h_n) = 0$ and \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (g_n + h_n) = 2$, gives \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} g_n = 1 = \mathbb{L}_+^0$ - $\lim_{n \rightarrow \infty} h_n$. \square

What follows is a convergence result regarding a sequence of numéraires of nonincreasing convex, closed and bounded subsets of \mathbb{L}_+^0 .

Proposition 2.4. *Fix $\mathbb{P} \in \Pi$. Let $(\mathcal{K}_n)_{n \in \mathbb{N}}$ be a nonincreasing sequence of closed, convex and bounded subsets of \mathbb{L}_+^0 , and let $\mathcal{K} := \bigcap_{n \in \mathbb{N}} \mathcal{K}_n$. For each $n \in \mathbb{N}$, let \widehat{f}_n be the numéraire in \mathcal{K}_n under \mathbb{P} . Also, let \widehat{f} be the numéraire in \mathcal{K} under \mathbb{P} . (These numéraires exist in view of Theorem 2.2.) Assume that $\{\widehat{f} = 0\}$ is Π -null. Then, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \widehat{f}_n = \widehat{f}$.*

Proof. We shall actually establish the existence of a subsequence $(\widehat{f}_{m_n})_{n \in \mathbb{N}}$ of $(\widehat{f}_n)_{n \in \mathbb{N}}$ such that \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \widehat{f}_{m_n} = \widehat{f}$. By the same argument, it will follow that *any* subsequence of $(\widehat{f}_n)_{n \in \mathbb{N}}$ has a *further* subsequence that \mathbb{L}_+^0 -converges to \widehat{f} . Since \mathbb{L}_+^0 is equipped with a metric topology, it will follow that the whole sequence $(\widehat{f}_n)_{n \in \mathbb{N}}$ converges to \widehat{f} .

First, one applies Lemma A.1 from [5] to get the existence of a sequence $(\widetilde{f}_n)_{n \in \mathbb{N}}$ such that $\widetilde{f}_n \in \text{conv}(\{\widehat{f}_n, \dots, \widehat{f}_{\ell_n}\})$, where $n \leq \ell_n \in \mathbb{N}$, and such that $\widetilde{f} := \mathbb{L}_+^0$ - $\lim_{n \rightarrow \infty} \widetilde{f}_n$ exists. We may assume that $(\ell_n)_{n \in \mathbb{N}}$ is an increasing sequence, forcing it to be so if necessary. The fact that $\{\widehat{f} = 0\}$ is Π -null implies that $\{\widehat{f}_n = 0\}$ is Π -null for all $n \in \mathbb{N}$. Therefore, $\{\widetilde{f}_n = 0\}$ is Π -null for all $n \in \mathbb{N}$. Since $\mathbb{E}_{\mathbb{P}}[f/\widehat{f}_n] \leq 1$ holds for all $f \in \mathcal{K}$ and $n \in \mathbb{N}$, the convexity of function $\mathbb{R}_{++} \ni x \mapsto 1/x \in \mathbb{R}_+$ gives $\mathbb{E}_{\mathbb{P}}[f/\widetilde{f}_n] \leq 1$, for all $f \in \mathcal{K}$ and $n \in \mathbb{N}$. By Fatou's lemma, $\mathbb{E}_{\mathbb{P}}[f/\widetilde{f}] \leq 1$ holds for all $n \in \mathbb{N}$; in particular, $\{\widetilde{f} = 0\}$ is Π -null. As $\widetilde{f} \in \mathcal{K}$ and \widehat{f} is the numéraire in \mathcal{K} , we have $\widetilde{f} = \widehat{f}$.

Define $m_0 = 1$ and a \mathbb{N} -valued strictly increasing sequence $(m_n)_{n \in \mathbb{N}}$ inductively via $m_n = \ell_{m_{n-1}}$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, $\mathbb{E}_{\mathbb{P}}[\widetilde{f}_{m_n}/\widehat{f}_{m_n}] \leq 1$ holds by definition and $\mathbb{E}_{\mathbb{P}}[\widetilde{f}_{m_n}/\widetilde{f}_{m_{n-1}}] \leq 1$ holds because $\widetilde{f}_{m_{n-1}} \in \text{conv}(\{\widehat{f}_{m_{n-1}}, \dots, \widehat{f}_{m_n}\})$. Letting $g_n := \widehat{f}_{m_n}/\widetilde{f}_{m_{n-1}}$ and $h_n := \widetilde{f}_{m_n}/\widehat{f}_{m_n}$ for all $n \in \mathbb{N}$, we have $\mathbb{E}_{\mathbb{P}}[g_n] \leq 1$, $\mathbb{E}_{\mathbb{P}}[h_n] \leq 1$, and \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (g_n h_n) = \mathbb{L}_+^0$ - $\lim_{n \rightarrow \infty} (\widetilde{f}_{m_n}/\widetilde{f}_{m_{n-1}}) = 1$, which follows because \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \widetilde{f}_{m_n} = \widehat{f}$ and $\mathbb{P}[\widehat{f} = 0] = 0$. Therefore, the conditions of the

statement of Lemma 2.3 are satisfied for $(g_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$. It follows that \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} h_n = 1$, which, in view of \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \tilde{f}_{m_n} = \hat{f}$, gives \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \hat{f}_{m_n} = \hat{f}$. \square

Remark 2.5. The result of Proposition 2.4 does not necessarily hold if $\{\hat{f} = 0\}$ is not Π -null. Indeed, let $\Omega = (0, 1]$, \mathcal{F} be the Borel σ -field on Ω and \mathbb{P} be Lebesgue measure on (Ω, \mathcal{F}) . Define two *nonincreasing* sequences $(\hat{f}_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ via $\hat{f}_n := (1/2)\mathbb{I}_{(0, 1/3]} + (1/n)\mathbb{I}_{(1/3, n/(n+1)]} + \mathbb{I}_{(n/(n+1), 1]}$ and $g_n := \mathbb{I}_{(0, 1/3]} + (1/(5n))\mathbb{I}_{(1/3, 1]}$. Let \mathcal{K}_n be the solid hull of $\text{conv}(\{\hat{f}_n, g_n\})$. Then, $\mathcal{K} = \bigcap_{n \in \mathbb{N}} \mathcal{K}_n = \{h\mathbb{I}_{(0, 1/3]} \mid 0 \leq h \leq 1\}$, and $(\mathcal{K}_n)_{n \in \mathbb{N}}$ is a nonincreasing sequence of closed, convex and \mathbb{L}_+^0 -bounded sets. Evidently, $\hat{f} = \mathbb{I}_{(0, 1/3]}$ is the numéraire in \mathcal{K} under \mathbb{P} . Also,

$$\mathbb{E}_{\mathbb{P}} \left[\frac{g_n}{\hat{f}_n} \right] = 2 \left(\frac{1}{3} \right) + \frac{1}{5} \left(\frac{n}{n+1} - \frac{1}{3} \right) + \frac{1}{5n} \left(\frac{1}{n+1} \right) \leq \frac{2}{3} + \frac{2}{15} + \frac{1}{5} = 1,$$

from which it easily follows that \hat{f}_n is the numéraire in \mathcal{K}_n under \mathbb{P} for all $n \in \mathbb{N}$. However, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \hat{f}_n = (1/2)\mathbb{I}_{(0, 1/3]} \neq \hat{f}$.

Since $f = 1 \in \mathcal{C}^{\text{num}}$, there exists $\mathbb{P} \in \Pi$ such that $\mathbb{E}_{\mathbb{P}}[h] \leq 1$ for all $h \in \mathcal{C}$. *Until the end of the proof of (4) \Rightarrow (1) we shall keep this $\mathbb{P} \in \Pi$ fixed.* Since each \mathcal{C}_n , $n \in \mathbb{N}$, is \mathbb{L}_+^0 -bounded and $f = 1$ is the numéraire in \mathcal{C} under \mathbb{P} , a combination of Theorem 2.2 and Proposition 2.4 imply that the numéraire \hat{f}_n in \mathcal{C}_n under \mathbb{P} exists for all $n \in \mathbb{N}$, and \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \hat{f}_n = 1$.

We finally state two more helpful results before we establish implication (4) \Rightarrow (1). In both of them, we tacitly assume that $f = 1$ and that \hat{f}_n is the numéraire in \mathcal{C}_n under \mathbb{P} for each $n \in \mathbb{N}$.

Lemma 2.6. *Let $(h_n)_{n \in \mathbb{N}}$ be any sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ such that \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} h_n = 1$. Then, $\lim_{n \rightarrow \infty} \sup_{\mathbb{N} \ni k \geq n} \mathbb{E}_{\mathbb{P}}[|h_k/\hat{f}_n - 1|] = 0$.*

Proof. Since \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} h_n = 1$, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \hat{f}_n = 1$ and $\{\hat{f}_n = 0\}$ is Π -null for all $n \in \mathbb{N}$, we have \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (h_n/\hat{f}_n) = 1$. If $\lim_{n \rightarrow \infty} \sup_{\mathbb{N} \ni k \geq n} \mathbb{E}_{\mathbb{P}}[|h_k/\hat{f}_n - 1|] = 0$ fails, then there exists $\epsilon > 0$ and two \mathbb{N} -valued sequences $(n_\ell)_{\ell \in \mathbb{N}}$ and $(k_\ell)_{\ell \in \mathbb{N}}$ with $\uparrow \lim_{\ell \rightarrow \infty} n_\ell = \infty$ and $n_\ell \leq k_\ell$ for all $\ell \in \mathbb{N}$, such that $\mathbb{E}_{\mathbb{P}}[|h_{k_\ell}/\hat{f}_{n_\ell} - 1|] \geq \epsilon$ for all $\ell \in \mathbb{N}$. In particular, this means that the sequence $(h_{k_\ell}/\hat{f}_{n_\ell})_{\ell \in \mathbb{N}}$ cannot $\mathbb{L}_+^1(\mathbb{P})$ -converge to 1. We shall however show in the next paragraph that $\mathbb{L}_+^1(\mathbb{P})$ - $\lim_{\ell \rightarrow \infty} (h_{k_\ell}/\hat{f}_{n_\ell}) = 1$, reaching a contradiction and establishing the claim of Lemma 2.6.

Note that \mathbb{L}_+^0 - $\lim_{\ell \rightarrow \infty} (h_{k_\ell}/\hat{f}_{n_\ell}) = 1$, since $h_{k_\ell}/\hat{f}_{n_\ell} = (h_{k_\ell}/\hat{f}_{k_\ell})(\hat{f}_{k_\ell}/\hat{f}_{n_\ell})$ and both sequences $(h_{k_\ell}/\hat{f}_{k_\ell})_{\ell \in \mathbb{N}}$ and $(\hat{f}_{k_\ell}/\hat{f}_{n_\ell})_{\ell \in \mathbb{N}}$ are \mathbb{L}_+^0 -convergent to $f = 1$. Furthermore, since \hat{f}_{n_ℓ} is the numéraire in \mathcal{C}_{n_ℓ} under \mathbb{P} and $h_{k_\ell} \in \mathcal{C}_{n_\ell}$, $\mathbb{E}_{\mathbb{P}}[h_{k_\ell}/\hat{f}_{n_\ell}] \leq 1$ holds for all $\ell \in \mathbb{N}$. By Fatou's lemma, \mathbb{L}_+^0 - $\lim_{\ell \rightarrow \infty} (h_{k_\ell}/\hat{f}_{n_\ell}) = 1$ translates to $\lim_{\ell \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[h_{k_\ell}/\hat{f}_{n_\ell}] = 1$. Therefore, $(h_{k_\ell}/\hat{f}_{n_\ell})_{\ell \in \mathbb{N}}$ is $\mathbb{L}_+^1(\mathbb{P})$ -convergent to 1, which completes the argument. \square

Lemma 2.7. *Any sequence of forward convex combinations of $(\hat{f}_n)_{n \in \mathbb{N}}$ \mathbb{L}_+^0 -converges to $f = 1$.*

Proof. Let $(\tilde{f}_n)_{n \in \mathbb{N}}$ be sequence of forward convex combinations of $(\hat{f}_n)_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$, $\tilde{f}_n \in \text{conv}(\{\hat{f}_n, \dots, \hat{f}_{\ell_n}\})$, for some $\ell_n \in \mathbb{N}$ with $\ell_n \geq n$. Since $f = 1$, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (\hat{f}_{\ell_n}/\hat{f}_n) = 1$. Furthermore, $\mathbb{E}_{\mathbb{P}}[\tilde{f}_n/\hat{f}_n] \leq 1$ and $\mathbb{E}_{\mathbb{P}}[\hat{f}_{\ell_n}/\hat{f}_n] \leq 1$ hold for all $n \in \mathbb{N}$. Letting $g_n := \tilde{f}_n/\hat{f}_n$ and

$h_n := \widehat{f}_n / \widetilde{f}_n$ for all $n \in \mathbb{N}$, the conditions of the statement of Lemma 2.3 are satisfied. Therefore, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} g_n = 1$, which implies that \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \widetilde{f}_n = \mathbb{L}_+^0$ - $\lim_{n \rightarrow \infty} \widehat{f}_n = 1$. \square

We can now finish the proof of implication (4) \Rightarrow (1) of Theorem 1.4. Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$, and write $h_n = \sum_{k=n}^{\ell_n} \alpha_{n,k} f_k$, where $n \leq \ell_n \in \mathbb{N}$, $\alpha_{n,k} \geq 0$ for all $n \in \mathbb{N}$ and $n \leq k \leq \ell_n$, as well as $\sum_{k=n}^{\ell_n} \alpha_{n,k} = 1$. Let $(\widetilde{f}_n)_{n \in \mathbb{N}}$ be the sequence of forward convex combinations of $(\widehat{f}_n)_{n \in \mathbb{N}}$ defined via $\widetilde{f}_n := \sum_{k=n}^{\ell_n} \alpha_{n,k} \widehat{f}_k$ for each $n \in \mathbb{N}$. Then,

$$\mathbb{E}_{\mathbb{P}} \left[\frac{|h_n - \widetilde{f}_n|}{\widehat{f}_n} \right] \leq \sum_{k=n}^{\ell_n} \alpha_{n,k} \mathbb{E}_{\mathbb{P}} \left[\frac{|f_k - \widehat{f}_k|}{\widehat{f}_n} \right] \leq \sup_{\mathbb{N} \ni k \geq n} \left(\mathbb{E}_{\mathbb{P}} \left[\frac{|f_k - \widehat{f}_k|}{\widehat{f}_n} \right] \right) + \sup_{\mathbb{N} \ni k \geq n} \left(\mathbb{E}_{\mathbb{P}} \left[\frac{|\widehat{f}_k - \widehat{f}_n|}{\widehat{f}_n} \right] \right).$$

A double use of Lemma 2.6, with f and \widehat{f} respectively in place of h , gives \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (|h_n - \widetilde{f}_n| / \widehat{f}_n) = 0$. Since $(\widehat{f}_n)_{n \in \mathbb{N}}$ is \mathbb{L}_+^0 -convergent to $f = 1$, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} (h_n - \widetilde{f}_n) = 0$ follows. Now, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \widetilde{f}_n = 1$ holds in view of Lemma 2.7. Therefore, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} h_n = 1$, which concludes the proof of implication (4) \Rightarrow (1).

2.3.2. *Proof of (1) \Rightarrow (5).* Given the equivalence of (1) and (4), implication (1) \Rightarrow (5) can be proved by a reduction to the already-treated special case $f = 0$, via the following result.

Lemma 2.8. *The following statements are equivalent:*

- (1) *Every sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ \mathbb{L}_+^0 -converges to f .*
- (2) *Every sequence of forward convex combinations of $(|f_n - f|)_{n \in \mathbb{N}}$ \mathbb{L}_+^0 -converges to zero.*

Proof. As (2) \Rightarrow (1) is immediate, we only treat implication (1) \Rightarrow (2). Start by defining the sequence $(\widetilde{f}_n)_{n \in \mathbb{N}}$ via $\widetilde{f}_n = f_n \wedge f$ for $n \in \mathbb{N}$. Then, \mathbb{L}_+^0 - $\lim_{n \rightarrow \infty} \widetilde{f}_n = f$. If $\widetilde{\mathcal{C}}$ is the equivalent of the set \mathcal{C} with $(\widetilde{f}_n)_{n \in \mathbb{N}}$ in place of $(f_n)_{n \in \mathbb{N}}$, we have $f \in \widetilde{\mathcal{C}}$ and that $g \leq f$ for all other $g \in \widetilde{\mathcal{C}}$. By the already-established implication (4) \Rightarrow (1), it follows that $\widetilde{\mathcal{C}} \subseteq \mathbb{L}_+^0$ and $f \in \widetilde{\mathcal{C}}^{\text{num}}$; therefore, every sequence of forward convex combinations of $(\widetilde{f}_n)_{n \in \mathbb{N}}$ \mathbb{L}_+^0 -converges to f . As $(f_n - f) \vee 0 = f_n - (f_n \wedge f)$ for all $n \in \mathbb{N}$, we obtain that every sequence of forward convex combinations of $((f_n - f) \vee 0)_{n \in \mathbb{N}}$ \mathbb{L}_+^0 -converges to zero. Furthermore, as $(f - f_n) \vee 0 = (f_n - f) \vee 0 - (f_n - f)$ for all $n \in \mathbb{N}$, every sequence of forward convex combinations of $((f - f_n) \vee 0)_{n \in \mathbb{N}}$ \mathbb{L}_+^0 -converges to zero. Finally, since $|f_n - f| = (f_n - f) \vee 0 + (f - f_n) \vee 0$ for all $n \in \mathbb{N}$, every sequence of forward convex combinations of $(|f_n - f|)_{n \in \mathbb{N}}$ \mathbb{L}_+^0 -converges to zero. \square

In view of the result of Lemma 2.8 and the treatment in §2.2, we obtain the existence of $\mathbb{Q} \in \Pi$ such that $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[|f_n - f|] < \infty$ and $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[|f_n - f|] = 0$. Replacing \mathbb{Q} , if necessary, by $\mathbb{Q}' \in \Pi$ defined via $d\mathbb{Q}'/d\mathbb{P} = c(1 + f)^{-1}$ where $c = (\mathbb{E}_{\mathbb{Q}}[(1 + f)^{-1}])^{-1}$, we may further assume that $\mathbb{E}_{\mathbb{Q}}[f] < \infty$; in other words, $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[f_n] < \infty$ and $\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[|f_n - f|] = 0$.

Remark 2.9. In the proof of Lemma 2.8, implication of (4) \Rightarrow (1) of Theorem 1.4 under (CONV) is used. This is the reason we went through all the trouble of first establishing the implication (4) \Rightarrow (1) of Theorem 1.4. Unfortunately, there does not seem to be any nontrivial way to obtain the interesting implication (1) \Rightarrow (2) of Lemma 2.8.

2.4. Proof of claims after the equivalences. The following result will be the key to establishing all the properties of \mathcal{C}_1 that are mentioned after the five equivalences in Theorem 1.4.

Lemma 2.10. *Let $\mathcal{C}'_1 \subseteq \mathbb{L}_+$ be the set on the right-hand-side of (1.1). If $\mathbb{Q} \in \Pi$ is such that condition (5) of Theorem 1.4 holds, then \mathcal{C}'_1 is $\mathbb{L}_+^1(\mathbb{Q})$ -compact.*

Proof. First of all, since $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[f_n] < \infty$, which in particular implies that $\mathbb{E}_{\mathbb{Q}}[f] < \infty$ by Fatou's lemma, it is clear that $\sup_{g \in \mathcal{C}'_1} \mathbb{E}_{\mathbb{Q}}[g] < \infty$ — in particular, $\mathcal{C}'_1 \subseteq \mathbb{L}_+^0$.

We shall show that *any* sequence $(g_k)_{k \in \mathbb{N}}$ in \mathcal{C}'_1 has an $\mathbb{L}_+^1(\mathbb{Q})$ -convergent subsequence. For all $k \in \mathbb{N}$, write $g_k = \sum_{n \in \mathbb{N}} \alpha_{k,n} f_n + (1 - \sum_{n \in \mathbb{N}} \alpha_{k,n}) f$, where $\alpha_k = (\alpha_{k,n})_{n \in \mathbb{N}} \in \Delta^{\mathbb{N}}$. By a diagonalization argument, we can find a subsequence of $(g_k)_{k \in \mathbb{N}}$, which we shall still denote by $(g_k)_{k \in \mathbb{N}}$, such that $\alpha_n := \lim_{k \rightarrow \infty} \alpha_{k,n}$ exists for all $n \in \mathbb{N}$. Fatou's lemma implies that $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in \Delta^{\mathbb{N}}$. Let $g := \sum_{n \in \mathbb{N}} \alpha_n f_n + (1 - \sum_{n \in \mathbb{N}} \alpha_n) f$. We shall show that $\lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[|g_k - g|] = 0$. For $\epsilon > 0$, pick $N = N(\epsilon) \in \mathbb{N}$ such that $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[|f_{N+n} - f|] \leq \epsilon/2$. Define $g^{(N)} := \sum_{n=1}^N \alpha_n f_n + (1 - \sum_{n=1}^N \alpha_n) f$, as well as $g_k^{(N)} := \sum_{n=1}^N \alpha_{k,n} f_n + (1 - \sum_{n=1}^N \alpha_{k,n}) f$ for all $k \in \mathbb{N}$. Observe that

$$\mathbb{E}_{\mathbb{Q}}[|g^{(N)} - g|] = \mathbb{E}_{\mathbb{Q}}\left[\left|\sum_{n \in \mathbb{N}} \alpha_{N+n} (f_{N+n} - f)\right|\right] \leq \sum_{n \in \mathbb{N}} \alpha_{N+n} \mathbb{E}_{\mathbb{Q}}[|f_{N+n} - f|] \leq \frac{\epsilon}{2}$$

Similarly, $\mathbb{E}_{\mathbb{Q}}[|g_k^{(N)} - g_k|] \leq \epsilon/2$ holds for all $k \in \mathbb{N}$. Furthermore,

$$\limsup_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[|g_k^{(N)} - g^{(N)}|] \leq \limsup_{k \rightarrow \infty} \left(\sum_{n=1}^N |\alpha_{k,n} - \alpha_n| \mathbb{E}_{\mathbb{Q}}[|f_n - f|] \right) = 0.$$

It follows that $\limsup_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[|g_k - g|] \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, $\lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[|g_k - g|] = 0$. \square

To finish the proof of Theorem 1.4, it remains to show that $\mathcal{C}_1 = \mathcal{C}'_1$ and that the \mathbb{L}_+^0 -topology coincides with the $\mathbb{L}_+^1(\mathbb{Q})$ -topology on \mathcal{C}_1 . First of all, since $f \in \mathcal{C}_1$, $f_n \in \mathcal{C}_1$ for all $n \in \mathbb{N}$, and \mathcal{C}_1 is closed, we have $\mathcal{C}'_1 \subseteq \mathcal{C}_1$. On the other hand, $\text{conv}(\{f_1, f_2, \dots\}) \subseteq \mathcal{C}'_1$; since \mathcal{C}'_1 is \mathbb{L}_+^0 -closed by Lemma 2.10, $\mathcal{C}_1 = \overline{\text{conv}}(\{f_1, f_2, \dots\}) \subseteq \mathcal{C}'_1$. Therefore, $\mathcal{C}_1 = \mathcal{C}'_1$. Finally, let $(g_k)_{k \in \mathbb{N}}$ be a \mathcal{C}_1 -valued and \mathbb{L}_+^0 -convergent sequence, and call $g := \mathbb{L}_+^0\text{-}\lim_{k \rightarrow \infty} g_k \in \mathcal{C}_1$. Lemma 1.1 implies that every subsequence of $(g_k)_{k \in \mathbb{N}}$ has a further subsequence that is $\mathbb{L}_+^1(\mathbb{Q})$ -convergent. All the latter subsequences have to $\mathbb{L}_+^1(\mathbb{Q})$ -converge to g , which means that $(g_k)_{k \in \mathbb{N}}$ $\mathbb{L}_+^1(\mathbb{Q})$ -converges at g .

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CONSTANTINOS KARDARAS, MATHEMATICS AND STATISTICS DEPARTMENT, BOSTON UNIVERSITY, 111 CUMMINGTON STREET, BOSTON, MA 02215, USA.

E-mail address: kardaras@bu.edu

GORDAN ŽITKOVIĆ, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN, 1 UNIVERSITY STATION, C1200, AUSTIN, TX 78712, USA

E-mail address: gordanz@math.utexas.edu