Poisson limit of an inhomogeneous nearly critical INAR(1) model *

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Abstract

An inhomogeneous first–order integer–valued autoregressive (INAR(1)) process is investigated, where the autoregressive type coefficient slowly converges to one. It is shown that the process converges weakly to a Poisson or a compound Poisson distribution.

Keywords: Integer–valued time series; INAR(1) model; nearly unstable model; Poisson approximation; Galton–Watson process

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1 Introduction

A zero start inhomogeneous first order integer-valued autoregressive (INAR(1)) time series $(X_n)_{n\in\mathbb{Z}_+}$ is defined as

$$\begin{cases} X_n = \sum\limits_{j=1}^{X_{n-1}} \xi_{n,j} + \varepsilon_n, , \quad n \in \mathbb{N}, \\ X_0 = 0, \end{cases} \tag{1.1}$$

where $\{\xi_{n,j}, \varepsilon_n : n, j \in \mathbb{N}\}$ are independent non-negative integer-valued random variables such that $\{\xi_{n,j} : j \in \mathbb{N}\}$ are identically distributed and $\mathsf{P}(\xi_{n,1} \in \{0,1\}) = 1$ for each $n \in \mathbb{N}$. In fact, $(X_n)_{n\in\mathbb{Z}_+}$ is a special Galton-Watson branching process with immigration such that the offspring distributions are Bernoulli distributions. We can interpret X_n as the size of the n^{th} generation of a population, $\xi_{n,j}$ is the number of offspring produced

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by the j^{th} individual belonging to the $(n-1)^{\text{th}}$ generation, and ε_n is the number of immigrants in the n^{th} generation.

The process (1.1) is called INAR(1) since it may also be written in the form

$$\begin{cases} X_n = \varrho_n \circ X_{n-1} + \varepsilon_n, & n \in \mathbb{N}, \\ X_0 = 0, \end{cases}$$

where

$$\varrho_n := \mathsf{E}\xi_{n,1}$$

denotes the mean of the Bernoulli offspring distribution in the n^{th} generation, and we use the *Steutel and van Harn operator* $\varrho \circ$ which is defined for $\varrho \in [0,1]$ and for a non-negative integer-valued random variable X by

$$\varrho \circ X := \begin{cases} \sum\limits_{j=1}^{X} \xi_j, & X > 0, \\ 0, & X = 0, \end{cases}$$

where the counting sequence $(\xi_j)_{j\in\mathbb{N}}$ consists of independent and identically distributed Bernoulli random variables with mean ϱ, independent of X (see Steutel and van Harn [19]), and the counting sequences involved in $\varrho_n \circ X_{n-1}$, $n \in \mathbb{N}$, are mutually independent and independent of $(\varepsilon_n)_{n\in\mathbb{N}}$.

Let us denote the factorial moments of the immigration distributions by

$$m_{n,k} := \mathsf{E}\varepsilon_n(\varepsilon_n - 1)\cdots(\varepsilon_n - k + 1), \qquad n, k \in \mathbb{N}.$$

If $m_{n,1} < \infty$ for all $n \in \mathbb{N}$ then we have the recursion

$$\mathsf{E}X_n = \varrho_n \mathsf{E}X_{n-1} + m_{n,1}, \qquad n \in \mathbb{N},$$

since

$$\mathsf{E}(X_n \mid X_{n-1}) = \mathsf{E}\left(\sum_{j=1}^{X_{n-1}} \xi_{n,j} + \varepsilon_n \,\middle|\, X_{n-1} \right) = \sum_{j=1}^{X_{n-1}} \mathsf{E}\xi_{n,j} + \mathsf{E}\varepsilon_n = X_{n-1}\varrho_n + m_{n,1}.$$

Consequently, the sequence $(\varrho_n)_{n\in\mathbb{N}}$ of the offspring means plays a crucial role in the asymptotic behavior of the sequence $(X_n)_{n\in\mathbb{Z}_+}$ as $n \to \infty$. The INAR(1) process $(X_n)_{n\in\mathbb{Z}_+}$ is called *nearly critical* if $\varrho_n \to 1$ as $n \to \infty$. We will investigate the asymptotic behavior of nearly critical INAR(1) processes.

Non–negative integer–valued time series, known as counting processes, arise in several fields of medicine (see, e.g., Cardinal et al. [8] and Franke and Seligmann [13]). To model counting processes Al–Osh and Alzaid [4] proposed the INAR(1) model. Ispány et al. [14] investigated the asymptotic inference for nearly unstable INAR(1) models. Later on Al–Osh and Alzaid [5] and Du and Li [10] generalized this model by introducing the INAR(p) model.

The INAR models are special branching processes where the offspring distributions are Bernoulli distributions. The theory of branching processes has been developed for a long time, see Athreya and Ney [6], and it can be applied in various fields. Branching processes are well-known models of binary search trees, see Devroye [9]. A recent application of them is the domain of peer-to-peer file sharing networks. Traffic measurements show that the workload generated by P2P applications is the dominant part of most of the Internet segments. The file population dynamics can be described by these mathematical models which also make possible the design and control of peer-to-peer systems, see Adar and Huberman [2], Zhao et al. [20]. Space-time processes are standard models in seismology, see Lise and Stella [18]. One of these is the Epidemic-type Aftershock Sequence (ETAS) model and they serve for surveillance of infections diseases as well, see Farrington et al. [12]. The theory of branching processes can also be applied to data on different aspects of biodiversity or macroevolution by the help of using phylogenetic trees, see, e.g., Aldous and Popovic [3] and Haccou and Iwasa [16]. An inhomogeneous branching mechanism has been considered in Ispány et al. [15]. Drost et al. [11] proved that the limit experiment of a homogeneous INAR(1) model has a Poisson distribution.

The present paper seems to be the first attempt to deal with the so–called nearly unstable inhomogeneous INAR(1) model. The paper is organized as follows. In Section 2 two basic lemmas are proved for inhomogeneous INAR(1) process. In Section 3 the case of Bernoulli immigrations, in Section 4 the case of non-Bernoulli immigrations with Poisson limit distribution are considered. Section 5 is devoted to the general case when the limit distribution is a compound Poisson distribution. The results are extended for triangular system of mixtures of binomial distributions. In the Appendix at the end of paper some technical lemmas are gathered.

2 Preliminaries

Let $\mathrm{Be}(p)$ denote a Bernoulli distribution with mean $p \in [0,1]$. The distribution of a random variable ξ will be denoted by $\mathcal{L}(\xi)$. Consider the unit disk $D := \{z \in \mathbb{C} : |z| \leqslant 1\}$ of the complex plane $\mathbb{C}$. The (probability) generating function of a non-negative integer-valued random variable ξ is given by $z \mapsto \mathsf{E}(z^{\xi})$ for $z \in D$, and we have $\mathsf{E}(z^{\xi}) \in D$ for all $z \in D$. Introduce the generating functions

$$F_n(z) := \mathsf{E}(z^{X_n}), \qquad G_n(z) := \mathsf{E}(z^{\xi_{n,1}}), \qquad H_n(z) := \mathsf{E}(z^{\varepsilon_n}), \qquad z \in D.$$

Lemma 1 *For an arbitrary inhomogeneous INAR(1) process* $(X_n)_{n\in\mathbb{Z}_+}$ *we have*

$$F_n(z) = \prod_{k=1}^{n} H_k\big(1 + \varrho_{[k,n]}(z-1)\big), \qquad n \in \mathbb{N},$$

for all $z \in D$, *where*

$$\varrho_{[k,n]} := \begin{cases} \prod\limits_{\ell=k+1}^{n} \varrho_\ell & \text{for } 1 \leqslant k \leqslant n-1, \\ 1 & \text{for } k = n. \end{cases}$$

Proof. The basic recursion for the generating functions F_n, $n \in \mathbb{N}$, is

$$\begin{aligned} F_n(z) =& \mathsf{E}\Big(z^{\sum_{j=1}^{X_{n-1}} \xi_{n,j}+\varepsilon_n}\Big) = \mathsf{E}\Big(\mathsf{E}\Big(z^{\sum_{j=1}^{X_{n-1}} \xi_{n,j}+\varepsilon_n} \,\Big|\, X_{n-1}\Big)\Big) \\ =& \mathsf{E}\big(G_n(z)^{X_{n-1}}\big) H_n(z) = F_{n-1}(G_n(z)) H_n(z), \end{aligned} \tag{2.1}$$

valid for all $z \in \mathbb{C}$ with $z \in D$ and $G_n(z) \in D$, see Athreya and Ney [6, p. 263]. Clearly $z \in D$ implies $G_n(z) \in D$, hence (2.1) is valid for all $z \in D$. Since $\mathcal{L}(\xi_{n,1}) = \mathrm{Be}(\varrho_n)$, we have

$$G_n(z) = 1 - \varrho_n + \varrho_n z = 1 + \varrho_n(z-1)$$

for all $z \in \mathbb{C}$. We prove the statement of the lemma by induction. For $n = 1$, we have $F_1(z) = H_1(z) = H_1(1 + (z-1))$. By the recursion (2.1), we obtain for $n \geqslant 2$

$$\begin{aligned} F_n(z) &= F_{n-1}\big(1 + \varrho_n(z-1)\big) H_n(z) = H_n(z) \prod_{k=1}^{n-1} H_k\big(1 + \varrho_n \varrho_{[k,n-1]}(z-1)\big) \\ &= \prod_{k=1}^{n} H_k\big(1 + \varrho_{[k,n]}(z-1)\big), \end{aligned}$$

and the proof is complete. □

In fact, X_n can be considered as a sum of independent Galton-Watson processes without immigration. Namely,

$$X_n = \sum_{k=1}^{n} Y_{n,k}, \qquad n \in \mathbb{N}, \tag{2.2}$$

where

$$Y_{n,k} := \begin{cases} 0 & \text{for } k = 0, \\ \displaystyle\sum_{j=Y_{n-1,\,1}+\cdots+Y_{n-1,\,k-1}+1}^{Y_{n-1,\,1}+\cdots+Y_{n-1,\,k}} \xi_{n,j} & \text{for } 1 \leqslant k \leqslant n-1, \\ \varepsilon_n & \text{for } k = n. \end{cases} \tag{2.3}$$

The distribution of $Y_{n,k}$ is a mixture of binomial distributions with a common probability parameter ϱ_n, since the number $Y_{n-1,k}$ of Bernoulli random variables in the sum (2.3) is a random variable as well. For a probability measure μ on $\mathbb{Z}_+$ and for a number $p \in [0,1]$, the mixture $\mathrm{Bi}(\mu, p)$ of binomial distributions with parameters μ and p is a probability measure on $\mathbb{Z}_+$ defined by

$$\mathrm{Bi}(\mu,p)\{j\} := \sum_{\ell=j}^{\infty} \binom{\ell}{j} p^j (1-p)^{\ell-j} \mu\{\ell\} \qquad \text{for } j \in \mathbb{Z}_+.$$

It is a particular example for mixture of distributions, see Johnson and Kotz [17, Section I.7.3], because the common method for mixture of binomial distributions is to use different values of probability parameter, see Johnson and Kotz [17, Section III.11]. Note that $\mathrm{Bi}(\mu, 1) = \mu$.

Lemma 2 *For all $n \in \mathbb{N}$, $1 \leqslant k \leqslant n$, the distribution of $Y_{n,k}$ is a mixture of binomial distributions with parameters ε_k and $\varrho_{[k,n]}$. Thus*

$$\mathcal{L}(X_n) = \mathop{*}_{k=1}^{n} \mathrm{Bi}\big(\mathcal{L}(\varepsilon_k), \varrho_{[k,n]}\big),$$

where $*$ denotes convolution of probability measures.

Proof. First we check that $\mathrm{Bi}\big(\mathrm{Bi}(\mu,p),q\big)=\mathrm{Bi}(\mu,pq)$ for an arbitrary probability measure μ on $\mathbb{Z}_+$ and for all $p,q\in[0,1]$. Indeed, for all $j\in\mathbb{Z}_+$,

$$\begin{aligned}
\mathrm{Bi}\big(\mathrm{Bi}(\mu,p),q\big)\{j\} &= \sum_{\ell=j}^{\infty}\binom{\ell}{j}q^j(1-q)^{\ell-j}\,\mathrm{Bi}(\mu,p)\{\ell\}\\
&= \sum_{\ell=j}^{\infty}\binom{\ell}{j}q^j(1-q)^{\ell-j}\sum_{k=\ell}^{\infty}\binom{k}{\ell}p^\ell(1-p)^{k-\ell}\,\mu\{k\}\\
&= \sum_{k=j}^{\infty}(pq)^j\,\mu\{k\}\sum_{\ell=j}^{k}\binom{\ell}{j}\binom{k}{\ell}(p(1-q))^{\ell-j}(1-p)^{k-\ell}\\
&= \sum_{k=j}^{\infty}\binom{k}{j}(pq)^j\,\mu\{k\}\sum_{\ell=j}^{k}\binom{k-j}{k-\ell}(p(1-q))^{\ell-j}(1-p)^{k-\ell}\\
&= \sum_{k=j}^{\infty}\binom{k}{j}(pq)^j(1-pq)^{k-j}\,\mu\{k\}\\
&= \mathrm{Bi}(\mu,pq)\{j\}.
\end{aligned}$$

Since $Y_{1,1}=\varepsilon_1$, thus $\mathcal{L}(Y_{1,1})=\mathrm{Bi}(\mathcal{L}(\varepsilon_1),1)$, and $\mathcal{L}(Y_{n,k})=\mathrm{Bi}(\mathcal{L}(Y_{n,k-1}),\varrho_n)$ for all $n\geqslant 2$ and all $k=1,\dots,n$, we obtain the statement of the lemma by induction using the previous argument. □

Remark that Lemma 2 implies the formula given for the generating function of X_n in Lemma 1, since the generating function of a distribution $\mathrm{Bi}(\mu,p)$ is $z\mapsto H(1+(z-1)p)$, where H denotes the generating function of μ.

3 Poisson limit distribution: the case of Bernoulli immigrations

First consider the simplest case, when $\mathcal{L}(\varepsilon_n)=\mathrm{Be}(m_{n,1})$, $n\in\mathbb{N}$.

Theorem 1 *Let $(X_n)_{n\in\mathbb{Z}_+}$ be an INAR(1) process such that $\mathsf{P}(\varepsilon_n\in\{0,1\})=1$ for all $n\in\mathbb{N}$. Assume that*

(i) $\varrho_n<1$ *for all* $n\in\mathbb{N}$, $\lim\limits_{n\to\infty}\varrho_n=1$, $\sum\limits_{n=1}^{\infty}(1-\varrho_n)=\infty$,

(ii) $\lim\limits_{n\to\infty}\frac{m_{n,1}}{1-\varrho_n}=\lambda\in[0,\infty)$.

Then

$$X_n\stackrel{\mathcal{D}}{\longrightarrow}\mathrm{Po}(\lambda)\quad as\quad n\to\infty. \tag{3.1}$$

(Here and in the sequel $\mathrm{Po}(0)$ is understood as a Dirac measure concentrated at the point 1.)

Remark that the condition $\sum_{n=1}^{\infty}(1-\varrho_n)=\infty$ may be replaced by $\prod_{n=1}^{\infty}\varrho_n=0$. Moreover, if $\lambda>0$ then the condition $\lim_{n\to\infty}\varrho_n=1$ may be replaced by $\lim_{n\to\infty}m_{n,1}=0$.

Proof. In order to prove the statement, we will show that

$$\lim_{n\to\infty}F_n(z)=\mathrm{e}^{\lambda(z-1)}$$

for all $z\in D$. Since $\mathcal{L}(\varepsilon_j)=\mathrm{Be}(m_{j,1})$, we have

$$H_k(z)=1+m_{k,1}(z-1),\qquad z\in D,\quad k\in\mathbb{N}.$$

Applying Lemma 1, we can write

$$F_n(z)=\prod_{k=1}^{n}\left[1+m_{k,1}\varrho_{[k,n]}(z-1)\right],\qquad z\in D,\quad n\in\mathbb{N}.\tag{3.2}$$

Consider the functions $\widetilde{F}_n:\mathbb{C}\to\mathbb{C}$, $n\in\mathbb{N}$, defined by

$$\widetilde{F}_n(z):=\prod_{k=1}^{n}\mathrm{e}^{m_{k,1}\varrho_{[k,n]}(z-1)}.\tag{3.3}$$

In fact, (3.3) is the generating function of a Poisson distribution. The terms in the products in (3.2) and (3.3) are generating functions of probability distributions, hence Lemma 3 is applicable, and we obtain

$$|\widetilde{F}_n(z)-F_n(z)|\leqslant\sum_{k=1}^{n}\left|\mathrm{e}^{m_{k,1}\varrho_{[k,n]}(z-1)}-1-m_{k,1}\varrho_{[k,n]}(z-1)\right|$$

for $z\in D$, $n\in\mathbb{N}$. An application of the inequality $|\mathrm{e}^u-1-u|\leqslant|u|^2$ valid for all $u\in\mathbb{C}$ with $|u|\leqslant 1/2$ implies

$$\left|\mathrm{e}^{m_{k,1}\varrho_{[k,n]}(z-1)}-1-m_{k,1}\varrho_{[k,n]}(z-1)\right|\leqslant m_{k,1}^2\varrho_{[k,n]}^2|z-1|^2\tag{3.4}$$

for $z\in\mathbb{C}$ with $m_{k,1}\varrho_{[k,n]}|z-1|\leqslant 1/2$. By Lemma 5 and taking into account assumption $\lim_{n\to\infty}\frac{m_{n,1}}{1-\varrho_n}=\lambda\in[0,\infty)$, we have

$$\max_{1\leqslant k\leqslant n}m_{k,1}\varrho_{[k,n]}=\max_{1\leqslant k\leqslant n}\frac{m_{k,1}}{1-\varrho_k}a_{n,k}^{(1)}\to 0\qquad\text{as }\ n\to\infty.\tag{3.5}$$

Thus, the estimate (3.4) is valid for all $z\in D$, for sufficiently large n and for all $k=1,\dots,n$, and we obtain

$$|\widetilde{F}_n(z)-F_n(z)|\leqslant|z-1|^2\sum_{k=1}^{n}m_{k,1}^2\varrho_{[k,n]}^2.$$

By $\lim_{n\to\infty}\frac{m_{n,1}^2}{1-\varrho_n}=0$ and by Lemma 5 we obtain

$$\sum_{k=1}^{n}m_{k,1}^2\varrho_{[k,n]}^2=\sum_{k=1}^{n}\frac{m_{k,1}^2}{1-\varrho_k}a_{n,k}^{(2)}\to 0\qquad\text{as }\ n\to\infty.\tag{3.6}$$

Consequently,

$$\lim_{n\to\infty} |\widetilde{F}_n(z) - F_n(z)| = 0 \qquad \text{for all} \;\; z \in D.$$

An application of Lemma 5 yields

$$\sum_{k=1}^{n} m_{k,1}\varrho_{[k,n]} = \sum_{k=1}^{n} \frac{m_{k,1}}{1-\varrho_k} a_{n,k}^{(1)} \to \lambda \qquad \text{as} \;\; n \to \infty. \tag{3.7}$$

Consequently,

$$\lim_{n\to\infty} \widetilde{F}_n(z) = \mathrm{e}^{\lambda(z-1)} \qquad \text{for all} \;\; z \in D,$$

and we obtain $F_n(z) \to \mathrm{e}^{\lambda(z-1)}$ as $n \to \infty$ for all $z \in D$. □

Second proof of Theorem 1 by Poisson approximation. We may prove the theorem by Poisson approximation as well. The total variation distance between two probability measures μ and ν on $\mathbb{Z}_+$ equals

$$d(\mu,\nu) = \frac{1}{2} \sum_{j=0}^{\infty} \big|\mu\{j\} - \nu\{j\}\big|.$$

A sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{Z}_+$ converges weakly to a probability measure μ on $\mathbb{Z}_+$ if and only if $d(\mu_n,\mu) \to 0$. We prove (3.1) by showing that

$$d\big(\mathcal{L}(X_n), \mathrm{Po}(\lambda)\big) \to 0 \qquad \text{as} \;\; n \to \infty. \tag{3.8}$$

One can easily check that $\mathrm{Bi}\big(\mathrm{Be}(p), q\big) = \mathrm{Be}(pq)$ for arbitrary $p, q \in [0,1]$, hence by Lemma 2 we obtain $\mathcal{L}(X_n) = \mathop{\ast}\limits_{k=1}^{n} \mathrm{Be}\left(m_{k,1}\varrho_{[k,n]}\right)$. By Lemma 4,

$$d\Big(\mathcal{L}(X_n), \mathop{\ast}\limits_{k=1}^{n} \mathrm{Po}(m_{k,1}\varrho_{[k,n]})\Big) \leqslant \sum_{k=1}^{n} d\big(\mathrm{Be}(m_{k,1}\varrho_{[k,n]}), \mathrm{Po}(m_{k,1}\varrho_{[k,n]})\big).$$

We show that

$$d\big(\mathrm{Be}(p), \mathrm{Po}(p)\big) \leqslant p^2 \tag{3.9}$$

for all $p \in [0,1]$. Indeed,

$$d\big(\mathrm{Be}(p), \mathrm{Po}(p)\big) = \frac{1}{2}(\mathrm{e}^{-p} - 1 + p) + \frac{1}{2}(p - p\,\mathrm{e}^{-p}) + \frac{1}{2}(1 - \mathrm{e}^{-p} - p\,\mathrm{e}^{-p}) = p(1 - \mathrm{e}^{-p}) \leqslant p^2.$$

Applying (3.9) and (3.6), we conclude

$$d\Big(\mathcal{L}(X_n), \mathop{\ast}\limits_{k=1}^{n} \mathrm{Po}(m_{k,1}\varrho_{[k,n]})\Big) \leqslant \sum_{k=1}^{n} m_{k,1}^2 \varrho_{[k,n]}^2 \to 0.$$

Clearly,

$$\mathop{\ast}\limits_{k=1}^{n} \mathrm{Po}(m_{k,1}\varrho_{[k,n]}) = \mathrm{Po}\left(\sum_{k=1}^{n} m_{k,1}\varrho_{[k,n]}\right) \to \mathrm{Po}(\lambda)$$

in law by (3.7), and we obtain $X_n \xrightarrow{\mathcal{D}} \mathrm{Po}(\lambda)$. □

4 Poisson limit distribution: the case of non-Bernoulli immigrations

Theorem 2 *Let* $(X_n)_{n\in\mathbb{Z}_+}$ *be an inhomogeneous INAR(1) process. Assume that*

(i) $\varrho_n < 1$ *for all* $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} \varrho_n = 1$, $\sum\limits_{n=1}^{\infty}(1-\varrho_n) = \infty$,

(ii) $\lim\limits_{n\to\infty} \frac{m_{n,1}}{1-\varrho_n} = \lambda \in [0,\infty)$, $\lim\limits_{n\to\infty} \frac{m_{n,2}}{1-\varrho_n} = 0$.

Then

$$X_n \xrightarrow{\mathcal{D}} \mathrm{Po}(\lambda) \quad \text{as} \quad n \to \infty.$$

Remark 1 Since

$$m_{n,1} = \sum_{j=1}^{\infty} \mathsf{P}(\varepsilon_n \geqslant j), \qquad m_{n,2} = 2\sum_{j=1}^{\infty} j\mathsf{P}(\varepsilon_n > j),$$

assumption (ii) implies

$$\lim_{n\to\infty} \frac{\mathsf{P}(\varepsilon_n \geqslant 1)}{1-\varrho_n} = \lambda, \qquad \lim_{n\to\infty} \frac{\mathsf{P}(\varepsilon_n \geqslant 2)}{1-\varrho_n} = 0. \tag{4.1}$$

In general the converse is not true. However, if there exists a sequence $(b_j)_{j\in\mathbb{N}}$ of non-negative real numbers such that $\sum_{j=1}^{\infty} jb_j < \infty$ and $\frac{\mathsf{P}(\varepsilon_n > j)}{1-\varrho_n} \leqslant b_j$ for all $j, n \in \mathbb{N}$, then (4.1) implies (ii) by the dominated convergence theorem.

Proof. By Lemma 1, we can write

$$F_n(z) = \prod_{k=1}^{n} H_k\big(1 + \varrho_{[k,n]}(z-1)\big), \qquad z \in D, \quad n \in \mathbb{N},$$

Consider the functions $\widetilde{F}_n : \mathbb{C} \to \mathbb{C}$, $n \in \mathbb{N}$, defined by

$$\widetilde{F}_n(z) = \prod_{k=1}^{n} \left[1 + m_{k,1}\varrho_{[k,n]}(z-1)\right].$$

By Lemma 3, we obtain

$$|F_n(z) - \widetilde{F}_n(z)| \leqslant \sum_{k=1}^{n} \Big| H_k\big(1 + \varrho_{[k,n]}(z-1)\big) - 1 - m_{k,1}\,\varrho_{[k,n]}(z-1) \Big|$$

for $z \in D$, $n \in \mathbb{N}$. Applying Lemma 6, we have

$$|H_k(u) - 1 - m_{k,1}(u-1)| \leqslant \frac{1}{2} m_{k,2}\,|u-1|^2 \qquad u \in D, \quad k \in \mathbb{N}.$$

Thus

$$\Big| H_k\big(1 + \varrho_{[k,n]}(z-1)\big) - 1 - m_{k,1}\,\varrho_{[k,n]}(z-1) \Big| \leqslant \frac{1}{2} m_{k,2}\,\varrho_{[k,n]}^2 |z-1|^2$$

for all $z \in D$, since $z \in D$ implies $1 + \varrho_{[k,n]}(z-1) \in D$. Consequently,

$$|F_n(z) - \widetilde{F}_n(z)| \leqslant \frac{1}{2}|z-1|^2 \sum_{k=1}^n m_{k,2}\, \varrho_{[k,n]}^2 \to 0$$

as $n \to \infty$ for all $z \in D$ by Lemma 5 taking into account assumption $\lim\limits_{n\to\infty} \frac{m_{n,2}}{1-\varrho_n} = 0$. Theorem 1 clearly implies $\widetilde{F}_n(z) \to \mathrm{e}^{\lambda(z-1)}$ for all $z \in D$, hence we conclude $F_n(z) \to \mathrm{e}^{\lambda(z-1)}$ as $n \to \infty$ for all $z \in D$. □

Second proof of Theorem 2 by Poisson approximation. Note that $m_{k,1}\varrho_{[k,n]} \leqslant 1$ for sufficiently large n and for all $1 \leqslant k \leqslant n$ by (3.5). By Lemmas 2 and 4, we have, for sufficiently large $n \in \mathbb{N}$,

$$d\left(\mathcal{L}(X_n), \mathop{*}_{k=1}^{n} \mathrm{Be}(m_{k,1}\varrho_{[k,n]})\right) \leqslant \sum_{k=1}^n d\big(\mathrm{Bi}(\varepsilon_k, \varrho_{[k,n]}), \mathrm{Be}(m_{k,1}\varrho_{[k,n]})\big).$$

We prove that

$$d\big(\mathrm{Bi}(\varepsilon, p), \mathrm{Be}(p\mathsf{E}\varepsilon)\big) \leqslant \frac{3}{2} p^2 \mathsf{E}\varepsilon(\varepsilon-1), \tag{4.2}$$

where $p \in [0,1]$ and ε is a non-negative integer-valued random variable such that $p\mathsf{E}\varepsilon \leqslant 1$. We have

$$d\big(\mathrm{Bi}(\varepsilon, p), \mathrm{Be}(p\mathsf{E}\varepsilon)\big) \leqslant \frac{1}{2}(A+B+C),$$

where

$$\begin{aligned}
A &:= \left|\sum_{\ell=0}^\infty (1-p)^\ell\, \mathsf{P}(\varepsilon=\ell) - (1-p\mathsf{E}\varepsilon)\right| \leqslant \sum_{\ell=0}^\infty |(1-p)^\ell - 1 + \ell p|\, \mathsf{P}(\varepsilon=\ell),\\
B &:= \left|\sum_{\ell=1}^\infty \ell p(1-p)^{\ell-1}\, \mathsf{P}(\varepsilon=\ell) - p\mathsf{E}\varepsilon\right| \leqslant p\sum_{\ell=1}^\infty \ell|(1-p)^{\ell-1} - 1|\, \mathsf{P}(\varepsilon=\ell),\\
C &:= \sum_{j=2}^\infty \sum_{\ell=j}^\infty \binom{\ell}{j} p^j (1-p)^{\ell-j}\, \mathsf{P}(\varepsilon=\ell) = \sum_{\ell=2}^\infty \mathsf{P}(\varepsilon=\ell) \sum_{j=2}^{\ell} \binom{\ell}{j} p^j(1-p)^{\ell-j}\\
&= \sum_{\ell=2}^\infty \mathsf{P}(\varepsilon=\ell)\big(1-(1-p)^\ell - \ell p(1-p)^{\ell-1}\big)\\
&\leqslant \sum_{\ell=2}^\infty \mathsf{P}(\varepsilon=\ell)\big(|1-\ell p-(1-p)^\ell| + \ell p|1-(1-p)^{\ell-1}|\big).
\end{aligned}$$

By Taylor's formula for the function $p \mapsto (1-p)^k$ we get

$$|1-\ell p-(1-p)^\ell| \leqslant \frac{1}{2}\ell(\ell-1)p^2 \sup_{\theta\in[0,1]} (1-\theta p)^{\ell-2} \leqslant \frac{1}{2}\ell(\ell-1)p^2.$$

Finally, since

$$|(1-p)^{\ell-1} - 1| \leqslant p(\ell-1),$$

we obtain (4.2). Thus, we have

$$d\left(\mathcal{L}(X_n), \mathop{*}_{k=1}^{n} \mathrm{Be}(m_{k,1}\varrho_{[k,n]})\right) \leqslant \frac{3}{2}\sum_{k=1}^n m_{k,2}\, \varrho_{[k,n]}^2,$$

where the right hand side tends to 0 by the assumption $\lim\limits_{n\to\infty} \frac{m_{n,2}}{1-\varrho_n} = 0$. Obviously, Theorem 1 implies

$$d\left(\mathop{*}_{k=1}^{n} \mathrm{Be}(m_{k,1}\varrho_{[k,n]}),\, \mathrm{Po}(\lambda)\right) \to 0 \qquad \text{as} \ \ n\to\infty,$$

hence we obtain

$$d\big(\mathcal{L}(X_n), \mathrm{Po}(\lambda)\big) \to 0 \qquad \text{as} \ \ n\to\infty,$$

which completes the proof. □

In fact, a similar theorem holds for triangular system of mixtures of binomial distributions.

Theorem 3 *Let* $k_n \in \mathbb{N}$ *for all* $n\in\mathbb{N}$, *and* $\{\zeta_{n,k} : 1 \leqslant k \leqslant k_n,\ n\in\mathbb{N}\}$ *be non-negative integer-valued random variables. Moreover, let* $p_{n,k}\in[0,1]$, $1\leqslant k\leqslant k_n$, $n\in\mathbb{N}$. *Assume that*

(i) $\sum\limits_{k=1}^{k_n} p_{n,k}\mathsf{E}\zeta_{n,k} \to \lambda$ *for some* $\lambda \geqslant 0$*;*

(ii) $\sum\limits_{k=1}^{k_n} (p_{n,k}\mathsf{E}\zeta_{n,k})^2 \to 0$*;*

(iii) $\sum\limits_{k=1}^{k_n} p_{n,k}^2\mathsf{E}\zeta_{n,k}(\zeta_{n,k}-1) \to 0$

as $n\to\infty$. *Then*

$$\mathop{*}_{k=1}^{k_n} \mathrm{Bi}\big(\mathcal{L}(\zeta_{n,k}), p_{n,k}\big) \to \mathrm{Po}(\lambda)$$

in law as $n\to\infty$.

5 Compound Poisson limit distribution

Recall that if μ is a finite measure on $\mathbb{Z}_+$ then the *compound Poisson distribution* $\mathrm{CP}(\mu)$ with intensity measure μ is the probability measure on $\mathbb{Z}_+$ with generating function

$$z \mapsto \exp\left\{\sum_{j=1}^{\infty} \mu\{j\}(z^j-1)\right\} \qquad \text{for} \ \ z\in D.$$

In fact, $\mathrm{CP}(\mu)$ is an infinitely divisible distribution on $\mathbb{Z}_+$ with Lévy measure μ restricted onto $\mathbb{N}$, and, for an arbitrary infinitely divisible distribution ν on $\mathbb{Z}_+$, there exists a finite measure μ on $\mathbb{N}$ such that $\nu = \mathrm{CP}(\mu)$. Moreover, $\mathrm{CP}(\mu)$ is the distribution of the random sum

$$\sum_{j=1}^{\Pi} \zeta_j,$$

where $\{\Pi,\, \zeta_j : j\in\mathbb{N}\}$ are independent random variables, $\mathcal{L}(\Pi) = \mathrm{Po}(\|\mu\|)$ and $\mathcal{L}(\zeta_j) = \frac{\mu}{\|\mu\|}$ for $j\in\mathbb{N}$, where $\|\mu\| := \sum_{j=1}^{\infty}\mu\{j\}$. Further, $\mathrm{CP}(\mu)$ is the distribution of the weakly convergent infinite sum

$$\sum_{j=1}^{\infty} j\eta_j,$$

where $\{\eta_j : j \in \mathbb{N}\}$ are independent random variables with $\mathcal{L}(\eta_j) = \mathrm{Po}(\mu\{j\})$ for $j \in \mathbb{N}$. (See Barbour et al. [7, Section 10.4].)

First we consider the case when the intensity measure μ of the limiting compound Poisson distribution $\mathrm{CP}(\mu)$ has bounded support.

Theorem 4 *Let $(X_n)_{n\in\mathbb{Z}_+}$ be an inhomogeneous INAR(1) process. Assume that*

(i) $\varrho_n < 1$ *for all* $n \in \mathbb{N}$, $\lim_{n\to\infty} \varrho_n = 1$, $\sum_{n=1}^{\infty} (1 - \varrho_n) = \infty$,

(ii) $\lim_{n\to\infty} \frac{m_{n,j}}{j(1-\varrho_n)} = \lambda_j \in [0, \infty)$ *for* $j = 1, \ldots, J$ *with* $\lambda_J = 0$.

Then

$$X_n \stackrel{\mathcal{D}}{\longrightarrow} \mathrm{CP}(\mu) \quad \text{as} \quad n \to \infty,$$

where μ *is a finite measure on* $\{1, \ldots, J-1\}$ *given by*

$$\mu\{j\} := \frac{1}{j!} \sum_{i=0}^{J-j-1} \frac{(-1)^i}{i!} \lambda_{j+i}, \qquad j = 1, \ldots, J-1. \tag{5.1}$$

Remark 2 One can easily check that λ_j, $j = 1, \ldots, J-1$, are the first $J-1$ factorial moments of the measure μ, i.e.,

$$\lambda_j = \sum_{i=j}^{J-1} i(i-1)\cdots(i-j+1)\mu\{i\}, \qquad j = 1, \ldots, J-1.$$

Moreover, since

$$m_{n,j} = j! \sum_{i=j}^{\infty} \binom{i-1}{j-1} \mathsf{P}(\varepsilon_n \geqslant i), \qquad j \in \mathbb{N},$$

assumption (ii) implies

(ii)′ $\lim_{n\to\infty} \frac{\mathsf{P}(\varepsilon_n \geqslant i)}{i(1-\varrho_n)} = \mu\{i\} \in [0, \infty)$ for $i = 1, \ldots, J$ with $\mu\{J\} = 0$.

On the other hand, (ii)′ and additional domination assumption, see Remark 1, imply (ii).

Proof. By Lemma 1, we can write

$$F_n(z) = \prod_{k=1}^{n} H_k\big(1 + \varrho_{[k,n]}(z-1)\big), \qquad z \in D, \quad n \in \mathbb{N}.$$

Consider the functions

$$\widetilde{F}_n(z) := \prod_{k=1}^{n} \mathrm{e}^{H_k(1+\varrho_{[k,n]}(z-1))-1}, \qquad z \in D, \quad n \in \mathbb{N}.$$

By Lemma 3, we obtain

$$|\widetilde{F}_n(z) - F_n(z)| \leqslant \sum_{k=1}^{n} \Big| \mathrm{e}^{H_k(1+\varrho_{[k,n]}(z-1))-1} - H_k\big(1 + \varrho_{[k,n]}(z-1)\big) \Big|$$

for $z \in D$, $n \in \mathbb{N}$. An application of the inequality $|\mathrm{e}^u - 1 - u| \leqslant |u|^2$ valid for all $u \in \mathbb{C}$ with $|u| \leqslant 1/2$ implies

$$\left|\mathrm{e}^{H_k(1+\varrho_{[k,n]}(z-1))-1} - H_k\big(1 + \varrho_{[k,n]}(z-1)\big)\right| \leqslant \left|H_k\big(1 + \varrho_{[k,n]}(z-1)\big) - 1\right|^2 \tag{5.2}$$

for $z \in \mathbb{C}$ with $\left|H_k\big(1 + \varrho_{[k,n]}(z-1)\big) - 1\right| \leqslant 1/2$. Applying Lemma 6, we have

$$|H_k(u) - 1| \leqslant m_{k,1}\,|u-1|, \qquad u \in D, \quad k \in \mathbb{N}.$$

Thus

$$\left|H_k\big(1 + \varrho_{[k,n]}(z-1)\big) - 1\right| \leqslant m_{k,1}\,\varrho_{[k,n]}|z-1|$$

for all $z \in D$, since $z \in D$ implies $1 + \varrho_{[k,n]}(z-1) \in D$. By Lemma 5 and taking into account assumption $\lim\limits_{n\to\infty} \frac{m_{n,1}}{1-\varrho_n} = \lambda_1 \in [0,\infty)$, we obtain (3.5). Thus, the estimate (5.2) is valid for all $z \in D$, for sufficiently large n and for all $k = 1, \ldots, n$, and we obtain

$$|\widetilde{F}_n(z) - F_n(z)| \leqslant |z-1|^2 \sum_{k=1}^n m_{k,1}^2\,\varrho_{[k,n]}^2 \to 0 \qquad \text{as } n \to \infty \text{ for all } z \in D$$

by (3.6). Clearly

$$\widetilde{F}_n(z) = \exp\left\{\sum_{k=1}^n \Big[H_k\big(1 + \varrho_{[k,n]}(z-1)\big) - 1\Big]\right\}.$$

Again by Lemma 6, we have

$$H_k\big(1 + \varrho_{[k,n]}(z-1)\big) - 1 = \sum_{j=1}^{J-1} \frac{m_{k,j}}{j!}\varrho_{[k,n]}^j (z-1)^j + R_{n,k,J}(z),$$

for all $z \in D$, for sufficiently large n and for all $k = 1, \ldots, n$, where

$$|R_{n,k,J}(z)| \leqslant \frac{m_{k,J}}{J!}\varrho_{[k,n]}^J |z-1|^J.$$

An application of Lemma 5 yields

$$\sum_{k=1}^n m_{k,j}\varrho_{[k,n]}^j = \sum_{k=1}^n \frac{m_{k,j}}{1-\varrho_k} a_{n,k}^{(j)} \to \lambda_j \qquad \text{as } n \to \infty \tag{5.3}$$

for $j = 1, \ldots, J$. Moreover, by (5.3),

$$\sum_{k=1}^n |R_{n,k,J}(z)| \leqslant \frac{|z-1|^J}{J!} \sum_{k=1}^n m_{k,J}\varrho_{[k,n]}^J \to 0$$

as $n \to \infty$ for all $z \in D$ since $\lambda_J = 0$. Consequently,

$$\lim_{n\to\infty} F_n(z) = \exp\left\{\sum_{j=1}^{J-1} \frac{\lambda_j}{j!}(z-1)^j\right\} =: F(z) \qquad \text{for all } z \in D,$$

where F is the generating function of a probability distribution. Clearly

$$\begin{aligned}\sum_{j=1}^{J-1}\frac{\lambda_j}{j!}(z-1)^j &= \sum_{j=1}^{J-1}\frac{\lambda_j}{j!}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}z^i = \sum_{i=0}^{J-1}\frac{z^i}{i!}\sum_{j=i}^{J-1}\frac{(-1)^{j-i}}{(j-i)!}\lambda_j\\ &= \sum_{i=0}^{J-1}\frac{z^i}{i!}\sum_{j=0}^{J-i-1}\frac{(-1)^j}{j!}\lambda_{j+i} = \sum_{i=1}^{J-1}\mu\{i\}(z^i-1)\end{aligned}$$

since $\sum_{i=1}^{J-1}\mu\{i\} = \sum_{j=1}^{J-1}\frac{(-1)^j}{j!}\lambda_j$, and we obtain $X_n \stackrel{\mathcal{D}}{\longrightarrow} \mathrm{CP}(\mu)$ as $n\to\infty$. □

Second proof of Theorem 4 by Poisson approximation. By Lemmas 2 and 4, we have

$$d\left(\mathcal{L}(X_n),\, \mathop{*}_{k=1}^{n}\mathrm{CP}\big(\mathrm{Bi}(\varepsilon_k,\varrho_{[k,n]})\big)\right) \leqslant \sum_{k=1}^{n} d\Big(\mathrm{Bi}(\varepsilon_k,\varrho_{[k,n]}),\, \mathrm{CP}\big(\mathrm{Bi}(\varepsilon_k,\varrho_{[k,n]})\big)\Big).$$

We prove that

$$d\Big(\mathrm{Bi}(\varepsilon,p),\mathrm{CP}\big(\mathrm{Bi}(\varepsilon,p)\big)\Big) \leqslant p^2(\mathsf{E}\varepsilon)^2 \tag{5.4}$$

for all $p\in[0,1]$ and for all non-negative integer-valued random variable ε. By Barbour et al. [7, Corollary 10.L.1], we have

$$d\Big(\mathrm{Bi}(\varepsilon,p),\mathrm{CP}\big(\mathrm{Bi}(\varepsilon,p)\big)\Big) \leqslant \mathsf{P}\big(\mathrm{Bi}(\varepsilon,p)\geqslant 1\big)^2.$$

Now

$$\begin{aligned}\mathsf{P}(\mathrm{Bi}(\varepsilon,p)\geqslant 1) &= 1-\sum_{\ell=0}^{\infty}(1-p)^{\ell}\,\mathsf{P}(\varepsilon=\ell) = \sum_{\ell=0}^{\infty}\left[1-(1-p)^{\ell}\right]\mathsf{P}(\varepsilon=\ell)\\ &\leqslant \sum_{\ell=0}^{\infty}p\ell\,\mathsf{P}(\varepsilon=\ell) = p\,\mathsf{E}\varepsilon,\end{aligned} \tag{5.5}$$

hence we obtain (5.4). Applying (5.4), we conclude

$$d\left(\mathcal{L}(X_n),\, \mathop{*}_{k=1}^{n}\mathrm{CP}\big(\mathrm{Bi}(\varepsilon_k,\varrho_{[k,n]})\big)\right) \leqslant \sum_{k=1}^{n} m_{k,1}^2\varrho_{[k,n]}^2 \to 0$$

by (3.6). Clearly

$$\mathop{*}_{k=1}^{n}\mathrm{CP}\left(\mathrm{Bi}(\varepsilon_k,\varrho_{[k,n]})\right) = \mathrm{CP}\left(\sum_{k=1}^{n}\mathrm{Bi}(\varepsilon_k,\varrho_{[k,n]})\right),$$

hence, in order to prove the statement, it suffices to show

$$\sum_{k=1}^{n}\mathrm{Bi}(\varepsilon_k,\varrho_{[k,n]}) \to \mu.$$

We will check

$$\sum_{k=1}^{n}\mathsf{P}\left(\mathrm{Bi}\left(\varepsilon_k,\varrho_{[k,n]}\right)=j\right) \to \mu\{j\} \qquad \text{for all } j\in\mathbb{N}. \tag{5.6}$$

First note that by Taylor's formula, for all $p \in [0,1]$ and all $K, I \in \mathbb{N}$,

$$(1-p)^K = \sum_{i=0}^{I-1} \binom{K}{i} (-1)^i p^i + R_{K,I}(p),$$

where

$$|R_{K,I}(p)| \leqslant \binom{K}{I} p^I.$$

Hence

$$\begin{aligned}
\mathsf{P}\left(\mathrm{Bi}(\varepsilon_k, \varrho_{[k,n]}) = j\right) &= \sum_{\ell=j}^{\infty} \binom{\ell}{j} \varrho_{[k,n]}^j (1-\varrho_{[k,n]})^{\ell-j}\, \mathsf{P}(\varepsilon_k = \ell) \\
&= \sum_{\ell=j}^{\infty} \binom{\ell}{j} \varrho_{[k,n]}^j \mathsf{P}(\varepsilon_k = \ell) \left[\sum_{i=0}^{J-j-1} \binom{\ell-j}{i} (-1)^i \varrho_{[k,n]}^i + R_{\ell-j,J-j}(\varrho_{[k,n]}) \right] \\
&= \sum_{i=0}^{J-j-1} \sum_{\ell=j+i}^{\infty} \frac{(-1)^i\, \ell!}{j!\, i!\, (\ell-j-i)!}\, \varrho_{[k,n]}^{j+i}\, \mathsf{P}(\varepsilon_k = \ell) + \widetilde{R}_{n,k,j} \\
&= \frac{1}{j!} \sum_{i=0}^{J-j-1} \frac{(-1)^i}{i!} m_{k,j+i}\, \varrho_{[k,n]}^{j+i} + \widetilde{R}_{n,k,j},
\end{aligned}$$

where the sum is 0 if $j \geqslant J$ and

$$\widetilde{R}_{n,k,j} := \sum_{\ell=j}^{\infty} \binom{\ell}{j} \varrho_{[k,n]}^j \mathsf{P}(\varepsilon_k = \ell) R_{\ell-j,J-j}(\varrho_{[k,n]}).$$

Assumption (ii) implies (5.3) again and we have

$$\sum_{k=1}^{n} \frac{1}{j!} \sum_{i=0}^{J-j-1} \frac{(-1)^i}{i!} m_{k,j+i}\, \varrho_{[k,n]}^{j+i} = \frac{1}{j!} \sum_{i=0}^{J-j-1} \frac{(-1)^i}{i!} \sum_{k=1}^{n} m_{k,j+i}\, \varrho_{[k,n]}^{j+i} \to \frac{1}{j!} \sum_{i=0}^{J-j-1} \frac{(-1)^i}{i!} \lambda_{j+i} = \mu\{j\}$$

as $n \to \infty$ for $j = 1, \ldots, J-1$. Moreover, (5.3) implies

$$\sum_{k=1}^{n} |\widetilde{R}_{n,k,j}| \leqslant \sum_{k=1}^{n} \sum_{\ell=j}^{\infty} \binom{\ell}{j} \varrho_{[k,n]}^j \mathsf{P}(\varepsilon_k = \ell) \binom{\ell}{J-j} \varrho_{[k,n]}^{J-j} = \frac{1}{j!(J-j)!} \sum_{k=1}^{n} m_{k,J} \varrho_{[k,n]}^J \to \mu\{J\} = 0,$$

hence we conclude (5.6). □

Next we study the case when the intensity measure μ of the limiting compound Poisson distribution $\mathrm{CP}(\mu)$ may have unbounded support.

Theorem 5 *Let $(X_n)_{n \in \mathbb{Z}_+}$ be an inhomogeneous INAR(1) process. Assume that*

(i) $\varrho_n < 1$ *for all* $n \in \mathbb{N}$, $\lim_{n\to\infty} \varrho_n = 1$, $\sum_{n=1}^{\infty} (1-\varrho_n) = \infty$,

(ii) $\lim_{n\to\infty} \frac{m_{n,j}}{j(1-\varrho_n)} = \lambda_j \in [0,\infty)$ *for all* $j \in \mathbb{N}$ *such that the limits*

$$\mu\{j\} := \frac{1}{j!} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \lambda_{j+i}, \qquad j \in \mathbb{N}, \tag{5.7}$$

exist.

Then

$$X_n \xrightarrow{\mathcal{D}} \mathrm{CP}(\mu) \quad as \quad n \to \infty.$$

Proof. We follow the second proof of Theorem 4 by Poisson approximation. We have to show that μ is a finite measure on $\mathbb{N}$ and to check (5.6). First note that by Taylor's formula, for all $p \in [0,1]$ and all $K, I \in \mathbb{N}$,

$$\sum_{i=0}^{2I-1} \binom{K}{i} (-1)^i p^i \leqslant (1-p)^K \leqslant \sum_{i=0}^{2I} \binom{K}{i} (-1)^i p^i.$$

Hence for all $I \in \mathbb{N}$,

$$\begin{aligned}
\mathsf{P}\left(\mathrm{Bi}(\varepsilon_k, \varrho_{[k,n]}) = j\right) &= \sum_{\ell=j}^{\infty} \binom{\ell}{j} \varrho_{[k,n]}^{j} (1-\varrho_{[k,n]})^{\ell-j}\, \mathsf{P}(\varepsilon_k = \ell) \\
&\leqslant \sum_{\ell=j}^{\infty} \binom{\ell}{j} \varrho_{[k,n]}^{j} \mathsf{P}(\varepsilon_k = \ell) \sum_{i=0}^{2I} \binom{\ell-j}{i} (-1)^i \varrho_{[k,n]}^{i} \\
&= \sum_{i=0}^{2I} \sum_{\ell=j}^{\infty} \frac{(-1)^i\, \ell!}{j!\, i!\, (\ell-j-i)!}\, \varrho_{[k,n]}^{j+i}\, \mathsf{P}(\varepsilon_k = \ell) \\
&= \frac{1}{j!} \sum_{i=0}^{2I} \frac{(-1)^i}{i!} m_{k,j+i}\, \varrho_{[k,n]}^{j+i}.
\end{aligned}$$

One can easily check that (5.3) holds for all $j \in \mathbb{N}$, and we obtain

$$\limsup_{n\to\infty} \sum_{k=1}^{n} \mathsf{P}\left(\mathrm{Bi}(\varepsilon_k, \varrho_{[k,n]}) = j\right) \leqslant \frac{1}{j!} \sum_{i=0}^{2I} \frac{(-1)^i}{i!} \lambda_{j+i}.$$

In a similar way, for all $I \in \mathbb{N}$,

$$\liminf_{n\to\infty} \sum_{k=1}^{n} \mathsf{P}\left(\mathrm{Bi}(\varepsilon_k, \varrho_{[k,n]}) = j\right) \geqslant \frac{1}{j!} \sum_{i=0}^{2I-1} \frac{(-1)^i}{i!} \lambda_{j+i},$$

hence by the existence of the limits (5.7) we conclude (5.6).

Finally, for all $J \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{j=1}^{J} \mu\{j\} &= \sum_{j=1}^{J} \lim_{n\to\infty} \sum_{k=1}^{n} \mathsf{P}\left(\mathrm{Bi}(\varepsilon_k, \varrho_{[k,n]}) = j\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathsf{P}\left(1 \leqslant \mathrm{Bi}(\varepsilon_k, \varrho_{[k,n]}) \leqslant J\right) \\
&\leqslant \lim_{n\to\infty} \sum_{k=1}^{n} \mathsf{P}\left(\mathrm{Bi}(\varepsilon_k, \varrho_{[k,n]}) \geqslant 1\right) \leqslant \lim_{n\to\infty} \sum_{k=1}^{n} m_{k,1} \varrho_{[k,n]} = \lambda_1
\end{aligned}$$

using again (5.5). Consequently, $\sum_{j=1}^{\infty} \mu\{j\} \leqslant \lambda_1 < \infty$, hence the measure μ is finite. $\Box$

Remark 3 A possible limit measure $\mathrm{CP}(\mu)$ in Theorem 5 is a special compound Poisson measure, since its intensity measure μ has finite moments. Indeed, for all $J, \ell \in \mathbb{N}$, we have

$$\sum_{j=\ell}^{J} j(j-1)\cdots(j-\ell+1)\mu\{j\} = \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=\ell}^{J} j(j-1)\cdots(j-\ell+1)\mathsf{P}(\mathrm{Bi}\left(\varepsilon_k, \varrho_{[k,n]}\right) = j)$$

$$\leqslant \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=\ell}^{\infty} j(j-1)\cdots(j-\ell+1)\mathsf{P}(\mathrm{Bi}\left(\varepsilon_k, \varrho_{[k,n]}\right) = j)\,.$$

It is easy to check that for all $p \in [0,1]$ and for all non-negative integer-valued random variable ε we have

$$\sum_{j=\ell}^{\infty} j(j-1)\cdots(j-\ell+1)\,\mathsf{P}(\mathrm{Bi}\,(\varepsilon,p)=j) = p^{\ell}\,\mathsf{E}\varepsilon(\varepsilon-1)\cdots(\varepsilon-\ell+1).$$

Consequently,

$$\sum_{j=\ell}^{J} j(j-1)\cdots(j-\ell+1)\mu\{j\} \leqslant \lim_{n\to\infty}\sum_{k=1}^{n} m_{k,\ell}\varrho_{[k,n]}^{\ell} = \lambda_\ell < \infty$$

using again (5.3).

Example 1 For $n\in\mathbb{N}$, let $\varrho_n = 1-\frac{1}{n}$ and $\mathsf{P}(\varepsilon_n = j) = \frac{1}{nj(j+1)}$, $j\in\mathbb{N}$, $\mathsf{P}(\varepsilon_n=0) = 1-\frac{1}{n}$. Then $\mathsf{E}\varepsilon_n = \infty$ for all $n\in\mathbb{N}$, thus inequality (5.4) is not enough to prove the compound Poisson convergence. Moreover, $\frac{\mathsf{P}(\varepsilon_n\geq j)}{j(1-\varrho_n)} = \frac{1}{j^2}$ for all $j,n\in\mathbb{N}$. The measure μ on $\mathbb{N}$ defined by $\mu\{j\} := \frac{1}{j^2}$, $j\in\mathbb{N}$, is finite and the infinite series $\sum_{j=1}^{\infty} j\mu\{j\}$ diverges. We prove that $(X_n)_{n\in\mathbb{Z}_+}$ converges to $\mathrm{CP}(\mu)$ in spite of the fact that assumption (ii) of Theorem 5 does not hold. We have, for $n\in\mathbb{N}$ and $z\in\mathbb{C}$ with $|z|<1$ and $z\neq 0$,

$$H_n(z) = 1-\frac{1}{n}+\frac{1}{n}\sum_{j=1}^{\infty}\frac{z^j}{j(j+1)} = 1-\frac{1-z}{n}\left(1+\sum_{j=1}^{\infty}\frac{z^j}{j+1}\right) = 1+\frac{(1-z)\ln(1-z)}{nz},$$

which representation is valid on the whole D. By Lemma 1 we have

$$F_n(z) = \prod_{k=1}^{n}\left(1+\frac{(1-z)\ln(\frac{k}{n}(1-z))}{n(1-\frac{k}{n}(1-z))}\right), \qquad z\in D, \quad n\in\mathbb{N}.$$

Consider the functions $\widetilde{F}_n : D\to D$, $n\in\mathbb{N}$, defined by

$$\widetilde{F}_n(z) := \prod_{k=1}^{n}\mathrm{e}^{\frac{(1-z)\ln(\frac{k}{n}(1-z))}{n(1-\frac{k}{n}(1-z))}}.$$

We have

$$\widetilde{F}_n(z) = \exp\left\{\frac{1}{n}\sum_{k=1}^{n}\frac{(1-z)\ln(\frac{k}{n}(1-z))}{1-\frac{k}{n}(1-z)}\right\} \to \exp\left\{(1-z)\int_0^1\frac{\ln(t(1-z))}{1-t(1-z)}\,\mathrm{d}t\right\} \tag{5.8}$$

as $n\to\infty$. Since for the dilogarithm, see Abramowitz and Stegun [1, Section 27.7],

$$\mathrm{Li}_2(z) := \sum_{j=1}^{\infty}\frac{z^j}{j^2} = -\int_0^z\frac{\ln(1-u)}{u}\,\mathrm{d}u, \qquad z\in D,$$

holds, we have

$$\widetilde{F}_n(z) \to \exp\left\{\sum_{j=1}^{\infty} \frac{z^j - 1}{j^2}\right\} \qquad \text{as} \quad n \to \infty$$

for all $z \in D$. On the other hand, one can easily check that

$$\max_{1 \leqslant k \leqslant n} \left| \frac{\ln(\frac{k}{n}(1-z))}{n(1 - \frac{k}{n}(1-z))} \right| \to 0 \qquad \text{as } n \to \infty \text{ for all } z \in D \text{ with } z \neq 1. \tag{5.9}$$

Namely, for all $z \in D$ with $z \neq 1$, all $n \geqslant 2$ and all $1 \leqslant k \leqslant n$ we have

$$\left|1 - \frac{k}{n}(1-z)\right|^2 = 1 - \frac{2\alpha k}{n}\left(1 - \frac{k}{n}\right) - \frac{k^2}{n^2}(1 - |z|^2) \leqslant 1 - \frac{2\alpha(n-1)}{n^2} \leqslant 1 - \frac{\alpha}{n} < 1,$$

where $\alpha := 1 - \operatorname{Re} z \in (0, 2]$. Moreover,

$$\ln(1-u) = -\sum_{j=1}^{\infty} \frac{u^j}{j}, \qquad \text{for all } u \in \mathbb{C} \text{ with } |u| < 1.$$

Hence, for all $z \in D$ with $z \neq 1$, all $n \geqslant 2$, and all $1 \leqslant k \leqslant n$ we conclude

$$\begin{aligned}
\left| \frac{\ln\left(\frac{k}{n}(1-z)\right)}{n\left(1 - \frac{k}{n}(1-z)\right)} \right| &\leqslant \frac{1}{n}\sum_{j=1}^{\infty} \frac{1}{j}\left|1 - \frac{k}{n}(1-z)\right|^{j-1} \\
&\leqslant \frac{1}{n}\sum_{j=1}^{\infty} \frac{1}{j}\left(1 - \frac{\alpha}{n}\right)^{(j-1)/2} = -\frac{\ln\left(1 - \sqrt{1 - \frac{\alpha}{n}}\right)}{n\sqrt{1 - \frac{\alpha}{n}}} \to 0
\end{aligned}$$

as $n \to \infty$. An application of the inequality $|\mathrm{e}^u - 1 - u| \leqslant |u|^2$ valid for all $u \in \mathbb{C}$ with $|u| \leqslant 1/2$ implies

$$|\widetilde{F}_n(z) - F_n(z)| \leqslant \sum_{k=1}^{n} \left| \frac{(1-z)\ln(\frac{k}{n}(1-z))}{n(1 - \frac{k}{n}(1-z))} \right|^2 \to 0 \qquad \text{as} \quad n \to \infty$$

for all $z \in D$ by (5.8) and (5.9). Thus we finished the proof.

Open Problem. The above example shows that in Theorem 5 we do not exhaust the possible limiting compound Poisson distribution. We conjecture that every compound Poisson measure can appear as a limiting distribution of an inhomogeneous INAR(1) process.

Theorem 3 can also be extended for the case of limiting compound Poisson distribution.

Theorem 6 *Let* $k_n \in \mathbb{N}$ *for all* $n \in \mathbb{N}$, *and* $\{\zeta_{n,k} : 1 \leqslant k \leqslant k_n,\ n \in \mathbb{N}\}$ *be non-negative integer-valued random variables with factorial moments*

$$m_{n,k,j} := \mathsf{E}\zeta_{n,k}(\zeta_{n,k} - 1)\cdots(\zeta_{n,k} - j + 1), \qquad j \in \mathbb{N}.$$

Moreover, let $p_{n,k} \in [0,1]$, $1 \leqslant k \leqslant k_n$, $n \in \mathbb{N}$. *Assume that*

(i) $\sum_{k=1}^{k_n} p_{n,k}^j m_{n,k,j} \to \lambda_j \in [0, \infty)$ *for all* $j \in \mathbb{N}$ *such that the limits in* (5.7) *exist;*

(ii) $\sum_{k=1}^{k_n} (p_{n,k} \mathsf{E} \zeta_{n,k})^2 \to 0$;

as $n \to \infty$. *Then*

$$\mathop{*}_{k=1}^{k_n} \mathrm{Bi}\big(\mathcal{L}(\zeta_{n,k}), p_{n,k}\big) \to \mathrm{CP}(\mu)$$

in law as $n \to \infty$.

6 Appendix

Lemma 3 *If* $a_k, b_k \in D$, $k = 1, \ldots, n$, *then*

$$\left| \prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k \right| \leqslant \sum_{k=1}^{n} |a_k - b_k|.$$

Proof. The statement follows from

$$\prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k = \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} a_j \right) (a_k - b_k) \left(\prod_{j=k+1}^{n} b_j \right)$$

valid for arbitrary $a_k, b_k \in \mathbb{C}$, $k = 1, \ldots, n$. □

Lemma 4 *If* μ_k, ν_k, $k = 1, \ldots, n$, *are probability measures on* $\mathbb{Z}_+$ *then*

$$d\left(\mathop{*}_{k=1}^{n} \mu_k, \mathop{*}_{k=1}^{n} \nu_k \right) \leqslant \sum_{k=1}^{n} d(\mu_k, \nu_k).$$

Proof. The inequality

$$d\left(\mathop{*}_{k=1}^{n} \mu_k, \mathop{*}_{k=1}^{n} \nu_k \right) \leqslant d\left(\prod_{k=1}^{n} \mu_k, \prod_{k=1}^{n} \nu_k \right)$$

easily follows from the definition of the total variation distance, where $\prod$ denotes product of measures. By Barbour et al. [7, Proposition A.1.1], we have

$$d\left(\prod_{k=1}^{n} \mu_k, \prod_{k=1}^{n} \nu_k \right) \leqslant \sum_{k=1}^{n} d(\mu_k, \nu_k),$$

and we obtain the statement. □

In the proofs we use extensively the following lemma about some summability methods defined by the sequence $(\varrho_n)_{n \in \mathbb{N}}$ of the offspring means.

Lemma 5 *Let* $(\varrho_n)_{n \in \mathbb{N}}$ *be a sequence of real numbers such that* $\varrho_n \in [0, 1)$ *for all* $n \in \mathbb{N}$, $\lim_{n \to \infty} \varrho_n = 1$, *and* $\sum_{n=1}^{\infty} (1 - \varrho_n) = \infty$. *Put*

$$a_{n,j}^{(k)} := (1 - \varrho_j) \prod_{\ell=j+1}^{n} \varrho_\ell^k \qquad \text{for } n, j, k \in \mathbb{N} \text{ with } j \leqslant n.$$

Then $a_{n,j}^{(k)} \leqslant a_{n,j}^{(1)}$ *for all* $n, j, k \in \mathbb{N}$ *with* $j \leqslant n$,

$$\max_{1 \leqslant j \leqslant n} a_{n,j}^{(1)} \to 0 \qquad \text{as} \quad n \to \infty, \tag{6.1}$$

and for an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ *of real numbers with* $\lim_{n\to\infty} x_n = x \in \mathbb{R}$,

$$\sum_{j=1}^{n} a_{n,j}^{(k)} x_j \to \frac{x}{k} \qquad \text{as} \quad n \to \infty \quad \text{for all} \quad k \in \mathbb{N}. \tag{6.2}$$

Proof. For each $J \in \mathbb{N}$, we have the inequality

$$0 \leqslant \max_{1\leqslant j\leqslant n} a_{n,j}^{(1)} \leqslant \max_{j>J} a_{n,j}^{(1)} + \max_{1\leqslant j\leqslant J} a_{n,j}^{(1)} \leqslant \max_{j>J}(1-\varrho_j) + \max_{1\leqslant j\leqslant J}(1-\varrho_j) \prod_{\ell=J+1}^{n} \varrho_\ell,$$

hence letting $n \to \infty$, we obtain

$$0 \leqslant \limsup_{n\to\infty} \max_{1\leqslant j\leqslant n} a_{n,j}^{(1)} \leqslant \max_{j>J}(1-\varrho_j),$$

since

$$0 \leqslant \prod_{\ell=J+1}^{n} \varrho_\ell \leqslant \exp\left\{ -\sum_{\ell=J+1}^{n} (1-\varrho_\ell) \right\} \to 0 \qquad \text{as} \quad n \to \infty. \tag{6.3}$$

Now letting $J \to \infty$ we get

$$0 \leqslant \limsup_{n\to\infty} \max_{1\leqslant j\leqslant n} a_{n,j}^{(1)} \leqslant 0,$$

and we conclude (6.1).

By the Toeplitz theorem, in order to prove (6.2), we have to show

$$\lim_{n\to\infty} a_{n,j}^{(k)} = 0 \qquad \text{for all} \quad j \in \mathbb{N}, \tag{6.4}$$

$$\lim_{n\to\infty} \sum_{j=1}^{n} a_{n,j}^{(k)} = \frac{1}{k}, \tag{6.5}$$

$$\sup_{n\in\mathbb{N}} \sum_{j=1}^{n} |a_{n,j}^{(k)}| < \infty \tag{6.6}$$

for all $k \in \mathbb{N}$. By the assumptions,

$$0 \leqslant a_{n,j}^{(k)} \leqslant (1-\varrho_j) \prod_{\ell=j+1}^{n} \varrho_\ell \leqslant (1-\varrho_j) \exp\left\{ -\sum_{\ell=j+1}^{n} (1-\varrho_\ell) \right\} \to 0 \qquad \text{as} \quad n \to \infty,$$

and we obtain (6.4). Next we prove (6.5) and (6.6) for $k = 1$. We have

$$\sum_{j=1}^{n} a_{n,j}^{(1)} = \sum_{j=1}^{n} (1-\varrho_j) \prod_{\ell=j+1}^{n} \varrho_\ell = \sum_{j=1}^{n} \left(\prod_{\ell=j+1}^{n} \varrho_\ell - \prod_{\ell=j}^{n} \varrho_\ell \right) = 1 - \prod_{\ell=1}^{n} \varrho_\ell \to 1 \qquad \text{as} \quad n \to \infty$$

by (6.3), and $\lim_{n\to\infty} \sum_{j=1}^{n} a_{n,j}^{(1)} = 1$ also implies that $\sup_{n\in\mathbb{N}} \sum_{j=1}^{n} |a_{n,j}^{(1)}| = \sup_{n\in\mathbb{N}} \sum_{j=1}^{n} a_{n,j}^{(1)} < \infty$. Hence we finished the proof of the statement of the lemma in case $k = 1$.

The aim of the following discussion is to show (6.5) and (6.6) for all $k \geqslant 2$. Observe that

$$\begin{aligned} 1 - \prod_{\ell=1}^{n} \varrho_\ell^k &= \sum_{j=1}^{n} \left(\prod_{\ell=j+1}^{n} \varrho_\ell^k - \prod_{\ell=j}^{n} \varrho_\ell^k \right) = \sum_{j=1}^{n} (1 - \varrho_j^k) \prod_{\ell=j+1}^{n} \varrho_\ell^k \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{k} \binom{k}{i} (1 - \varrho_j)^i \right) \prod_{\ell=j+1}^{n} \varrho_\ell^k \\ &= k \sum_{j=1}^{n} a_{n,j}^{(k)} + \sum_{i=2}^{k} \binom{k}{i} \sum_{j=1}^{n} (1 - \varrho_j)^{i-1} a_{n,j}^{(k)}, \end{aligned}$$

where, by (6.3), $0 \leqslant \lim_{n\to\infty} \prod_{\ell=1}^{n} \varrho_\ell^k \leqslant \lim_{n\to\infty} \prod_{\ell=1}^{n} \varrho_\ell = 0$. Moreover,

$$0 \leqslant \sum_{j=1}^{n} (1 - \varrho_j)^{i-1} a_{n,j}^{(k)} \leqslant \sum_{j=1}^{n} (1 - \varrho_j)^{i-1} a_{n,j}^{(1)} \to 0 \qquad \text{as } n \to \infty \text{ for all } i \geqslant 2$$

by the lemma for $k = 1$ and by the assumption $\lim_{n\to\infty} \varrho_n = 1$. Consequently, we obtain (6.5) and hence (6.6) for all $k \geqslant 2$. □

Lemma 6 *Let ε be a nonnegative integer-valued random variable with factorial moments*

$$m_k := \mathsf{E}\varepsilon(\varepsilon - 1)\cdots(\varepsilon - k + 1), \qquad k \in \mathbb{N},$$

$m_0 := 1$, *and with generating function $H(z) = \mathsf{E}(z^\varepsilon)$, defined for $z \in D$. If $m_k < \infty$ for some $k \in \mathbb{N}$ then*

$$H(z) = \sum_{j=0}^{k-1} \frac{m_j}{j!}(z - 1)^j + R_k(z) \qquad \textit{for all } z \in D,$$

where

$$|R_k(z)| \leqslant \frac{m_k}{k!}|z - 1|^k \qquad \textit{for all } z \in D.$$

Proof. By $m_j = \sum_{\ell=0}^{\infty} \ell(\ell - 1) \cdot (\ell - j + 1)\,\mathsf{P}(\varepsilon = \ell)$,

$$R_k(z) = H(z) - \sum_{j=0}^{k-1} \frac{m_j}{j!}(z - 1)^j = \sum_{\ell=0}^{\infty} \left(z^\ell - \sum_{j=0}^{k-1} \binom{\ell}{j} (z - 1)^j \right) \mathsf{P}(\varepsilon = \ell),$$

and by Taylor's formula for the function $z \mapsto z^\ell$ we get

$$\begin{aligned} \left| z^\ell - \sum_{j=0}^{k-1} \binom{\ell}{j} (z - 1)^j \right| &\leqslant \frac{1}{k!}|z - 1|^k \sup_{\theta\in[0,1]} \left| \ell(\ell - 1)\cdots(\ell - k + 1)(1 + \theta(z - 1))^{\ell-k} \right| \\ &\leqslant \frac{1}{k!}\ell(\ell - 1)\cdots(\ell - k + 1)|z - 1|^k \end{aligned}$$

for all $z \in D$. □

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