Utility Maximization of an Indivisible Market with Transaction Costs

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Abstract

This work takes up the challenges of utility maximization problem when the market is indivisible and the transaction costs are included. First there is a so-called solvency region given by the minimum margin requirement in the problem formulation. Then the associated utility maximization is formulated as an optimal switching problem. The diffusion turns out to be degenerate and the boundary of domain is an unbounded set. One no longer has the continuity of the value function without posing further conditions due to the degeneracy and the dependence of the random terminal time on the initial data. This paper provides sufficient conditions under which the continuity of the value function is obtained. The essence of our approach is to find a sequence of continuous functions locally uniformly converging to the desired value function. Thanks to continuity, the value function can be characterized by using the notion of viscosity solution of certain quasi-variational inequality.

Key Words. Utility optimization, indivisible market, transaction cost, continuity of value function, quasi-variational inequality.

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1 Introduction

The study of utility optimization has a long history. Utility maximization under the setup of Black-Scholes type models can be traced back to [18]. By now, it is widely understood that in a complete market, the optimal strategy of this problem is attainable if an investor can make infinitesimally small adjustments of the position frequently. Recent study indicates that market imperfections such as transaction costs and asset indivisibility affect virtually every transaction and generate costs, which interfere with the trades that rational individuals would make in a complete market (see [9]). As was alluded to in the above, the two main assumptions, namely zero transaction costs and infinite divisibility of an asset, are crucial. Failure in either of the two assumptions results in an incomplete market, so Merton's optimal strategy becomes non-attainable.

From a practical point of view, although the technical advancement and the on-line trading make the transaction costs not significantly influential, the transaction costs can hardly be ignored. As for the other assumption, it is almost evident that an asset cannot be infinitely divisible in any practical situation.

Incorporating transaction cost in utility maximization has received much attentions from both researchers and practitioners in the past few decades. In fact, there is a vast literature on this subject; see for example, [6, 7, 13, 16, 17, 20, 24, 26], and the references therein. In contrast, there are relatively few works on asset indivisibility. Two of the exceptions are [23] and [25]. It should be noted that most existing works on asset indivisibility have focused on discrete-time models. Our goal in this paper is to take up the challenges in both parts. We will characterize the solution of the utility maximization problems of an indivisible market with transaction cost in continuous time.

To incorporate the asset indivisibility, the stock shares in the portfolio are restricted to a finite set of integers $\mathcal{K}$ (to be defined in (2.5)). In addition, there is a minimum maintenance margin requirement for the investigator; the corresponding condition is termed as a solvency region O (to be defined in (2.7)). The associated utility maximization is modeled as an optimal switching problem on degenerate diffusion in the restricted unbounded domain. It is noted that with nondegenerate diffusion, the value function can be shown to belongs to, for example, $W^{1,\infty}(O) \cap W^{2,\infty}_{\text{loc}}(O)$ for a bounded domain [10], and $W^{1,\infty}$ for a one-dimensional unbounded domain [21].

In our work, one cannot obtain the continuity of the value function V of (2.12) for free since the underlying process (X, Y, Z) is degenerate and the random terminal time τ of (2.11) depends on the initial condition (t, x, y, z); see the counterexample in [3, Example 4.1] with the absence of optimal switching. As a result, to characterize the value function, we use the notion of viscosity solution for quasi-variational inequality. It turns out to be crucial to show the continuity of the value function with some appropriate conditions.

The continuity of the value function in a bounded domain has been widely discussed within the framework of classical stochastic control theory without switching costs, known as stochastic exit problem. When the domain is bounded, a sufficient condition for the continuity of the value function is provided in [11, p. 205] by using a probabilistic approach, where the continuity was presented in terms of the drift of the underlying diffusion. In contrast, a generalization of the continuity in [3] gave a condition taking into consideration of both the drift and diffusion coefficients. Along another line, the stochastic exit-time control problem has been studied by using purely analytical methods in [1, 2, 14, 15] under various setups.

In the current work, we use a probabilistic approach similar to that of [3] and [11]. We focus on utility optimization for indivisible cost with transaction costs. The essence depends on the verification of a continuity condition. We note that the main effort of [3] is to find a sequence of continuous functions uniformly converging to the desired value function, taking into consideration of the sample path properties of the diffusion processes. In this procedure, Dini's theorem plays an essential role to obtain the uniform convergence. However, this approach is not directly applicable to our work. This is because the boundary of the domain ∂O is unbounded. Because of the domain being non-compact, Dini's theorem cannot be used. Therefore, one needs asymptotic properties of the approximating functions. Here we devise an approximation sequence V^ε (see Lemma 3.2), and obtain the continuity of V by local uniform estimates using V^ε. The details are in Theorem 3.1 in what follows.

The rest of the work is arranged as follows. The precise formulation of the problem is given in Section 2. Section 3 is devoted to continuity of the value function. Section 4 analyzes properties of the value functions. In particular, we show that the value function is the unique viscosity solution of the quasi-variational inequality (2.17) with boundary-terminal condition (2.18). Section 5 makes some further remarks to conclude the paper. At the end, supplemental results are included in an appendix in Section 6.

2 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a complete filtered probability space on which is defined a standard Brownian motion W, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$. We assume that the filtration $\mathbb{F}$ is generated by W, augmented by all the P-null sets as usual. For simplicity, we assume that the financial market consists of only two assets, a bank account with zero interest and a risky asset.

Suppose that $X^{t,x}$, the price of the risky asset, is given by

$$X(s) = x + \int_t^s b(r, X(r))dr + \int_t^s \sigma(r, X(r))dW(r), \tag{2.1}$$

where $x > 0$ denotes the initial price. A bank account with positive interest can be considered in the current setup. Other than notational complexity, such a formulation does not introduce essential difficulties as long as the interest rates are not stochastic. Therefore, for simplicity, we use zero risk-free interest rate in this paper. Throughout the paper, we use the following standing assumptions. The objective function is an expected utility with transaction costs taken into consideration, whose precise form will be given shortly.

Assumption 2.1.

1. There exists a $C_1 > 0$ such that the drift b and the volatility σ satisfy

$$b(s,0) = \sigma(s,0) = 0, \text{ and } |b(s,x_1) - b(s,x_2)| + |\sigma(s,x_1) - \sigma(s,x_2)| \leq C_1|x_1 - x_2|. \tag{2.2}$$

2. The transaction cost function $c : \mathbb{Z} \mapsto \mathbb{R}$ satisfies

$$c(0) = 0, \quad c(z) > 0 \ \forall z \neq 0, \quad \text{and} \quad c(z_1) + c(z_2) \geq c(z_1 + z_2).$$

3. The risk-averse utility function $U : [0,\infty) \to [0,\infty)$ satisfies

$$U(0) = 0,\ U'(x) > 0,\ U''(x) < 0,\ \lim_{x\to\infty} U'(x) = 0, \quad \text{and} \quad \lim_{x\to 0} U'(x) = \infty, \tag{2.3}$$

where U' and U'' denote the first and the second derivatives of U with respect to x, respectively.

With condition (2.2), the price $X(s)$ stays nonnegative for all $s \geq t$. Note that (2.2) also implies linear growth of the functions b and σ in the variable x, and hence (2.1) has a unique strong solution. For a fixed time duration $[t,T]$, an investor has an initial wealth y and holds z shares of stock at price x, and hence $y - zx$ is the initial amount in the bank. We denote the ith nonzero trading occurs at time τ_i, and assume at most one transaction occurs at each time, i.e.,

$$t^- = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_N \leq T, \quad \text{for some } N. \tag{2.4}$$

We use $Z(s) = \sum_{i=0}^{N-1} Z(\tau_i) \mathbb{1}_{[\tau_i,\tau_{i+1})}(s)$ to denote the position of the risky asset in the portfolio at time s, and use $\Delta Z(s) = Z(s) - Z(s^-)$ denote the amount of transaction traded at time s. Therefore, the associated transaction cost at the ith transaction is $c(\Delta Z(\tau_i))$.

In practice, the risky asset traded in the market is indivisible. As a result, we restrict the investor's position in the risky asset to a set of finite integers $\mathcal{K}$, i.e., for some positive integer C_2 and C_3

$$\mathcal{K} \triangleq \{-C_2, -C_2+1, \ldots, 0, \ldots, C_3-1, C_3\}. \tag{2.5}$$

Then, with the initial investment y, the total wealth $\{Y^{t,x,y,z,Z}(s) : t \le s \le T\}$ follows

$$\begin{aligned} dY(s) &= Z(s)b(s, X(s))ds + Z(s)\sigma(s, X(s))dW(s), \quad s \in (\tau_i, \tau_{i+1}), \\ Y(\tau_i) &= Y(\tau_i^-) - c(\Delta Z(\tau_i)). \end{aligned}$$

One can rewrite the wealth process as

$$Y(s) = y + \int_t^s Z(r)b(r, X(r))dr + \int_t^s Z(r)\sigma(r, X(r))dW(r) - \sum_{\tau_i \le s} c(\Delta Z(\tau_i)). \tag{2.6}$$

Let the minimum maintenance margin requirement for the investor's account be $c(-Z(s))$, i.e., $Y(s) > c(-Z(s))$. The investor will receive a margin call at $\hat{\tau} = \inf\{s : Y(s) \le c(-Z(s))\}$, if $\hat{\tau} < T$ occurs. Under the self-financing rule, we assume no additional capital is available, and the investor has to clear the risky asset (zero capital remaining after clearance) at $\hat{\tau}$. In other words, we define the *solvency region* as

$$O = \{(x, y, z) : x > 0, y > c(-z), z \in \mathcal{K}\}. \tag{2.7}$$

Thus the state process $(X(s), Y(s), Z(s))$ satisfies the state constraint

$$(X(s), Y(s), Z(s)) \in O, \quad \text{Lebesgue-a.e. } s \in [t, T], \ \mathbb{P} - a.s.\ \omega \in \Omega. \tag{2.8}$$

In this work, $Z(s)$ is a control variable. Note that due to the state constraint (2.8), the control $Z(s)$ belongs to a state-dependent set $Z(s^-) + \Gamma(Y(s^-), Z(s^-))$, where $\Gamma(\cdot, \cdot)$ is a set-valued function given by (2.10), and $z + \Gamma(y, z)$ is understood as a set translation.

Definition 2.1 (Admissible control space)**.** Given $t \in [0, T]$, the set of admissible strategies, denoted as $\mathcal{Z}(t, x, y, z)$, is the space of $\mathbb{F}$-adapted processes Z over $[t, T]$ such that

1. For any $s \in [t, T]$, $Z(s) \in \mathcal{K}$ has the following form. For a sequence of strictly increasing stopping times, (2.4)

$$Z(s) = \sum_{i=0}^{N-1} Z(\tau_i)\mathbb{1}_{[\tau_i, \tau_{i+1})}(s), \quad Z(t^-) = z. \tag{2.9}$$

2. For $i \ge 1$, $\Delta Z(\tau_i) \in \Gamma(Y(\tau_i^-), Z(\tau_i^-))$, where

$$\Gamma(y, z) = \{\tilde{z} \in \mathcal{K} : c(\tilde{z} - z) + c(-\tilde{z}) \le y, \tilde{z} \ne z\}. \tag{2.10}$$

Remark 2.1. Definition 2.1 means the investor will trade only finitely many times during $[t, T]$ almost surely. If not, $Y(T) = -\infty$ almost surely due to $\min_{\mathcal{K}\setminus\{0\}} c(z) > 0$. Also, (2.9) implies $Z(T) = 0$, i.e., the investor will always clear his or her stock position at T and will hold only cash in the bank. Such an assumption is not unusual; see for example, [4] and [5]. On the other hand, the amount trading $\Delta Z(\tau_i)$ is required to take a value in a state-dependent set $\Gamma(Y(\tau_i^-), Z(\tau_i^-))$. This is the minimum requirement to keep the state, $(X(\tau_i), Y(\tau_i), Z(\tau_i))$, belonging to $\bar{O}$ (the closure of O) right after the transaction, and prevents the investor quits the market with negative wealth.

Let the stopping time τ be

$$\tau = \inf\{s : Y(s) \leq c(-Z(s))\} \wedge T. \tag{2.11}$$

For a given initial state (t, x, y, z), the investor's goal is to maximize the expected utility of the total wealth

$$J(t, x, y, z, Z) = \mathbb{E}[U(Y^{t,x,y,z,Z}(\tau))]$$

over all admissible strategy space $\mathcal{Z}(t, x, y, z)$. Therefore, the value function of our problem is

$$V(t, x, y, z) = \sup_{Z \in \mathcal{Z}(t,x,y,z)} J(t, x, y, z, Z) = \sup_{Z \in \mathcal{Z}(t,x,y,z)} \mathbb{E}[U(Y^{t,x,y,z,Z}(\tau))]. \tag{2.12}$$

Remark 2.2 (Discussions on assumptions)**.** There are two key assumptions in the formulation of the problem. One is the transaction cost $c(\cdot)$ being subadditive, and the other is that there is at most one transaction at any time, and thus the representation of $Z(\cdot)$ as a piecewise constant process is well defined. These are reasonable assumptions from the following point of view. Suppose there are multiple nonzero transactions occurred at time s, i.e.,

$$\tau_i = \tau_{i+1} = \cdots = \tau_{i+m} = s \text{ for some } i, m \geq 1, \text{ and } t \leq s \leq T,$$

and the transaction cost $c(\cdot)$ is not necessarily subadditive. Denote by ΔZ_k the number of shares traded at the kth transaction. The investor is obliged to pay the total transaction cost $\sum_{j=0}^m c(\Delta Z_{i+j})$ at time s. Then, we can always construct another function $\tilde{c}(\cdot)$ by

$$\tilde{c}(z) = \min\Big\{ \sum_{i=1}^n c(z_i) : z_1 + z_2 + \cdots + z_n = z \text{ for some } n \Big\}.$$

and such a $\tilde{c}(\cdot)$ turns out to be a subadditive function. Therefore, the multiple transactions at time s can always be replaced by a *single transaction* of the amount $\sum_{j=0}^m \Delta Z_{i+j}$ shares in terms of the new subadditive transaction cost function $\tilde{c}(\cdot)$. As a result, the strategy remains the same as before, while the transaction cost becomes less under $\tilde{c}(\cdot)$, i.e., $\sum_{j=0}^m c(\Delta Z_{i+j}) \geq \tilde{c}(\sum_{j=0}^m \Delta Z_{i+j})$; see [17] for a more general discussion.

We define two operators

$$\mathcal{L}\varphi(t,x,y,z) = b\varphi_x + \frac{1}{2}\sigma^2\varphi_{xx} + zb\varphi_y + \frac{1}{2}z^2\sigma^2\varphi_{yy} + z\sigma^2\varphi_{xy}, \tag{2.13}$$

and

$$\mathcal{S}\varphi(t,x,y,z) = \max_{\tilde{z}\in\Gamma(y,z)} \varphi(t,x,y-c(\tilde{z}-z),\tilde{z}). \tag{2.14}$$

In the above, we used $\max_{z\in\emptyset}\varphi(t,x,y-c(\tilde{z}-z),\tilde{z}) = -\infty = -\min_{z\in\emptyset}\varphi(t,x,y-c(\tilde{z}-z),\tilde{z})$ as convention. In the definition of $\mathcal{S}$, we used max instead of sup owing to the finite cardinality of $\Gamma(y,z)$. Also note that, the operator $\mathcal{L}$ of (2.13) is degenerate. In other words, the diffusion (X,Y,Z) is always degenerate, even if X is non-degenerate.

Provided that V is a continuous function, we can proceed with the dynamic programming principle (DPP) and obtain

$$V(t,x,y,z) = \sup_{Z\in\mathcal{Z}(t,x,y,z)} \mathbb{E}[V(\theta, X^{t,x}(\theta), Y^{t,x,y,z,Z}(\theta), Z(\theta))], \ \forall\theta\le\tau.$$

The general discussions of DPP is referred to [11, 22]. If we appeal DPP on instantaneous transaction strategy with $\tau_1 = t$, then it follows that

$$V(t,x,y,z) \ge \mathcal{S}V(t,x,y,z), \ (t,x,y,z)\in[0,T]\times O. \tag{2.15}$$

Define an operator $\mathcal{A}$ that maps from measurable functions $\varphi : (0,T)\times O\to\mathbb{R}$ to set-valued functions $\mathcal{A}[\phi]$ on $\mathcal{K}$ given by

$$\mathcal{A}[\varphi](z) = \{(t,x,y) : \varphi(t,x,y,z) > \mathcal{S}\varphi(t,x,y,z)\}. \tag{2.16}$$

Note that $\mathcal{A}[V](z)$ is a *no-action region* associated with $z\in\mathcal{K}$. DPP implies that for the initial data $(t,x,y)\in\mathcal{A}[V](z)$, the value process $V(s,X^{t,x}(s),Y^{t,x,y,z}(s),z)$ is a martingale in $A[V](z)$, whose generator is given by $\frac{\partial}{\partial t}+\mathcal{L}$. Moreover, a heuristic derivation leads to that V satisfies the following quasi-variational inequality

$$\min\{-u_t - \mathcal{L}u, u - \mathcal{S}u\} = 0, \ \text{on } [0,T)\times O. \tag{2.17}$$

We aim to show the value function V is the unique viscosity solution of the quasi-variational inequality (2.17) with Cauchy-Dirichlet data

$$u(t,x,y,z) = U(y-c(-z)), \ \text{on } \partial^*([0,T)\times O), \tag{2.18}$$

where $\partial^*([0,T)\times O)$ is the parabolic boundary. It turns out to be crucial to know the continuity of V *a priori*.

For later use in the uniqueness proof, we define the function F as

$$\begin{aligned} F(t,x,y,z,q,p,A) \;&= -q-(b(t,x)p_1+\tfrac{1}{2}\sigma^2(t,x)A_{11}+zb(t,x)p_2 \\ &\quad +\tfrac{1}{2}z^2\sigma^2(t,x)A_{22}+z\sigma^2(t,x,)A_{22}+z\sigma^2(t,x)A_{12}). \end{aligned} \tag{2.19}$$

Then, (2.17) can be rewritten as

$$\min\{F(t,x,y,z,u_t,Du,D^2u),u-\mathcal{S}u\}=0. \tag{2.20}$$

3 Continuity

Continuity is crucial to characterize the value function as the unique viscosity solution. The difficulty to show the continuity of $V(\cdot)$ stems from the following:

1. the stopping time τ of (2.11) depends on the initial state (x,y);
2. the boundary $\partial^*([0,T)\times O)$ is an unbounded set;
3. the control space $\mathcal{Z}(t,x,y,z)$ depends on the initial state (x,y).

To prove the continuity of $V(\cdot)$, we introduce another value function $V^\varepsilon(\cdot)$ in what follows, which avoids the above two issues of $V(\cdot)$. Let the strategy space $\mathcal{Z}(t,z)$ be defined as a strategy space without constraint (2.8), so that the space does not depend on the initial state (x,y), i.e.,

$$\mathcal{Z}(t,z)=\{Z:Z(t^-)=z,\mathcal{K}\ni Z(s)=\textstyle\sum_{i=0}^{N-1}Z(\tau_i)\mathbb{1}_{[\tau_i,\tau_{i+1})}(s)\text{ for some }N,Z(T)=0\}.$$

Recall that τ of (2.11) is defined as the first exit time of the random process $(t,X^{t,x,},Y^{t,x,y,z},Z)$ from the domain $[0,T)\times O$. Thus, one can rewrite V of (2.12) as,

$$V(t,x,y,z)=\sup_{Z\in\mathcal{Z}(t,z)}\mathbb{E}[U(Y^{t,x,y,z,Z}(T))\mathbb{1}_{\{\tau=T\}}].$$

We also define Λ^ε as a penalty function of the form

$$\Lambda^\varepsilon(t,s,Y,Z)=\exp\Big\{-\frac{1}{\varepsilon}\int_t^s\Big(c(-Z(r))-Y(r)\Big)^+dr\Big\}, \tag{3.1}$$

where $c(z)^+$ denotes the positive part of $c(z)$ as usual. Finally, we define V^ε as

$$V^\varepsilon(t,x,y,z)=\sup_{Z\in\mathcal{Z}(t,z)}\mathbb{E}[\Lambda^\varepsilon(t,T,Y^{t,x,y,z,Z},Z)U(Y^{t,x,y,z,Z}(T))]. \tag{3.2}$$

In the above, we extend the function $U(\cdot)$ to $(-\infty,\infty)$ by $U(x)=0$ for any $x<0$. Since $\Lambda^\varepsilon\equiv 1$ on the set $\{\tau=T\}$, it leads to

$$V^\varepsilon(t,x,y,z)\ge V(t,x,y,z),\ \forall(t,x,y,z).$$

The $V^\varepsilon(t,x,y,z)$ can be thought of as a penalized or regularized "value function." We use V^ε to establish the desired properties of V. The tasks to be performed are:

1. to show that $V^\varepsilon(\cdot,\cdot,\cdot,z)$ is continuous for each ε;

2. to show that V^ε converges monotonically to V in $[0,T)\times O$; and

3. to show that V^ε converges locally uniformly to V.

3.1 Preliminary Results

Proposition 3.1 (Properties of $\mathcal{S}$)**.** *The following properties hold for the operator $\mathcal{S}$:*

1. (Monotonicity) $\mathcal{S}u \geq \mathcal{S}v$ *whenever* $u \geq v$.

2. (sub-distributivity) $\mathcal{S}(u+v) \leq \mathcal{S}u + \mathcal{S}v$.

3. (Preservation of continuity) $\mathcal{S}u$ *is continuous in* (t,x,y) *whenever* u *is continuous in* (t,x,y).

Proof. 1. (Monotonicity) If $u \geq v$, then by definition (2.14)

$$\begin{aligned}
&\mathcal{S}u(t,x,y,z) - \mathcal{S}v(t,x,y,z)\\
&\quad = \max\nolimits_{\tilde z\in\Gamma(y,z)} u(t,x,y-c(\tilde z - z),\tilde z) - \max\nolimits_{\tilde z\in\Gamma(y,z)} v(t,x,y-c(\tilde z - z),\tilde z)\\
&\quad = \max\nolimits_{\tilde z\in\Gamma(y,z)} u(t,x,y-c(\tilde z - z),\tilde z) + \min\nolimits_{\tilde z\in\Gamma(y,z)} (-v)(t,x,y-c(\tilde z - z),\tilde z)\\
&\quad \geq \min\nolimits_{\tilde z\in\Gamma(y,z)} (u-v)(t,x,y-c(\tilde z - z),\tilde z) \geq 0.
\end{aligned}$$

2. (sub-distributivity) The proof is obvious and thus omitted.

3. (Preservation of continuity) For each pair $(z,\tilde z)$, $u(t,x,y-c(\tilde z - z),\tilde z)$ is continuous in (t,x,y). Also, note that $\Gamma(y,z)$ is a finite set. Thus, $\max_{\tilde z\in\Gamma(y,z)} u(t,x,y-c(\tilde z - z),\tilde z)$ is also continuous.

□

Lemma 3.1. *Let $Z \in \mathcal{Z}(t,z)$. For any $m \geq 1$, the wealth process Y given by* (2.6) *satisfies*

$$\mathbb{E}\Big[\sup_{t\leq s\leq T} |Y^{t,x,y,z,Z}(s) - y + \sum_{\tau_i\leq s} c(\Delta Z_{\tau_i})|^m\Big] \leq C_{m,T}|x|^m, \tag{3.3}$$

and

$$\mathbb{E}\Big[\sup_{t\leq s\leq T} |Y^{t,x_1,y_1,z,Z}(s) - Y^{t,x_2,y_2,z,Z}(s)|^m\Big] \leq C_{m,T}(|x_1-x_2|^m + |y_1-y_2|^m). \tag{3.4}$$

Proof. We denote $Y^Z \triangleq Y^{t,x,y,z,Z}$ and $X \triangleq X^{t,x}$. Using the Burkholder-Davis-Gundy (BDG) and Hölder inequalities multiple times combined with linear growth and Lipschitz conditions in (2.2), we compute

$$\begin{aligned}
&\mathbb{E}\Big[(\sup_{t\le s\le T}|Y^{t,x,y,z,Z}(s)-y+\textstyle\sum_{\tau_i\le s}c(\Delta Z_{\tau_i})|^m\Big]\\
&\le C_m\mathbb{E}\Big[\sup_s\Big|\int_t^s Z(r)b(r,X(r))dr\Big|^m\Big]+C_m\mathbb{E}\Big[\sup_s\Big|\int_t^s Z(r)\sigma(r,X(r))dW(r)\Big|^m\Big]\\
&\le C_m\mathbb{E}\Big[\sup_s\int_t^s |Z(r)b(r,X(r))|^m dr\Big]+C_m\mathbb{E}\Big[\Big(\int_t^T Z^2(r)\sigma^2(r,X(r))dr\Big)^{m/2}\Big]\\
&\le C_m\mathbb{E}\Big[\int_t^T |b(r,X(r))|^m dr\Big]+C_m\mathbb{E}\Big[\Big(\int_t^T \sigma^2(r,X(r))dr\Big)^{m/2}\Big]\\
&\le C_m\mathbb{E}\Big[\int_t^T |X(r)|^m dr\Big]+C_m\mathbb{E}\Big[\Big(\int_t^T |X(r)|^2 dr\Big)^{m/2}\Big]\\
&\le C_{m,T}|x|^m.
\end{aligned}$$

Then (3.3) follows. For convenience, we also denote $Y^{i,Z}\triangleq Y^{t,x_i,y_i,z,Z}$ and $X^i\triangleq X^{t,x_i}$ for $i=1,2$. Similar arguments lead to

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\le s\le T}|Y^{1,Z}(s)-Y^{2,Z}(s)|^m\Big]\\
&\le C_m|y_1-y_2|^m+C_m\mathbb{E}\Big[\sup_s\Big|\int_t^s Z(r)(b(r,X^1(r))-b(r,X^2(r)))dr\Big|^m\Big]\\
&\qquad\qquad +C_m\mathbb{E}\Big[\sup_s\Big|\int_t^s Z(r)(\sigma(r,X^1(r))-\sigma(r,X^2(r)))dW(r)\Big|^m\Big]\\
&\le C_m|y_1-y_2|^m+C_m\mathbb{E}\Big[\sup_s\int_t^s |Z(r)(b(r,X^1(r))-b(r,X^2(r)))|^m dr\Big]\\
&\qquad\qquad +C_m\mathbb{E}\Big[\Big(\int_t^T Z^2(r)(\sigma(r,X^1(r))-\sigma(r,X^2(r)))^2dr\Big)^{m/2}\Big]\\
&\le C_m|y_1-y_2|^m+C_m\mathbb{E}\Big[\int_t^T |X^1(r)-X^2(r)|^m dr\Big]+C_m\mathbb{E}\Big(\int_t^T |X^1(r)-X^2(r)|^2dr\Big)^{m/2}\\
&\le C_m|y_1-y_2|^m+C_{m,T}|x_1-x_2|^m.
\end{aligned}$$

□

3.2 Properties of V^ε

Lemma 3.2. *$V^\varepsilon(t,x,y,z)$ is increasing in y, and continuous in (t,x,y). Furthermore, V^ε satisfies*

$$\lim_{x\to\infty}\sup_{t,\varepsilon}\frac{V^\varepsilon(t,x,y,z)}{x}=0,\ \forall(y,z), \tag{3.5}$$

and

$$\lim_{y\to\infty}\sup_{t,\varepsilon}\frac{V^\varepsilon(t,x,y,z)}{y}=0,\ \forall(x,z), \tag{3.6}$$

Proof. It is clear that V^ε is increasing in y.

1. In this part, we prove V^ε is continuous in (x,y). Given that $(t,z)\in[0,T]$ and $(x_i,y_i)\in\mathbb{R}^+\times\mathbb{R}$ with $i=1,2$, we denote

$$Y^{i,Z,+}\triangleq\max\{Y^{i,Z},0\},\quad i=1,2.$$

Then we have

$$
\begin{aligned}
&|V^\varepsilon(t,x_1,y_1,z) - V^\varepsilon(t,x_2,y_2,z)| \\
&= \Big| \sup_{Z\in\mathcal{Z}(t,z)} \mathbb{E}[\Lambda^\varepsilon(t,T,Y^{1,Z},Z)U(Y^{1,Z,+}(T))] \\
&\qquad\qquad - \sup_{Z\in\mathcal{Z}(t,z)} \mathbb{E}[\Lambda^\varepsilon(t,T,Y^{2,Z},Z)U(Y^{2,Z,+}(T))]\Big| \\
&= \sup_{Z\in\mathcal{Z}(t,z)} \mathbb{E}\Big|\Lambda^\varepsilon(t,T,Y^{1,Z},Z)U(Y^{1,Z,+}(T)) - \Lambda^\varepsilon(t,T,Y^{2,Z},Z)U(Y^{2,Z,+}(T))\Big| \\
&\le \sup_{Z\in\mathcal{Z}(t,z)} \mathbb{E}\Big|(\Lambda^\varepsilon(t,T,Y^{1,Z},Z) - \Lambda^\varepsilon(t,T,Y^{2,Z},Z))U(Y^{1,Z,+}(T))\Big| \\
&\qquad\qquad + \sup_{Z\in\mathcal{Z}(t,z)} \mathbb{E}\Big|\Lambda^\varepsilon(t,T,Y^{2,Z},Z)(U(Y^{1,Z,+}(T)) - U(Y^{2,Z,+}(T)))\Big| \\
&\le \sup_{Z\in\mathcal{Z}(t,z)} \|\Lambda^\varepsilon(t,T,Y^{1,Z},Z) - \Lambda^\varepsilon(t,T,Y^{2,Z},Z)\|_2 \|U(Y^{1,Z,+}(T))\|_2 \\
&\qquad\qquad\qquad + \sup_{Z\in\mathcal{Z}(t,z)} \mathbb{E}\Big|(U(Y^{1,Z,+}(T)) - U(Y^{2,Z,+}(T)))\Big|.
\end{aligned}
\tag{3.7}
$$

The last inequality of (3.7) follows from Hölder's inequality and the fact $|\Lambda^\varepsilon| \le 1$. In the above and what follows, we use $\|\cdot\|_2$ to denote the norm in the space $L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R})$. To proceed, we examine each of the terms after the last inequality sign in (3.7).

Since we have

$$
\begin{aligned}
&|\Lambda^\varepsilon(t,T,Y^{1,Z},Z) - \Lambda^\varepsilon(t,T,Y^{2,Z},Z)| \\
&= \Big|\exp\{-\frac{1}{\varepsilon}\int_t^T (c(-Z(r)) - Y^{1,Z}(r))^+ dr\} \\
&\qquad\qquad - \exp\{-\frac{1}{\varepsilon}\int_t^T (c(-Z(r)) - Y^{2,Z}(r))^+ dr\}\Big| \\
&\le \frac{1}{\varepsilon}\Big|\int_t^T \Big((c(-Z(r)) - Y^{1,Z}(r))^+ - (c(-Z(r)) - Y^{2,Z}(r))^+\Big)dr\Big| \\
&\le \frac{1}{\varepsilon}\int_t^T |Y^{1,Z}(r) - Y^{2,Z}(r)|dr \\
&\le C_{\varepsilon,T} \sup_{r\in[t,T]} |Y^{1,Z}(r) - Y^{2,Z}(r)| \quad \text{a.s.},
\end{aligned}
\tag{3.8}
$$

the first factor in the next to the last row of (3.7) is

$$
\begin{aligned}
&\|\Lambda^\varepsilon(t,T,Y^{1,Z},Z) - \Lambda^\varepsilon(t,T,Y^{2,Z},Z)\|_2 \\
&= \Big(\mathbb{E}|\Lambda^\varepsilon(t,T,Y^{1,Z},Z) - \Lambda^\varepsilon(t,T,Y^{2,Z},Z)|^2\Big)^{1/2} \\
&\le C_{\varepsilon,T}\Big(\mathbb{E}\sup_{r\in[t,T]} |Y^{1,Z}(r) - Y^{2,Z}(r)|^2\Big)^{1/2} \\
&\le C_{\varepsilon,T}(|x_1 - x_2| + |y_1 - y_2|),
\end{aligned}
\tag{3.9}
$$

where the last inequality follows from (3.4).

For the second factor in the next to the last row in (3.7), we utilize the fact $U^2(x) \le C(1+x^2)$ for some large C due to concavity of U

$$
\|U(Y^{1,Z,+}(T))\|_2 = \Big(\mathbb{E}[U^2(Y^{1,Z,+}(T))]\Big)^{1/2} \le C + C\|Y^{1,Z,+}(T)\|_2. \tag{3.10}
$$

Note that $|Y^{1,Z,+}(T)| \le |Y^{1,Z}(T) + \sum_{\tau_i \le T} c(\Delta Z_{\tau_i})|$, one can use the result of (3.3) to obtain

$$\|Y^{1,Z,+}(T)\|_2 \le (\mathbb{E}|Y^{t,x_1,y_1,Z}(s) + \sum_{\tau_i \le s} c(\Delta Z_{\tau_i})|^2)^{1/2} \le C_T(|x_1| + |y_1|). \tag{3.11}$$

Combining the inequalities (3.10) and (3.11), we have

$$\|U(Y^{1,Z,+}(T))\|_2 \le C_T(|x_1| + |y_1| + 1). \tag{3.12}$$

For the last term of (3.7), we use $|U(x_1) - U(x_2)| \le U(|x_1 - x_2|)$ and Jensen's inequality to obtain

$$\mathbb{E}\Big|(U(Y^{1,Z,+}(T)) - U(Y^{2,Z,+}(T)))\Big| \le U(\mathbb{E}|Y^{1,Z,+}(T) - Y^{2,Z,+}(T)|).$$

Also, thanks to (3.4), we further obtain

$$\mathbb{E}\Big|(U(Y^{1,Z,+}(T)) - U(Y^{2,Z,+}(T)))\Big| \le U(C_T(|x_1 - x_2| + |y_1 - y_2|)). \tag{3.13}$$

Coming back to (3.7) with the estimates (3.9), (3.12), and (3.13), we have

$$\begin{aligned} &|V^\varepsilon(t, x_1, y_1, z) - V^\varepsilon(t, x_2, y_2, z)| \\ &\le C_{\varepsilon,T}(|x_1 - x_2| + |y_1 - y_2|)(|x_1| + |y_1| + 1) + U(C_T(|x_1 - x_2| + |y_1 - y_2|)). \end{aligned} \tag{3.14}$$

Therefore, V^ε is continuous in (x, y).

2. With the continuity of V^ε in (x, y), we are now ready to establish the continuity of V^ε in t. We assume $t_1 < t_2$ and fix (x, y). By the definition of V^ε in (3.2), for any $Z_1 \in \mathscr{Z}(t_1, z)$

$$\begin{aligned} &V^\varepsilon(t_1, x, y, z) - V^\varepsilon(t_2, x, y, z) \ge \\ &\mathbb{E}_{t_1}[\Lambda^\varepsilon(t_1, t_2, Y^{t,x,y,z,Z_1}, Z_1)V^\varepsilon(t_2, X^{t_1,x}(t_2), Y^{t_1,x,y,z,Z_1}(t_2), Z_1(t_2))] - V^\varepsilon(t_2, x, y, z). \end{aligned} \tag{3.15}$$

If we restrict sup of (3.15) in $Z_1 \in \mathscr{Z}(t_1, z) : Z_1(s) = z \ \forall s \in [t_1, t_2]$, then it gives a one-sided estimate

$$\begin{aligned} &V^\varepsilon(t_1, x, y, z) - V^\varepsilon(t_2, x, y, z) \\ &\quad \ge \mathbb{E}_{t_1}[\Lambda^\varepsilon(t_1, t_2, Y^{t,x,y,z}, z)V^\varepsilon(t_2, X^{t_1,x}(t_2), Y^{t_1,x,y,z}(t_2), z)] - V^\varepsilon(t_2, x, y, z) \\ &\quad = \mathbb{E}_{t_1}[\Lambda^\varepsilon(t_1, t_2, Y^{t,x,y,z}, z)(V^\varepsilon(t_2, X^1(t_2), Y^1(t_2), z) - V^\varepsilon(t_2, x, y, z))] \\ &\qquad\qquad - \mathbb{E}_{t_1}[(1 - \Lambda^\varepsilon(t_1, t_2, Y^1, z))V^\varepsilon(t_2, x, y, z)]. \end{aligned} \tag{3.16}$$

The last term of (3.16) vanishes as $t_2 \to t_1$ by the dominated convergence theorem. The term on the next to the last line also goes to zero as $t_2 \to t_1$, due to

(a) estimation of (3.14) on V^ε in (x, y)

(b) the inequality

$$\mathbb{E}[\sup_{t_1 \le s t_2}(|X^1(t_2)-x|^m + |Y^1(t_2)-y|^m] \le C_m(1+|x|^m)(t_2-t_1)^{m/2}; \text{ and}$$

(c) $|\Lambda^\varepsilon| \le 1$.

Therefore, $\lim_{t_2\to t_1}(V^\varepsilon(t_1,x,y,z) - V^\varepsilon(t_2,x,y,z)) \ge 0$, and V^ε is left upper semicontinuous. For any $Z \in Z(t_1,z)$, we design $\hat{Z}(s) = Z(s)$ for all $s \ge t_2$, and $\hat{Z}(t_2^-) = z$. Then $\hat{Z} \in \mathcal{Z}(t_2,z)$. Thus,

$$\begin{aligned}
&V^\varepsilon(t_1,x,y,z) - V^\varepsilon(t_2,x,y,z)\\
&\le \sup_{Z\in\mathcal{Z}(t_1,z)} \Big\{\mathbb{E}_{t_1}[\Lambda^\varepsilon(t_1,t_2,Y^{1,Z},Z)V^\varepsilon(t_2,X^1(t_2),Y^{1,Z}(t_2),Z(t_2))]\\
&\qquad\qquad\qquad\qquad -\mathbb{E}_{t_1}[J^\varepsilon(t_2,x,y,z,\hat{Z})]\Big\}\\
&\le \sup_{Z\in\mathcal{Z}(t_1,z)} \Big\{\mathbb{E}_{t_1}[\Lambda^\varepsilon(t_1,t_2,Y^{1,Z},Z)V^\varepsilon(t_2,X^1(t_2),Y^{1,Z}(t_2),Z(t_2))]\\
&\qquad\qquad\qquad\qquad -\mathbb{E}_{t_1}[V^\varepsilon(t_2,x,y-c(Z(t_2)-z),Z(t_2))]\Big\}\\
&\le \sup_{Z\in\mathcal{Z}(t_1,z)} \mathbb{E}_{t_1}\Big[V^\varepsilon(t_2,X^1(t_2),Y^{1,Z}(t_2),Z(t_2)) - V^\varepsilon(t_2,x,y-c(Z(t_2)-z),Z(t_2))\Big].
\end{aligned}$$

Observe that by the sub-additivity of $c(\cdot)$,

$$\begin{aligned}
Y^{1,Z}(t_2) &= y + \int_{t_1}^{t_2} Z(s)dX(s) - \sum_{\tau_i\le t_2} c(\Delta Z(\tau_i))\\
&\le y + \int_{t_1}^{t_2} Z(s)dX(s) - c(Z(t_2)-z).
\end{aligned}$$

Together with monotonicity of V^ε in y, we obtain the desired estimate

$$\lim_{t_2\to t_1}(V^\varepsilon(t_1,x,y,z) - V^\varepsilon(t_2,x,y,z)) \le 0.$$

In other words, V^ε is left lower semicontinuous in t. Right continuity can be similarly shown along the above lines by forcing the limit $t_1 \to t_2$.

3. Note that by virtue of (3.3),

$$\begin{aligned}
V^\varepsilon(t,x,y,z) &\le \sup\nolimits_{Z\in\mathcal{Z}(t,z)} \mathbb{E}[U(Y^{t,x,y,z,Z}(T))]\\
&\le \sup\nolimits_{Z\in\mathcal{Z}(t,z)} U(\mathbb{E}[Y^{t,x,y,z,Z}(T)])\\
&\le \sup\nolimits_{Z\in\mathcal{Z}(t,z)} U(y+Cx).
\end{aligned}$$

This, together with (2.3), implies (3.5) and (3.6).

□

3.3 Continuity of V

Assumption 3.1. *For any $(t,x) \in (0,T) \times \mathbb{R}^+$ and $0 \neq z \in \mathcal{K}$, either $zb(t,x) < 0$ or $\sigma(t,x) \neq 0$.*

Remark 3.1. If $\mathcal{K}$ includes both negative and positive integers, then $zb(t,x) < 0$ is meaningless. But if $\mathcal{K}$ only contains nonnegative integers (that is, short position is prohibited), then $zb(t,x) < 0$ leads to $b(t,x) < 0$.

Define the effective boundary of the domain as follows:

$$\partial^1 O = \{(x,y,z) : x > 0, y = c(-z), z \in \mathcal{K}\}. \tag{3.17}$$

Lemma 3.3. *For arbitrarily given initial data $(t,x,y,z) \in [0,T) \times \partial^1 O \cap \{z \neq 0\}$ and $Z \in \mathcal{Z}(t,z)$, let $Y \triangleq Y^{t,x,y,z,Z}$ be a process of (2.6). Under Assumption 3.1, we have*

$$\inf\{s > t : Y(s) < C(-Z(s))\} = t \quad \mathbb{P} - a.s.$$

Proof. Given $Z \in \mathcal{Z}(t,z)$, we define $A = \{\omega : Z(t,\omega) = z\}$. For any $\omega \notin A$, one can see

$$Y(t,\omega) = c(-z) - c(Z(t,\omega) - z) < c(-z),$$

and thus,

$$\inf\{s > t : Y(s) < c(-Z(s))\} = t \quad \mathbb{P} - a.s. \text{ in } \Omega \setminus A.$$

Next, we want to show

$$\inf\{s > t : Y(s) < c(-Z(s))\} = t \quad \mathbb{P} - a.s. \text{ in } A.$$

Let $\rho(y,z) = c(-z) - y$. Consider $Z^1 \in \mathcal{Z}(t,z)$ given by

$$Z^1(s,\omega) = Z(s,\omega)\mathbb{1}_A(\omega) + z\mathbb{1}_{A^c}(\omega), \ \forall s \in [t,T).$$

In other words, Z^1 is constructed so that if there is a jump at t, then Z^1 follows exactly the sample path as Z, and if not Z^1 just takes constant z before clear all risky asset at time T.

We denote its associated state process with initial data (t,x,y,z) by $(X^1(s), Y^1(s), Z^1(s))$. Then, because of the existence and uniqueness of the strong solution of (2.1),

$$(X^1, Y^1, Z^1) \equiv (X, Y, Z), \ \mathbb{P} - a.s. \text{ in } A.$$

Therefore, it is enough to show that

$$\inf\{s > t : Y^1(s) < c(-Z^1(s))\} = t, \ \mathbb{P} - a.s.$$

By Itô's formula, for all $s < \tau_1$ of (2.4)

$$d\rho(Y^1(s), Z^1(s)) = d\rho(Y^1(s), z) = -zb(s, X^1(s))ds + z\sigma(s, X^1(s))dW(s).$$

By Proposition 6.2, $\inf\{s > t : \rho(Y^1(s), Z^1(s)) > 0\} = t$ under Assumption 3.1. ∎

Theorem 3.1 (Continuity of V)**.** *Assume Assumption* 3.1*. Then the value function* V *given in* (2.12) *is continuous in* (t,x,y).

Proof. Fix the initial data $(t,x,y,z)\in[0,T)\times\partial^1 O\cap\{z\neq 0\}$ and arbitrary $Z\in\mathcal{Z}(t,z)$. Let $Y\triangleq Y^{t,x,y,z,Z}$ be a process of (2.6). By Lemma 3.3, for any $s\in[t,T)$

$$\int_t^s \Big(c(-Z(r))-Y(r)\Big)^+ dr>0 \quad \mathbb{P}-a.s.$$

Hence, by definition (3.1),

$$\lim_{\varepsilon\to 0^+}\Lambda^\varepsilon(t,s,Y,Z)=0 \quad \mathbb{P}-a.s.$$

Fix a small $\delta>0$. Let $Z^\varepsilon\in\mathcal{Z}(t,z)$ be a δ-optimal control. That is,

$$\begin{aligned} V^\varepsilon(t,x,y,z) &\le \mathbb{E}_t[U(\Lambda^\varepsilon(t,T,Y^{t,x,y,z,Z^\varepsilon},Z^\varepsilon)Y^{t,x,y,z,Z^\varepsilon}(T))]+\delta\\ &\triangleq \mathbb{E}_t[U(\Lambda^\varepsilon(t,T,Y^\varepsilon,Z^\varepsilon)Y^\varepsilon(T))]+\delta, \end{aligned}$$

with the notation $Y^\varepsilon\triangleq Y^{t,x,y,z,Z^\varepsilon}$. Such a δ-optimal control Z^ε always exists for each ε. Since V^ε is monotone in ε and nonnegative, $\lim_{\varepsilon\to 0^+}V^\varepsilon(t,x,y,z)$ is well-defined. In addition, utilizing the fact $\lambda U(y)\le U(\lambda y)$ for any $\lambda\in(0,1)$

$$\begin{aligned} \lim_{\varepsilon\to 0^+}V^\varepsilon(t,x,y,z) &\le \lim_{\varepsilon\to 0^+}\mathbb{E}_t[\Lambda^\varepsilon(t,T,Y^\varepsilon,Z^\varepsilon)U(Y^\varepsilon(T))]+\delta\\ &\le \lim_{\varepsilon\to 0^+}\mathbb{E}_t[U(\Lambda^\varepsilon(t,T,Y^\varepsilon,Z^\varepsilon)Y^{\varepsilon,+}(T))]+\delta\\ &\le \lim_\varepsilon U(\mathbb{E}_t[\Lambda^\varepsilon(t,T,Y^\varepsilon,Z^\varepsilon)Y^{\varepsilon,+}(T)])+\delta\\ &= U(\lim_\varepsilon \mathbb{E}_t[\Lambda^\varepsilon(t,T,Y^\varepsilon,Z^\varepsilon)Y^{\varepsilon,+}(T)])+\delta\\ &= U(\mathbb{E}_t[\lim_\varepsilon \Lambda^\varepsilon(t,T,Y^\varepsilon,Z^\varepsilon)Y^{\varepsilon,+}(T)])+\delta\\ &= \delta. \end{aligned}$$

Note that $V^\varepsilon\ge 0$ and $\delta>0$ is arbitrary. These imply the pointwise convergence of

$$\lim_{\varepsilon\to 0}V^\varepsilon(t,x,y,z)=0=V(t,x,y,z),\quad \forall(t,x,y,z)\in[0,T)\times\partial^1 O\cap\{z\neq 0\}. \tag{3.18}$$

It is immediate to show by definition that

$$V^\varepsilon(t,0,c(-z),z)=0=V(t,0,c(-z),z),\ \text{and}\ V^\varepsilon(T,x,c(-z),z)=0=V(T,x,c(-z),z).$$

In addition, we can show $\lim_{\varepsilon\to 0}V^\varepsilon(t,x,0,0)=0$ since if τ_1 exists (otherwise trivial)

$$\begin{aligned} 0 &\le \lim_\varepsilon V^\varepsilon(t,x,0,0)\\ &\le \lim_\varepsilon \mathbb{E}[V^\varepsilon(\tau_1,X^{t,x}(\tau_1),-c(Z(\tau_1)),Z(\tau_1))]\\ &\le \mathbb{E}[\lim_\varepsilon V^\varepsilon(\tau_1,X^{t,x}(\tau_1),-c(Z(\tau_1)),Z(\tau_1))]\\ &\le \mathbb{E}[\lim_\varepsilon V^\varepsilon(\tau_1,X^{t,x}(\tau_1),c(-Z(\tau_1)),Z(\tau_1))]\\ &= 0. \end{aligned}$$

In the above, we used the dominated convergence theorem, and applied (3.18) together with the fact $Z(\tau_1) \neq 0$. Now, we can rewrite (3.18) as

$$\lim_{\varepsilon \to 0} V^\varepsilon(t,x,y,z) = 0 = V(t,x,y,z), \quad \forall (t,x,y,z) \in [0,T] \times \partial^2 O, \tag{3.19}$$

where $\partial^2 O$ is the closure of $\partial^1 O$, i.e.,

$$\partial^2 O = \{(x,y,z) : x \geq 0, y = c(-z), z \in \mathcal{K}\}.$$

Since $V^\varepsilon(t,x,y,z)$ is continuous on the compact set $([0,T] \times \partial^2 O) \cap \{x \leq \bar{x}\}$ for arbitrary given positive $\bar{x}$ and converges monotonically to the zero function by (3.19), Dini's theorem implies that

$$\lim_{\varepsilon \to 0^+} V^\varepsilon(t,x,y,z) = 0 \text{ uniformly on } ([0,T] \times \partial^2 O) \cap \{x \leq \bar{x}\}.$$

Due to the uniform convergence, we can set a real function $h^\varepsilon(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$h^\varepsilon(\bar{x}) \triangleq \sup\{V^\varepsilon(t,x,y,z) : (t,x,y,z) \in [0,T] \times \partial^2 O \cap \{x \leq \bar{x}\}\}.$$

Then

$$\lim_{\varepsilon \to 0} h^\varepsilon(\bar{x}) = 0 \text{ for any given } \bar{x} > 0. \tag{3.20}$$

From (3.5) of Lemma 3.2 and Lemma 6.1, we have

$$\lim_{x \to \infty} \frac{h^\varepsilon(x)}{x} = 0 \ \text{ uniformly in } \varepsilon,$$

and therefore there exists a large $x_0 > 0$ such that

$$\frac{h^\varepsilon(x)}{x} \leq \frac{h^\varepsilon(x_0)}{x_0} \ \text{ for all } x \geq x_0 \ \text{ and } \ \varepsilon > 0.$$

Therefore, we have for all $(t,x,y,z) \in [0,T] \times \partial^2 O$

$$V^\varepsilon(t,x,y,z) \leq x \frac{h^\varepsilon(x)}{x} \leq x \frac{h^\varepsilon(x_0)}{x_0} = C x h^\varepsilon(x_0).$$

Now we are ready to derive a bound of V in terms of V^ε in the domain $(t,x,y,z) \in [0,T] \times O$. First, we observe that, since $\Lambda^\varepsilon(t,s,Y^{t,x,y,z,Z},Z) \equiv 1$ for any stopping time $s \leq \tau$ of (2.11), we can write

$$V^\varepsilon(t,x,y,z) = \sup_{Z \in \mathcal{Z}(t,z)} \mathbb{E}[V^\varepsilon(\tau, X^{t,x}(\tau), Y^{t,x,y,z,Z}(\tau), Z(\tau))].$$

Also, the state $(X^{t,x}(\tau), Y^{t,x,y,z,Z}(\tau), Z(\tau))$ must fall in $\partial^2 O$, since $X^{t,x}(\tau) \neq 0$ almost surely. Therefore, for any $(t,x,y,z) \in [0,T] \times O$,

$$\begin{aligned}
V(t,x,y,z) &\leq V^\varepsilon(t,x,y,z) \\
&= \sup_{Z\in\mathcal{Z}(t,z)} \mathbb{E}[V^\varepsilon(\tau, X^{t,x}(\tau), Y^{t,x,y,z,Z}(\tau), Z(\tau))] \\
&\leq \sup_{Z\in\mathcal{Z}(t,z)} \mathbb{E}[h^\varepsilon(X^{t,x}(\tau))\mathbb{1}_{\{\tau<T\}} + V^\varepsilon(T, X^{t,x}(T), Y^{t,x,y,z,Z}(T), Z(T))\mathbb{1}_{\{\tau=T\}}] \\
&\leq Ch^\varepsilon(x_0)\mathbb{E}[X^{t,x}(\tau)] + \sup_{Z\in\mathcal{Z}(t,z)} \mathbb{E}[V^\varepsilon(T, X^{t,x}(T), Y^{t,x,y,z,Z}(T), Z(T))\mathbb{1}_{\{\tau=T\}}] \\
&\leq Cxh^\varepsilon(x_0) + V(t,x,y,z),
\end{aligned}$$

where τ is as in (2.11). The above inequalities imply that V^ε is a locally uniform estimate of V on the $[0,T] \times O$ in the sense of

$$|V^\varepsilon(t,x,y,z) - V(t,x,y,z)| \leq Cxh^\varepsilon(x_0), \ \forall (t,x,y,z) \in [0,T] \times O.$$

Finally, we can show continuity of V in (t,x,y). For any $(t_i, x_i, y_i, z) \in (0,T) \times O$ with $i = 1, 2$,

$$\begin{aligned}
&|V(t_1,x_1,y_1,z) - V(t_2,x_2,y_2,z)| \\
&\leq |(V - V^\varepsilon)(t_1,x_1,y_1,z)| + |(V - V^\varepsilon)(t_2,x_2,y_2,z)| + |V^\varepsilon(t_1,x_1,y_1,z) - V^\varepsilon(t_2,x_2,y_2,z)| \\
&\leq Ch^\varepsilon(x_0)(x_1 + x_2) + |V^\varepsilon(t_1,x_1,y_1,z) - V^\varepsilon(t_2,x_2,y_2,z)|.
\end{aligned}$$

Letting $(t_1,x_1,y_1) \to (t_2,x_2,y_2)$, the last term disappears by Lemma 3.2, and

$$\lim_{(t_1,x_1,y_1)\to(t_2,x_2,y_2)} |V(t_1,x_1,y_1,z) - V(t_2,x_2,y_2,z)| \leq Cx_2h^\varepsilon(x_0).$$

Thanks to (3.20), $\lim_{\varepsilon\to 0} h^\varepsilon(x_0) = 0$, and hence

$$\lim_{(t_1,x_1,y_1)\to(t_2,x_2,y_2)} |V(t_1,x_1,y_1,z) - V(t_2,x_2,y_2,z)| = 0.$$

□

3.4 Auxiliary Results Derived from Continuity

Thanks to the continuity of V, now we can show that the no-action region is an open set, which is crucial for the uniqueness. (see inequalities (4.12) and (4.13) with application of Ishii's lemma)

Proposition 3.2. *$\mathcal{A}[V](z)$ is open in $\mathbb{R}^3$ for any $z \in \mathcal{K}$.*

Proof. By the definition of $\mathcal{A}$ of (2.16), we write

$$\mathcal{A}[V](z) = \{(t,x,y) : V(t,x,y,z) > \mathcal{S}V(t,x,y,z)\} = \{(t,x,y) : (V - \mathcal{S}V)(t,x,y,z) > 0\}.$$

Note that $V(\cdot,\cdot,\cdot,z)$ is continuous by Theorem 3.1, so is $(V - \mathcal{S}V)(\cdot,\cdot,\cdot,z)$ by Proposition 3.1. This implies $\mathcal{A}[V](z)$ is an open set. □

Proposition 3.2 also enables us to characterize the optimal strategy by a $\mathbb{F}^X$-predictable process, where $\mathbb{F}^X$ is filtration generated by price process X. Practically, a trader can observe only the price process X (not the Brownian motion W), and $\mathbb{F}^X$-predictable strategy is more desirable. We briefly discuss the construction of the optimal strategy below.

By standard argument, the optimal strategy is essentially constructed by a series of optimal stopping time problem. Indeed, given initial state (t, x, y, z), using Y_1 to denote the process $Y^{t,x,y,Z}$ with constant control $Z \equiv z$, the first transaction is occurred at

$$\tau_1 = \inf\{s \geq t : Y_1 \notin \mathcal{A}[V](z)\} \tag{3.21}$$

and the size of transaction at τ_1 is

$$Z(\tau_1) - Z(\tau_1^-) = \arg\max_{\Delta z} V(\tau_1^-, X(\tau_1^-), Y_1(\tau_1^-) - c(\Delta z), z + \Delta z). \tag{3.22}$$

The subsequent transaction times and sizes are determined repeatedly by using the same procedure.

Note that, since $\mathcal{A}[V](z)$ is an open set by Proposition 3.2, τ_1 of (3.21) is an $\mathbb{F}^{Y_1}$-stopping time, where $\mathbb{F}^{Y_1}$ is the filtration generated by Y_1. Furthermore, together with the fact $\mathbb{F}^{Y_1} \subset \mathbb{F}^X$, this implies that τ_1 is an $\mathbb{F}^X$-stopping time. Also note that in (3.22), the jump size of $Z(\tau_1) - Z(\tau_1^-)$ is measurable with respect to $\mathbb{F}^X(\tau_1^-)$. Repeating above argument to the subsequent jump times, one can show that the above constructed process is $\mathbb{F}^X$-predictable.

4 Characterization of Value Function

In this section, we will show the value function is the unique viscosity solution of (2.17) with condition (2.18). First, we give definition of viscosity solution:

Definition 4.1. A function u is said to be a viscosity subsolution (resp. supersolution) of (2.17)-(2.18), if

1. for any $(t_0, x_0, y_0, z_0) \in (0, T) \times O$ and function $\varphi \in C^{1,2,2}((0, T) \times O, \mathbb{R})$ satisfying

   $$\varphi \geq \text{(resp. } \leq) \ u \text{ on } (0, T) \times O \ \text{ and } \varphi = u \text{ at } (t_0, x_0, y_0, z_0),$$

   the following inequality holds:

   $$\min\{(-\varphi_t - \mathcal{L}\varphi)(t_0, x_0, y_0, z_0), (\varphi - \mathcal{S}\varphi)(t_0, x_0, y_0, z_0)\} \leq \text{(resp. } \geq) 0,$$

   and

2. $u(t, x, y, z) \leq$ (resp. $\geq$) $U(y - c(-z))$, on $\partial^*([0, T) \times O)$.

The u is said to be a viscosity solution, if it is both a viscosity subsolution and a viscosity supersolution.

4.1 Viscosity Solution Properties

Next, we show the objective function V of (2.12) is a viscosity solution of quasi-variational inequality (2.17)-(2.18).

Theorem 4.1 (Viscosity properties)**.** *The objective function $V(t,x,y,z)$ of* (2.12) *is a viscosity solution of the quasi-variational inequality* (2.17) *with boundary-terminal condition* (2.18)*.*

Proof. The proof is divided into two steps.

1. First, we prove that V is a supersolution of (2.17). If not, there would exist $(t_0,\eta_0) \triangleq (t_0,x_0,y_0,z_0)$ and a function $\varphi \in C^{1,2,2}((0,T)\times O,\mathbb{R})$ with

$$\varphi \le V, \text{ and } \varphi(t_0,\eta_0) = V(t_0,\eta_0)$$

satisfying

$$\min\{(-\varphi_t - \mathcal{L}\varphi)(t_0,\eta_0), (\varphi - \mathcal{S}\varphi)(t_0,\eta_0)\} < 0. \tag{4.1}$$

Since by (2.15) and monotonicity of Proposition 3.1,

$$\varphi(t_0,\eta_0) = V(t_0,\eta_0) \ge \mathcal{S}V(t_0,\eta_0) \ge \mathcal{S}\varphi(t_0,\eta_0),$$

and (4.1) is equivalent to

$$(-\varphi_t - \mathcal{L}\varphi)(t_0,\eta_0) < 0. \tag{4.2}$$

We introduce a strict subtest function $\phi(\cdot)$ given by

$$\phi(t,x,y,z) = \varphi(t,x,y,z) - |t-t_0|^2 - |x-x_0|^4 - |y-y_0|^4.$$

One can check ϕ also satisfies inequality (4.2), i.e.,

$$(-\phi_t - \mathcal{L}\phi)(t_0,\eta_0) < 0.$$

Since $-\phi_t - \mathcal{L}\phi$ is continuous in (t,x,y),

$$\{(t,x,y) : (-\phi_t - \mathcal{L}\phi)(t,x,y,z_0) < 0\}$$

is an open set. Now, we can take a small open ball $B_r(t_0,x_0,y_0)\times\{z_0\} \subset (0,T)\times O$ such that

$$(-\phi_t - \mathcal{L}\phi)(t,\eta) < 0, \text{ in } B_r(t_0,x_0,y_0)\times\{z_0\}.$$

Observe that $\forall (t,\eta) \in \partial B_r(t_0,x_0,y_0)\times\{z_0\}$

$$\varphi(t,\eta) - \phi(t,\eta) = |t-t_0|^2 + |x-x_0|^4 + |y-y_0|^4 \ge 1 \wedge \frac{r^4}{3} \triangleq \varepsilon. \tag{4.3}$$

Consider the stopping time θ defined by

$$\theta = \{s \geq t_0 : (s, X^{t_0,x_0}(s), Y^{t_0,x_0,y_0,z_0}(s) \notin B_r(t_0, x_0, y_0)\}.$$

Applying Itô's formula on ϕ, with notations $X^{t_0,x_0} \triangleq X$, $Y^{t_0,x_0,y_0,Z_0} = Y$, and $Z_0(\cdot) \equiv z_0$, we have

$$\begin{aligned} V(t_0, \eta_0) &= \phi(t_0, \eta_0) \\ &= \mathbb{E}[\phi(\theta, X(\theta), Y(\theta), z_0) - \int_{t_0}^{\theta} (\phi_t + \mathcal{L}\phi)(s, X(s), Y(s), z_0) ds] \\ &\leq \mathbb{E}[\phi(\theta, X(\theta), Y(\theta), z_0)] \\ &\leq \mathbb{E}[\varphi(\theta, X(\theta), Y(\theta), z_0)] - \varepsilon \\ &\leq \mathbb{E}[V(\theta, X(\theta), Y(\theta), z_0)] - \varepsilon \\ &\leq V(t_0, \eta_0) - \varepsilon. \end{aligned}$$

This leads to a contradiction and completes the proof of viscosity supersolution property.

2. Next, we show the viscosity subsolution property. To the contrary, if there exists $(t_0, \eta_0) \triangleq (t_0, x_0, y_0, z_0)$ and a function $\varphi \in C^{1,2,2}((0, T) \times O, \mathbb{R})$ with

$$\varphi \geq V, \text{ and } \varphi(t_0, \eta_0) = V(t_0, \eta_0)$$

satisfying

$$\min\{(-\varphi_t - \mathcal{L}\varphi)(t_0, \eta_0), (\varphi - \mathcal{S}\varphi)(t_0, \eta_0)\} > 0.$$

One can rewrite the above inequality as

$$(-\varphi_t - \mathcal{L}\varphi)(t_0, \eta_0) > 0, \quad (\varphi - \mathcal{S}\varphi)(t_0, \eta_0) > 0. \tag{4.4}$$

The second inequality of (4.4), together with the monotonicity of $\mathcal{S}$ of Proposition 3.1, leads to

$$V(t_0, \eta_0) = \varphi(t_0, \eta_0) > \mathcal{S}\varphi(t_0, \eta_0) \geq \mathcal{S}V(t_0, \eta_0),$$

that is equivalent to

$$(t_0, x_0, y_0) \in \mathcal{A}[V](z_0), \tag{4.5}$$

Now, we consider a test function ϕ given by

$$\phi(t, x, y, z) = \varphi(t, x, y, z) + |t - t_0|^2 + |x - x_0|^4 + |y - y_0|^4.$$

One can check that, by (4.4)

$$(-\phi_t - \mathcal{L}\phi)(t_0, \eta_0) > 0.$$

Since $(-\phi_t - \mathcal{L}\phi)$ is continuous in (t, x, y),

$$\{(t, x, y) : (-\phi_t - \mathcal{L}\phi)(t, x, y, z_0) > 0\}$$

is an open set. Note also that (4.5) together with Proposition 3.2 implies $\mathcal{A}[V](z_0)$ is a non-empty open set. Thus,

$$\{(t, x, y) : (-\phi_t - \mathcal{L}\phi)(t, x, y, z_0) > 0\} \cap \mathcal{A}[V](z_0) \tag{4.6}$$

is also a non-empty set. We can take a small open ball $B_r(t_0, x_0, y_0) \times \{z_0\}$ contained in the open set of (4.6), i.e.,

$$(-\phi_t - \mathcal{L}\phi)(t, \eta) > 0, \quad V(t, \eta) > \mathcal{S}V(t, \eta), \quad \forall (t, \eta) \in B_r(t_0, x_0, y_0) \times \{z_0\}.$$

Similar to (4.3), we also have

$$\phi(t, \eta) - \varphi(t, \eta) = |t - t_0|^2 + |x - x_0|^4 + |y - y_0|^4 \geq 1 \wedge \frac{r^4}{3} \triangleq \varepsilon.$$

Define

$$\theta = \{s \geq t_0 : (s, X^{t_0,x_0}(s), Y^{t_0,x_0,y_0,z_0}(s) \notin B_r(t_0, x_0, y_0)\}.$$

Applying Itô's formula to ϕ, with notations $X^{t_0,x_0} \triangleq X$, $Y^{t_0,x_0,y_0,Z_0} = Y$, and $Z_0(\cdot) \equiv z_0$, we obtain

$$\begin{aligned} V(t_0, \eta_0) &= \phi(t_0, \eta_0) \\ &= \mathbb{E}[\phi(\theta, X(\theta), Y(\theta), z_0) - \int_{t_0}^{\theta} (\phi_t + \mathcal{L}\phi)(s, X(s), Y(s), z_0) ds] \\ &\geq \mathbb{E}[\phi(\theta, X(\theta), Y(\theta), z_0)] \\ &\geq \mathbb{E}[\varphi(\theta, X(\theta), Y(\theta), z_0)] + \varepsilon \\ &\geq \mathbb{E}[V(\theta, X(\theta), Y(\theta), z_0)] + \varepsilon. \end{aligned}$$

Since $V(t_0, \eta_0) = \mathbb{E}[V(\theta, X(\theta), Y(\theta), z_0)]$ in the no-action region $\mathcal{A}[V](z_0)$, this leads to a contradiction.

□

4.2 Uniqueness

In this part, we establish the uniqueness in the sense of viscosity solution for the quasi-variational inequality (2.17) with boundary-terminal condition (2.18).

Throughout this section, we assume that u and v are continuous sub- and supersolution of (2.17) and (2.18), respectively, satisfying sublinear growth of the form, for $\varphi = u, v$

$$\lim_{x \to \infty} \sup_t \frac{\varphi(t, x, y, z)}{x} = 0, \forall (y, z), \text{ and } \lim_{y \to \infty} \sup_t \frac{\varphi(t, x, y, z)}{y} = 0, \forall (x, z). \tag{4.7}$$

We are to show a comparison result

$$u \geq v$$

on the entire domain, which implies uniqueness.

Assumption 4.1. The b and σ are uniformly bounded, i.e., there exists a positive constant C_4 such that $\sup_{[0,T]\times[0,\infty)} |b(t,x)| + |\sigma(t,x)| < C_4$.

Define constants

$$\rho = \frac{1}{2}\min_{z\neq 0} c(z) > 0, \quad \text{and} \quad C_5 = \|b\|_\infty (C_2 \vee C_3 + 1) + 2\rho$$

and

$$v^\varepsilon(t,x,y,z) = v(t,x,y,z) + \varepsilon g(t,x,y,z)$$

where $g(t,x,y,z) = x + y + C_5(T-t)$.

Lemma 4.1. *v^ε is a strict supersolution, i.e., any smooth test function φ^ε with $\varphi^\varepsilon = v^\varepsilon$ at $(\bar{t},\bar{x},\bar{y},\bar{z}) \in (0,T)\times O$ satisfies*

$$(\varphi^\varepsilon - \mathcal{S}\varphi^\varepsilon)(\bar{t},\bar{x},\bar{y},\bar{z}) > \varepsilon\rho > 0, \tag{4.8}$$

and

$$(-\varphi^\varepsilon_t - \mathcal{L}\varphi^\varepsilon)(\bar{t},\bar{x},\bar{y},\bar{z}) > \varepsilon\rho > 0. \tag{4.9}$$

Proof. Note that $\varphi \triangleq \varphi^\varepsilon - \varepsilon g$ is a test function of v at $(\bar{t},\bar{x},\bar{y},\bar{z})$, and by viscosity supersolution property

$$\min\{(-\varphi_t - \mathcal{L}\varphi)(\bar{t},\bar{x},\bar{y},\bar{z}), (\varphi - \mathcal{S}\varphi)(\bar{t},\bar{x},\bar{y},\bar{z})\} \geq 0,$$

Using Proposition 3.1, (4.8) is obtained from

$$(\varphi^\varepsilon - \mathcal{S}\varphi^\varepsilon)(\bar{t},\bar{x},\bar{y},\bar{z}) \geq (\varphi - \mathcal{S}\varphi)(\bar{t},\bar{x},\bar{y},\bar{z}) + \varepsilon(g - \mathcal{S}g)(\bar{t},\bar{x},\bar{y},\bar{z}) \geq \varepsilon\rho.$$

Equation (4.9) is the result of viscosity supersolution property of v and

$$g_t + \mathcal{L}g < -\rho.$$

□

Lemma 4.2. *Let*

$$H(t,x,y,z) = u(t,x,y,z) - v^\varepsilon(t,x,y,z). \tag{4.10}$$

Then $H(t,x,y,z)$ attains its maximum in $[0,T]\times\bar{O}$, i.e., $\exists(\hat{t},\hat{x},\hat{y},\hat{z}) \in [0,T]\times\bar{O}$ such that

$$H(\hat{t},\hat{x},\hat{y},\hat{z}) = \max_{[0,T]\times\bar{O}} H(t,x,y,z).$$

Moreover,

$$(u - \mathcal{S}u)(\hat{t},\hat{x},\hat{y},\hat{z}) > 0.$$

Proof. Since εg grows at most linearly and $u - v$ has sublinear growth of the form (4.7) in (x, y), H satisfies $H(t, x, y, z) \to -\infty$ as $|x| + |y| \to \infty$. Thus, $H(\cdot)$ attains its maximum at some point in its domain, say $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$. To the contrary, if

$$(u - v)(\hat{t}, \hat{x}, \hat{y}, \hat{z}) \leq 0.$$

Then, $\exists z^* \neq \hat{z}$ such that

$$u(\hat{t}, \hat{x}, \hat{y}, \hat{z}) \leq u(\hat{t}, \hat{x}, \hat{y} - c(z^* - \hat{z}), z^*).$$

On the other hand, by Proposition 3.1- monotonicity and Lemma 4.1,

$$v^\varepsilon(\hat{t}, \hat{x}, \hat{y}, \hat{z}) > \mathcal{S}v^\varepsilon(\hat{t}, \hat{x}, \hat{y}, \hat{z}) \geq v^\varepsilon(\hat{t}, \hat{x}, \hat{y} - c(z^* - \hat{z}), z^*).$$

Combining the above two inequalities,

$$(u - v^\varepsilon)(\hat{t}, \hat{x}, \hat{y}, \hat{z}) < (u - v^\varepsilon)(\hat{t}, \hat{x}, \hat{y} - c(z^* - \hat{z}), z^*),$$

which is a contradiction to the definition of $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$ as a maximizer. □

For any $z \in \mathcal{K}$, define

$$O_z = \{(x, y) : (x, y, z) \in O\}. \tag{4.11}$$

Lemma 4.3. *Define* $\Phi_\alpha : [0, T] \times O_z^2 \times \mathcal{K} \to \mathbb{R}$ *as*

$$\Phi_\alpha(t, \zeta, \eta, z) = u(t, \zeta, z) - v^\varepsilon(t, \eta, z) - \frac{\alpha}{2}|\zeta - \eta|^2.$$

Then, the following assertions are true:

1. *For each* $z \in \mathcal{K}$, $\Phi_\alpha(\cdot, z)$ *achieves its maximum at a point in* $[0, T] \times \bar{O}_z^2$, *denoted by* $(t_\alpha^z, \zeta_\alpha^z, \eta_\alpha^z)$.

2. *There exists a convergent subsequence* $(t_\alpha^z, \zeta_\alpha^z) \to (t^z, \zeta^z) \in [0, T] \times O_z$ *such that*

$$H(t^z, \zeta^z, z) = \sup_{[0,T]\times\bar{O}_z} H(t, \zeta, z),$$

and

$$\lim_{\alpha\to\infty} \alpha|\zeta_\alpha^z - \eta_\alpha^z|^2 \to 0.$$

Proof. Note that $\Phi_\alpha(\cdot, z)$ achieves maximum, since $\Phi_\alpha(t, \zeta, \eta, z) \to -\infty$ as $|\zeta| + |\eta| \to \infty$. The rest of proof is an application of [8, Lemma 3.1] on function $\Phi_\alpha(\cdot, \cdot, \cdot, z)$. □

Theorem 4.2 (Comparison result)**.**

$$\sup_{[0,T]\times\bar{O}}(u-v)(t,x,y,z) = \sup_{\partial^*([0,T)\times O)}(u-v)(t,x,y,z).$$

Proof. It suffices to show that for an arbitrary given ε,

$$\sup_{[0,T]\times\bar{O}} H(t,x,y,z) = \sup_{\partial^*([0,T)\times O)} H(t,x,y,z).$$

To the contrary, we assume

$$H(\hat{t},\hat{x},\hat{y},\hat{z}) = \sup_{[0,T]\times\bar{O}} H(t,x,y,z) > \sup_{\partial^*([0,T)\times O)} H(t,x,y,z)$$

for some $\varepsilon > 0$. Then, we have $(\hat{t},\hat{x},\hat{y},\hat{z}) \in [0,T)\times O$ at the interior of the domain.

For notational convenience, we denote $(\hat{t}_\alpha,\hat{\zeta}_\alpha,\hat{\eta}_\alpha) = (t^{\hat{z}}_\alpha,\zeta^{\hat{z}}_\alpha,\eta^{\hat{z}}_\alpha)$. Also, we note that $(\hat{t},\hat{\zeta}) = (t^{\hat{z}},\zeta^{\hat{z}}) = \lim_{\alpha\to\infty}(\hat{t}_\alpha,\hat{\zeta}_\alpha)$ in Lemma 4.3. By Lemma 4.3, we have $\Phi_\alpha(\hat{t}_\alpha,\hat{\zeta}_\alpha,\hat{\eta}_\alpha,\hat{z}) \to H(\hat{t},\hat{\zeta},\hat{z})$ as $\alpha\to\infty$.

By Lemma 4.2, $(\hat{t},\hat{\zeta}) \in \mathcal{A}[u](\hat{z})$. Since $\mathcal{A}[u](\hat{z})$ is open by Proposition 3.2, there exists some $\alpha_1 > 0$ such that $(\hat{t}_\alpha,\hat{\zeta}_\alpha),(\hat{t}_\alpha,\hat{\eta}_\alpha) \in \mathcal{A}[u](\hat{z}) \subset [0,T)\times O_{\hat{z}}$ for all $\alpha > \alpha_1$. To proceed, we denote parabolic superjet (resp. subjet) by $D^{+(1,2)}$ (resp. $D^{-(1,2)}$), and its closure by $\bar{D}^{+(1,2)}$ (resp. $\bar{D}^{-(1,2)}$); see its definition and properties in [8]. Applying Ishii's lemma (also in [8]) on $u(\cdot,\hat{z}), v^\varepsilon(\cdot,\hat{z})$, and $w^\alpha(t,\zeta,\eta) = \frac{\alpha}{2}|\zeta-\eta|^2$, there exists $q,\tilde{q}\in\mathbb{R}$, $p,\tilde{p}\in\mathbb{R}^2$ and symmetric matrices A,B depending on α, such that

1. $(q,p,A) \in \bar{D}^{+(1,2)}u(\hat{t}_\alpha,\hat{\zeta}_\alpha,\hat{z})$, $p = D_\zeta w^\alpha(\hat{t}_\alpha,\hat{\zeta}_\alpha,\hat{\eta}_\alpha) = \alpha(\hat{\zeta}_\alpha-\hat{\eta}_\alpha)$;
2. $(\tilde{q},\tilde{p},\tilde{A}) \in \bar{D}^{-(1,2)}v^\varepsilon(\hat{t}_\alpha,\hat{\eta}_\alpha,\hat{z})$, $\tilde{p} = -D_\eta w^\alpha(\hat{t}_\alpha,\hat{\zeta}_\alpha,\hat{\eta}_\alpha) = \alpha(\hat{\zeta}_\alpha-\hat{\eta}_\alpha)$;
3. $q-\tilde{q} = 0$;
4. $-3\alpha I_4 \le \begin{bmatrix} A & 0 \\ 0 & -\tilde{A} \end{bmatrix} \le 3\alpha \begin{bmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{bmatrix}$.

By viscosity subsolution property of u, it yields

$$\min\{F(\hat{t}_\alpha,\hat{\zeta}_\alpha,\hat{z},q,p,A),(u-\mathcal{S}u)(t_\alpha,\hat{\zeta}_\alpha,\hat{z})\} \le 0.$$

Since $(\hat{t}_\alpha,\hat{\zeta}_\alpha) \in \mathcal{A}[u](\hat{z})$ and $\mathcal{A}[u](\hat{z})$ is open

$$F(\hat{t}_\alpha,\hat{\zeta}_\alpha,\hat{z},q,p,A) \le 0. \tag{4.12}$$

Also by (4.9),

$$F(\hat{t}_\alpha,\hat{\eta}_\alpha,\hat{z},\tilde{q},\tilde{p},\tilde{A}) > \varepsilon\rho > 0. \tag{4.13}$$

Using the result of Lemma 4.3, Lipschitz condition on b and σ, and Ishii's lemma, subtracting (4.13) from (4.12)

$$\varepsilon\rho < F(\hat{t}_\alpha, \hat{\eta}_\alpha, \hat{z}, \tilde{q}, \tilde{p}, \tilde{A}) - F(\hat{t}_\alpha, \hat{\zeta}_\alpha, \hat{z}, q, p, A) \to 0 \text{ as } \alpha \to \infty,$$

which leads to a contradiction. □

4.3 Summary of Results

Finally, we summarize what have been obtained so far. It is presented in the following characterization of the value function.

Theorem 4.3. *Given Assumption* 2.1*, Assumption* 3.1 *and Assumption* 4.1*, the value function* $V(\cdot,\cdot,\cdot,\cdot)$ *of* (2.12) *is the unique viscosity solution of the quasi-variational inequality* (2.17)-(2.18) *in the space of continuous functions with sublinear growth in* (x, y) *of the form* (4.7)*.*

5 Further Remark

In this work, we obtained the continuity of the value function, and further characterized the value function as the unique viscosity solution of a quasi-variational inequality with Cauchy-Dirichlet condition on $\partial^*([0,T)\times O)$ under some appropriate assumptions.

We have emphasized the continuity result in the current work. As a future study, we will consider viable uniqueness proofs with boundary conditions only on the effective boundary $\partial^*([0,T)\times O)\cap\{x>0\}$. The other consideration is to show the uniqueness without Assumption 4.1. One possible approach is to use domain transformation defined by $\bar{x} = \ln x$, and adjust the operators $\mathcal{L}$ and $\mathcal{S}$ appropriately, which is not included in the current paper due to notational complexity.

Another possible extension of the current work is to consider transaction cost of the form $c(x, z)$, with subadditive condition in z. More discussions are referred to [17]. It might also be interesting to study regime-switching models under optimal switching framework.

It is straightforward to generalize all the results to nonzero fixed risk-free rate $r > 0$ by usual normalization. However, it is nontrivial to consider similar utility maximization problems under various stochastic interest rate models.

6 Appendix

Next, for the sake of completeness, we show the sample path results on 1-D Itô's process. Proposition 6.1 is a generalized version of [3], and Proposition 6.2 is a special case of Proposition 6.1, which is needed in the proof of Lemma 3.3.

Consider the oem-dimensional Itô process

$$X(t,\omega) = \int_0^t b(s,\omega)ds + \sigma(s,\omega)dW(s), \tag{6.1}$$

where we assume

$$b(\cdot,\omega) \in L^1([0,T]), \sigma(\cdot,\omega) \in L^2([0,T]),\ \mathbb{P} - a.s. \tag{6.2}$$

The Itô process (6.1) is well defined under assumption (6.2); see Definition 4.1.1 of [19] and Problem 4.11 of [12].

Define the stopping times

$$\eta(\omega) \triangleq \inf\{t > 0 : \int_0^t \sigma^2(s,\omega)ds > 0\}, \tag{6.3}$$

and

$$\tau(\omega) \triangleq \inf\{t > 0 : X(t) > 0\}.$$

Proposition 6.1. *Assume (6.2) holds. For the Itô process given by (6.1), $\tau = 0$ $\mathbb{P}$-a.s. if one of the following two conditions is satisfied:*

1. *There exists stopping time $\theta > 0$ such that*

$$\int_0^t b(s,\omega)ds > 0,\ \forall t < \theta(\omega); \tag{6.4}$$

2. *$\eta = 0$, and there exists measurable function ψ satisfying*

$$b = \sigma\psi, \quad \text{and } \mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T \|\psi\|^2 ds\Big\}\Big] < \infty, \quad \text{for some } T > 0. \tag{6.5}$$

Proof. We divide the proof into three steps:

1. Suppose $b \equiv 0$ and $\eta = 0$ almost surely. Define

$$\eta_\varepsilon \triangleq \inf\{t > 0 : \int_0^t \sigma^2(s,\omega)ds > \varepsilon\}.$$

   Then, $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$ $\mathbb{P}$-a.s. Set $X_\varepsilon(t) = \int_0^t \sigma_\varepsilon(s,\omega)dW(s)$, where

$$\sigma_\varepsilon(s,\omega) = \begin{cases} \sigma(s,\omega), & s \le \eta_\varepsilon(\omega) \\ 1, & s > \eta_\varepsilon(\omega). \end{cases}$$

   Then, the quadratic variation $\langle X_\varepsilon\rangle(t) \to \infty$ as $t \to \infty$, and $X_\varepsilon(s,\omega) = X(s,\omega)$ for all $s < \eta_\varepsilon(\omega)$.

Let $T_\varepsilon(s) = \inf\{t > 0 : \langle X_\varepsilon\rangle(t) > s\}$. By Theorem 4.6 of [12], $B(s) \triangleq X_\varepsilon(T_\varepsilon(s))$ is a Brownian motion under $\mathbb{P}$ with time-changed filtration.

Note that $T_\varepsilon(\varepsilon) = \eta_\varepsilon$ by definition, and hence $B(\varepsilon) = X_\varepsilon(T_\varepsilon(\varepsilon)) = X_\varepsilon(\eta_\varepsilon)$ and

$$\sup_{0\le t\le \eta_\varepsilon} X(t,\omega) \equiv \sup_{0\le t\le \eta_\varepsilon} X_\varepsilon(t,\omega) = \sup_{0\le t\le \varepsilon} B(t,\omega) > 0,\ \mathbb{P}-a.s.\ \forall\varepsilon > 0.$$

Therefore, we obtain

$$\begin{aligned} 0 &\le \inf\{t > 0 : X(t,\omega) > 0\} \\ &\le \inf\{s > 0 : \sup_{0\le s\le t} X(t,\omega) > 0\} \le \eta_\varepsilon \to 0,\ \varepsilon \to 0,\ \mathbb{P}-a.s. \end{aligned}$$

and this implies $\tau = 0$ if $b \equiv 0$ and $\eta = 0$.

2. Now we assume existence of $\theta > 0$ satisfying (6.4). Set $A = \{\omega : \eta(\omega) > 0\}$, then

$$X(t,\omega) = \int_0^t b(s,\omega)ds,\ \forall s < \eta(\omega),\ \mathbb{P}-a.s.\ \omega\in A,$$

and thus $\tau = 0$ $\mathbb{P}$-a.s. $\omega \in A$. For $\omega \in A^c$, we have following inequality

$$X(t,\omega) \ge \int_0^t \sigma(s,\omega)dW(s),\ \forall t < \theta(\omega),$$

and by the proof of the first part, since $\eta = 0$ $\mathbb{P}$-a.s. in A^c,

$$\sup_{0\le s\le t} X(s,\omega) \ge \sup_{0\le s\le t}\int_0^s \sigma(r,\omega)dW(r) > 0,\ \forall t < \theta(\omega).$$

This implies $\tau = 0$, $\mathbb{P}$-a.s. in A^c. Thus, $\tau = 0$ with existence of $\theta > 0$ of (6.4).

3. With the assumption of (6.5), by Girsanov theorem, $X(t) = \int_0^t \sigma(s,\omega)d\widetilde{W}^Q$, where $Q \sim \mathbb{P}$. Then we can apply the result of first part to obtain $\tau = 0$ Q-a.s., and hence $\mathbb{P}$-a.s.

□

Example 6.1. Suppose $X(t) = -100t + W(t)$, then $\tau = 0$. This can be seen from (2) of Proposition 6.1.

Proposition 6.2. *Assuming that $\mathcal{F}_0$ is trivial σ-algebra. Consider Itô process* (6.1) *with continuous $b(\cdot,\omega)$ and $\sigma(\cdot,\omega)$. Then, $\tau = 0$, if either $b(0) > 0$ or $\sigma(0) > 0$.*

Proof. The proof is carried out in two steps.

1. If $b(0) > 0$, we can take $\theta = \inf\{t > 0 : b(t,\omega) \le \frac{1}{2}b(0).\}$. Because of the continuity of b, θ satisfies (6.4). Thus, $\tau = 0$.

2. By virtue of the continuity of σ, η in (6.3) is zero. Let $T_1 = \inf\{t > 0 : \sigma(t) \le \frac{1}{2}\sigma(0)\}$, and $T_2 = \inf\{t > 0 : |b(t)| \le |b(0)| + 1$. Set $T = T_1 \wedge T_2$, then $\psi = b/\sigma$ satisfies Nivikov condition (6.5). Thus, $\tau = 0$.

□

We need the following lemma to prove Theorem 3.1. The proof of Lemma 6.1 is elementary and is thus omitted.

Lemma 6.1. *Let* $f : [0, \infty) \mapsto \mathbb{R}^+$ *be a continuous function satisfying*

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0,$$

and denote $f^*(x) = \max\{f(y) : y \le x\}$. *Then*

$$\lim_{x\to\infty} \frac{f^*(x)}{x} = 0.$$

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