ASYMPTOTICS AND EXACT PRICING OF OPTIONS ON VARIANCE

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Abstract. We consider the pricing of derivatives written on the *discrete* realized variance of an underlying security. In the literature, the realized variance is usually approximated by its continuous-time limit, the quadratic variation of the underlying log-price. Here, we characterize the short-time limits of call options on both objects. We find that the difference strongly depends on whether or not the stock price process has jumps. To study the *exact* valuation of options on the discrete realized variance itself, we then propose a novel approach that allows to apply Fourier-Laplace techniques to price European-style options efficiently. To illustrate our results, we also present some numerical examples.

1. Introduction

Consider a discounted security $S = S_0 \exp(X)$ and let $d \in \mathbb{N}$ be the number of trading days per year, which is usually fixed to 252. For a time horizon of $T \in \frac{1}{d}\mathbb{N}$, the (annualized) *realized variance* on the period $[0,T]$ subdivided in dT business days is defined as

$$RV_T = \frac{1}{T}\sum_{k=1}^{dT} \log(S_{k/d}/S_{(k-1)/d})^2 = \frac{1}{T}\sum_{k=1}^{dT}\left(X_{k/d} - X_{(k-1)/d}\right)^2. \tag{1.1}$$

There exists a considerable number of financial instruments that are based on realized variance as an underlying (see e.g. [3, 6] or [10, Chapter 11] for an overview). Well-known examples are variance and volatility swaps, as well as *puts* and *calls on realized variance* with payoffs $(K - RV_T)^+$ resp. $(RV_T - K)^+$. Since the realized variance starts at zero today, the notion *At-The-Money* (henceforth *ATM*) is not defined by setting the strike K equal to the present value of the underlying. Instead it refers to choosing the strike K equal to the expectation $\mathbb{E}[RV_T]$ under the pricing measure, such that call and put prices coincide. In the following we focus on ATM call options for the sake of clarity, however most results can be generalized to other payoffs in a straightforward manner.

Given a stochastic model for S resp. X, the standard approach to pricing options on realized variance is to approximate realized variance by

$$RV_T \approx \frac{1}{T}[X,X]_T, \tag{1.2}$$

where $[X,X]$ is the quadratic variation of the log-price X. This approximation is motivated by the fact that realized variance (1.1) converges to $\frac{1}{T}[X,X]_T$ in probability as the mesh size $1/d$ tends to zero (cf. e.g. [11, I.4.47]). The advantage of this approach is that for many stochastic processes, the quadratic variation is a well-studied object. For example, as recently shown by [13], the characteristic

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function of the quadratic variation in any affine stochastic volatility model[1] can be computed as the solution of a generalized Riccati differential equation, such that in many cases methods based on Fourier-Laplace inversion (cf. [4, 15]) can be applied to compute option prices efficiently.

[3, 20] resp. [2] confirm via Monte-Carlo simulation resp. analytically that this approximation works very well for claims with linear payoffs, like variance swaps. On the other hand, it has been observed by Bühler [3] that "*while the approximation of realized variance via quadratic variation works very well for variance swaps, it is not sufficient for non-linear payoffs with short maturities. The effect is common to all variance curve models (or stochastic volatility models, for that matter).*" In particular, he presents some examples based on call options on realized variance in the Heston model, that indicate that the approximation by quadratic variation notably diverges from the true value for maturities shorter than 60 days (cf. [3, p. 128]). This leads to the following questions considered in the present study:

(1) To what extent is it indeed true that quadratic variation is not a good proxy for realized variance for short dated call-options? Put differently, is there a *discretization gap* resulting from this approximation?
(2) How can options on the realized variance itself be valuated exactly?

The remainder of the article is organized as follows. We first prove that Bühler's statement holds true for a large class of diffusion models. Afterwards, we turn to models with jumps. In Section 3, we show that in pure-jump exponential Lévy models, the short-time limits of call-prices on quadratic variation and realized variance coincide. In exponential Lévy models with both jumps and non-zero diffusion coefficient, however, the discretization gap does not vanish, and we quantify its exact size under mild conditions on the jump measure. Subsequently, in Section 4, we discuss how to compute prices of options on realized variance in exponential Lévy models using Fourier-Laplace methods. This leads to semi-explicit formulas of a similar complexity as for the approximation via quadratic variation in [5]. Afterwards, we present some numerical examples. Finally, we briefly sketch how our approach can potentially be extended to affine stochastic volatility models.

Notation. For a Lévy process X with Lévy-Khintchine triplet (b, σ^2, F), we denote by

$$\psi(u) = bu + \frac{\sigma^2}{2}u^2 + \int (e^{ux} - 1 - uh(x))F(dx),$$

the corresponding *Lévy exponent*, i.e. the continuous function $\psi : i\mathbb{R} \to \mathbb{C}$ such that $E(e^{uX_t}) = \exp(t\psi^X(u))$. Here, $h : \mathbb{R} \to \mathbb{R}$ is a suitable truncation function as e.g. $h(x) = x1_{\{|x|\leq 1\}}$.

2. Short-time limits in diffusion models

In this section, we assume that under a pricing measure, the log-price X follows an Itô process

$$dX_t = -\frac{\sigma_t^2}{2}dt + \sigma_t dW_t, \quad X_0 = 0,$$

for a standard Brownian motion W and a semimartingale σ. The following result shows that the resulting ATM call prices on quadratic variation indeed vanish as maturity tends to zero.

[1]The class of affine stochastic volatility models includes exponential Lévy models, the Heston model with and without jumps, and many stochastic time-change models.

Theorem 2.1. *Suppose σ_0 is constant and the mapping $T \mapsto \mathbb{E}\left[\sigma_T^2\right]$ is finite and continuous in some neighborhood $\mathcal{U}$ of zero. Then we have*

$$\lim_{T\to 0} \frac{1}{T}\mathbb{E}\left[([X,X]_T - \mathbb{E}\left[[X,X]_T\right])^+\right] = 0.$$

Proof. Since X is continuous, we have

$$[X,X]_T = \langle X,X\rangle_T = \int_0^T \sigma_t^2 dt,$$

for any $T > 0$ and hence, for $T \in \mathcal{U}$,

$$\begin{aligned}\frac{1}{T}\mathbb{E}\left[([X,X]_T - \mathbb{E}\left[[X,X]_T\right])^+\right] &= \frac{1}{T}\mathbb{E}\left[\left(\int_0^T (\sigma_t^2 - \mathbb{E}\left[\sigma_t^2\right])dt\right)^+\right] \\ &\leq \frac{1}{T}\mathbb{E}\left[\int_0^T (\sigma_t^2 - \mathbb{E}\left[\sigma_t^2\right])^+ dt\right] \\ &= \frac{1}{T}\int_0^T \mathbb{E}\left[(\sigma_t^2 - \mathbb{E}\left[\sigma_t^2\right])^+\right] dt,\end{aligned}$$

by Jensen's inequality and Fubini's theorem.

Now notice that the mapping $T \mapsto \mathbb{E}\left[(\sigma_T^2 - \mathbb{E}\left[\sigma_T^2\right])^+\right]$ is continuous on $\mathcal{U}$. Indeed, fix $T_0 \in \mathcal{U}$. Since $T \to \mathbb{E}\left[\sigma_T^2\right]$ is continuous on $\mathcal{U}$, we have $\sigma_T^2 \to \sigma_{T_0}^2$ in probability for $T \to T_0$ and the set $\{\sigma_T^2 : T \in \mathcal{U}\}$ is uniformly integrable. Consequently, $\lim_{T\to T_0}(\sigma_T^2, \mathbb{E}\left[\sigma_T^2\right]) = (\sigma_{T_0}^2, \mathbb{E}\left[\sigma_{T_0}^2\right])$ and hence also

$$\lim_{T\to T_0} \left(\sigma_T^2 - \mathbb{E}\left[\sigma_T^2\right]\right)^+ = \left(\sigma_{T_0}^2 - \mathbb{E}\left[\sigma_{T_0}^2\right]\right)^+ \quad \text{in probability,} \tag{2.1}$$

because the mapping $(x_1, x_2) \mapsto (x_1 - x_2)^+$ is continuous. Since $|(\sigma_T^2 - \mathbb{E}\left[\sigma_T^2\right])^+| \leq \sigma_T^2$, the uniform integrability of $\{\sigma_T^2 : T \in \mathcal{U}\}$ implies that $\{(\sigma_T^2 - \mathbb{E}\left[\sigma_T^2\right])^+ : T \in \mathcal{U}\}$ is uniformly integrable as well. Together with (2.1), this shows that $\lim_{T\to T_0} \mathbb{E}\left[(\sigma_T^2 - \mathbb{E}\left[\sigma_T^2\right])^+\right] = \mathbb{E}\left[(\sigma_{T_0}^2 - \mathbb{E}\left[\sigma_{T_0}^2\right])^+\right]$ as claimed.

Consequently, it follows from the fundamental theorem of calculus that

$$\begin{aligned}0 \leq \limsup_{T\to 0} \frac{1}{T}\mathbb{E}\left[([X,X]_T - \mathbb{E}\left[[X,X]_T\right])^+\right] &\leq \frac{\partial}{\partial T}\int_0^T \mathbb{E}\left[(\sigma_t^2 - \mathbb{E}\left[\sigma_t^2\right])^+\right] dt\bigg|_{T=0} \\ &= \mathbb{E}\left[(\sigma_0^2 - \mathbb{E}\left[\sigma_0^2\right])^+\right],\end{aligned}$$

which vanishes for constant σ_0. This proves the assertion. □

We now turn to the short-time behavior of options written on the discrete realized variance. For a fixed number d of trading days per year, the shortest dated nontrivial realized variance has maturity $T = 1/d$. In order to come up with a true short-time limit we therefore also let d go to infinity, i.e. we consider $\lim_{T\to 0} \mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+\right]$. The following result shows that unlike for calls on quadratic variation, this limit is strictly positive if the stochastic volatility σ and the Brownian motion W driving the stock price are independent. In particular, the *discretization gap* that results from comparing this limit to the corresponding limit for an ATM call on quadratic variation is the same as in the Black-Scholes model with constant volatility σ_0.

Theorem 2.2. *Suppose σ_0 is constant, $T \mapsto \mathbb{E}\left[\sigma_T^{4p}\right]$ is finite and continuous in some neighborhood $\mathcal{U}$ of zero for some $p > 1$, and let the volatility process σ be* independent *of the Brownian motion W driving the stock price. Then*

$$\lim_{T\to 0} \frac{1}{T}\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+\right] = \sqrt{\frac{2}{\pi e}}\sigma_0^2.$$

Proof. First notice that $\mathbb{E}\left[X_T^2\right] \leq \frac{1}{2}\int_0^T \mathbb{E}\left[\sigma_t^4\right] dt + 2\int_0^T \mathbb{E}\left[\sigma_t^2\right] dt$ by Jensen's inequality, the Itô isometry and Fubini's theorem. Hence X_T is square-integrable on $\mathcal{U}$, because $T \mapsto \mathbb{E}\left[\sigma_T^4\right]$ and $T \mapsto \mathbb{E}\left[\sigma_T^2\right]$ are bounded on any compact subset of $\mathcal{U}$ under the stated assumptions.

For $T \in \mathcal{U}$, let $\Sigma_T := \int_0^T \sigma_t^2 dt$. Due to the independence of σ and W, we have $\int_0^T \sigma_t dW_t \sim N(0, \Sigma_T)$ conditional on the σ-field $\mathcal{G}$ generated by σ. Since, moreover, $-\Sigma_T/2$ is $\mathcal{G}$-measurable, this implies $\mathbb{E}\left[X_T^2\right] = \mathbb{E}\left[\Sigma_T^2/4 + \Sigma_T\right]$ and, in turn,

$$\begin{aligned}\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+ |\mathcal{G}\right] &= \int_{-\infty}^{\infty} \left(x^2 - \Sigma_T x + \frac{\Sigma_T^2}{4} - \mathbb{E}\left[\frac{\Sigma_T^2}{4} - \Sigma_T\right]\right)^+ \frac{e^{-\frac{x^2}{2\Sigma_T}}}{\sqrt{2\pi\Sigma_T}} dx \\ &= \frac{\Sigma_T^2}{4} + \Sigma_T - \mathbb{E}\left[\frac{\Sigma_T^2}{4} + \Sigma_T\right] - I[T, x_+(T)] + I[T, x_-(T)],\end{aligned}$$

with $x_\pm(T) = \Sigma_T/2 \pm \sqrt{\mathbb{E}\left[\Sigma_T^2/4 + \Sigma_T\right]}$ and

$$I(T,x) = \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2\Sigma_T}}\right)\left(\frac{\Sigma_T^2}{4} + \Sigma_T - \mathbb{E}\left[\frac{\Sigma_T^2}{4} + \Sigma_T\right]\right) + \frac{e^{-\frac{x^2}{2\Sigma_T}}}{\sqrt{2\pi}}\left(\Sigma_T^{\frac{3}{2}} - x\Sigma_T^{\frac{1}{2}}\right),$$

for the Gaussian error function $\mathrm{erf}(\cdot)$. As $T \mapsto \sigma_T^2$ is right-continuous in zero, $T \mapsto \Sigma_T$ is differentiable in zero with derivative σ_0^2. Moreover, $T \mapsto \mathbb{E}\left[\Sigma_T\right]$ is differentiable in zero with derivative σ_0^2 by Fubini's theorem and the fundamental theorem of calculus. Finally, $\lim_{T\to 0} \frac{1}{T}\Sigma_T^2/4 = \Sigma_0\sigma_0^2/2 = 0$ implies that $\lim_{T\to 0}\frac{1}{T}\mathbb{E}\left[\Sigma_T^2/4\right] = 0$ by [12, Theorem 1.21], since $\Sigma_T^2 \leq \int_0^T \sigma_t^4 dt$ and $\lim_{T\to 0}\frac{1}{T}\int_0^T \sigma_t^4 dt = \sigma_0^4$ as well as $\lim_{T\to 0}\frac{1}{T}\mathbb{E}\left[\int_0^T \sigma_t^4 dt\right] = \sigma_0^4$ again by the fundamental theorem of calculus and Fubini's theorem. Hence l'Hospital's rule shows after differentiation that

$$\begin{aligned}\lim_{T\to 0}\frac{1}{T}\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+|\mathcal{G}\right] &= \frac{\partial}{\partial T}\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+|\mathcal{G}\right]\big|_{T=0} \\ &= \sqrt{\frac{2}{\pi e}}\sigma_0^2, \quad \text{a.s.}\end{aligned}$$

and therefore also in probability. Consequently, it remains to show that the set $\{\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+|\mathcal{G}\right] : T \in \mathcal{K}\}$ is uniformly integrable for some compact subset $\mathcal{K}$ of $\mathcal{U}$. In view of

$$|\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+|\mathcal{G}\right]| \leq \mathbb{E}\left[X_T^2|\mathcal{G}\right] \leq \frac{1}{4}\int_0^T \sigma_t^4 dt + \int_0^T \sigma_t^2 dt,$$

this follows the from finiteness and continuity of $T \mapsto \mathbb{E}\left[\sigma_t^{4p}\right]$ on $\mathcal{K}$, which combined with Jensen's inequality shows that $\{\int_0^T \sigma_t^2 dt : T \in \mathcal{K}\}$ and $\{\int_0^T \sigma_t^4 dt : T \in \mathcal{K}\}$ are bounded in L^p. □

By the dominated convergence theorem, the prerequisites of Theorems 2.1 and 2.2 are satisfied if σ is uniformly bounded in a neighborhood of zero. While this sufficient condition does not hold in most concrete models, it is typically straightforward to verify the necessary regularity directly:

Example 2.3. Theorems 2.1 and 2.2 are applicable in the following cases.

(1) *Lognormal volatility*: Let

$$d\sigma_t = \eta\sigma_t dt + \theta\sigma_t dZ_t,$$

for $\eta \in \mathbb{R}$, $\theta \geq 0$ and a standard Brownian motion Z. Then we have

$$\mathbb{E}\left[\sigma_T^2\right] = \exp[T(2\eta + \theta^2)], \quad \mathbb{E}\left[\sigma_T^{4p}\right] = \exp[T(4p\eta + (8p^2 - 2p)\theta^2)],$$

for any $p > 1$. Therefore $\mathbb{E}\left[\sigma_T^2\right]$ and $\mathbb{E}\left[\sigma_T^{4p}\right]$ are continuous in T.

(2) *Lévy-driven Ornstein-Uhlenbeck processes*: Suppose that σ^2 is given by

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dZ_t,$$

where $\lambda \in \mathbb{R}$ and Z is a subordinator, i.e. an increasing Lévy process. By e.g. [7, Section 15.3.1], the continuity in T of

$$\mathbb{E}\left[\sigma_T^2\right] = \sigma_0^2 e^{-\lambda T} + \frac{1-e^{-\lambda T}}{\lambda}\mathbb{E}\left[Z_1\right]$$

is assured, provided that $\mathbb{E}[Z_1]$ is finite. Likewise, [7, Proposition 15.1] and straightforward but tedious calculations yield that $T \mapsto \mathbb{E}\left[\sigma_T^6\right]$ is continuous, if $\mathbb{E}\left[Z_1^3\right] < \infty$.

(3) *Square-root processes*: Suppose that

$$d\sigma_t^2 = \lambda(\eta - \sigma_t^2)dt + \theta\sigma_t dZ_t,$$

for a standard Brownian motion Z and $\eta, \theta \geq 0$, $\lambda \in \mathbb{R}$. Then $\mathbb{E}\left[\sigma_T^2\right] = \eta + (\sigma_0^2 - \eta)e^{-\lambda T}$ is continuous in T (cf. e.g. [7, Section 15.1.2]). The continuity of $T \mapsto \mathbb{E}\left[\sigma_T^{4p}\right]$ in a neighborhood of zero follows for any $p > 1$ by differentiating the characteristic function of σ^2.

3. Short-time limits in exponential Lévy models

We now consider the short-time behavior of options written on the quadratic variation resp. the realized variance in *exponential Lévy models*, where the log-price X under a pricing measure is supposed to be a Lévy process with Lévy-Khintchine triplet (b, σ^2, F) (cf. e.g. [7, 18] for an overview). To ensure expectations of and hence call prices on both quadratic variation and realized variance exist, we assume throughout that

$$\int x^2 F(dx) < \infty,$$

which is equivalent to $\mathbb{E}\left[X_T^2\right] < \infty$ and $\mathbb{E}\left[[X,X]_T\right] < \infty$ for all $T > 0$ by [13, Lemma 4.1] and [7, Proposition 3.13].

Our first result shows that unlike in diffusion models, the prices of call options on normalized quadratic variation do not converge to zero in the short-time limit for exponential Lévy models with jumps.

Lemma 3.1. *We have*

$$\lim_{T\to 0}\frac{1}{T}\mathbb{E}\left[([X,X]_T - \mathbb{E}\left[[X,X]_T\right])^+\right] = \int x^2 F(dx).$$

Proof. Since $[X,X]_T = \sigma^2 T + \sum_{t\leq T}\Delta X_t^2$ by [11, I.4.52, II.2.6 and II.4.19], it follows that

$$\mathbb{E}\left[([X,X]_T - \mathbb{E}\left[[X,X]_T\right])^+\right] \leq \mathbb{E}\left[\sum_{t\leq T}\Delta X_t^2\right] = T\int x^2 F(dx),$$

by [11, II.1.8,II.2.6 and II.4.19]. This in turn yields

$$\limsup_{T\to 0}\frac{1}{T}\mathbb{E}\left[([X,X]_T - \mathbb{E}\left[[X,X]_T\right])^+\right] \leq \int x^2 F(dx). \tag{3.1}$$

For $r \in (1,2)$, define the function $f^{(r)} : \mathbb{R} \to \mathbb{R}$ via

$$f^{(r)}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x^r, & \text{if } x \in (0,1), \\ x, & \text{if } x \geq 1. \end{cases}$$

By [13, Lemma 4.1] and [11, II.4.19], $([X,X]_t - \mathbb{E}[[X,X]_t])_{t\in\mathbb{R}_+}$ is a Lévy process with Lévy-Khintchine triplet $(0,0,\int 1.(x^2)F(dx))$ relative to the truncation function $h(x)=0$. Since $f^{(r)}$ is continuous, bounded by the subadditive function $f(x)=x^+$ and satisfies $f^{(r)}(x)=O(|x|^r)$ for $x\to 0$, [9, Theorem 1.1] implies

$$\lim_{T\to 0}\frac{1}{T}\mathbb{E}\left[f^{(r)}([X,X]_T - \mathbb{E}[[X,X]_T])\right] = \int f^{(r)}(x^2)F(dx).$$

Since $f\geq f^{(r)}$, this yields

$$\liminf_{T\to 0}\frac{1}{T}\mathbb{E}\left[([X,X]_T - \mathbb{E}[[X,X]_T])^+\right] \geq \int f^{(r)}(x^2)F(dx).$$

In view of $f^{(r)}\uparrow f$ for $r\downarrow 1$ and because $\int x^2F(dx)<\infty$ by assumption, monotone convergence yields

$$\liminf_{T\to 0}\frac{1}{T}\mathbb{E}\left[([X,X]_T - \mathbb{E}[[X,X]_T])^+\right] \geq \int x^2F(dx),$$

which combined with (3.1) proves the assertion. □

Next, we turn to the corresponding short-time limit for ATM calls written on the realized variance.

Lemma 3.2. *We have*

$$\int x^2F(dx) \leq \liminf_{T\to 0}\frac{1}{T}\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+\right]$$
$$\leq \limsup_{T\to 0}\frac{1}{T}\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+\right] \leq \sigma^2 + \int x^2F(dx).$$

Proof. First notice that since $\mathbb{E}\left[X_T^2\right]\geq 0$ and

$$\mathbb{E}\left[X_T^2\right] = T\left(\sigma^2 + \int x^2F(dx)\right) + T^2\left(b + \int (x-h(x))F(dx)\right),$$

by e.g. [7, Proposition 3.13], it follows that

$$\limsup_{T\to 0}\frac{1}{T}\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+\right] \leq \limsup_{T\to 0}\frac{1}{T}\mathbb{E}\left[X_T^2\right] = \sigma^2 + \int x^2F(dx).$$

On the other hand, for any $K>0$,

$$\frac{1}{T}\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+\right] \geq \frac{1}{T}\mathbb{E}\left[(X_T^2 - K)^+\right] \tag{3.2}$$

for sufficiently small T. By [9, Theorem 1.1],

$$\lim_{T\to 0}\frac{1}{T}\mathbb{E}\left[(X_T^2 - K)^+\right] = \int (x^2-K)^+F(dx),$$

which together with (3.2) implies

$$\liminf_{T\to 0}\frac{1}{T}\mathbb{E}\left[(X_T^2 - \mathbb{E}\left[X_T^2\right])^+\right] \geq \int (x^2-K)^+F(dx).$$

The claim now follows from using monotone convergence for $K\to 0$. □

In view of the preceding two lemmas, there is *no discretization gap* if the driving Lévy process is of *pure jump type*. This includes several popular models from the literature as e.g. normal inverse Gaussian, variance Gamma and CGMY processes, see [7, 18]. If X is Brownian motion, Theorem 2.2 yields a limit of $\sigma^2\sqrt{\frac{2}{\pi e}}\approx 0.48\sigma^2$ showing that neither bound has to be sharp when the Lévy process has a non-zero diffusion component. However, with some additional effort and by imposing a mild condition on the jump component of the Lévy process, we are able to identify the exact short-time limit of a call option on discrete realized variance. The condition we impose is related to the small-time fluctuation of the pure jump component L, which can be quantified by its *Blumenthal-Getoor index*

$$\beta := \inf\left\{\alpha > 0 : \int_{|x|<1} |x|^\alpha \, F(dx) < \infty\right\}$$

introduced in [1]. Note that β is always contained in the interval $[0,2]$. Loosely speaking, the Blumenthal-Getoor index quantifies how diffusion-like the jump process L behaves at a small time-scale, with values close to 2 indicating more diffusion-like behavior. In case of an α-stable process, for example, $\beta = \alpha$. For precise results and applications see e.g. [1] and [17, Section 47].

Theorem 3.3. *Suppose that the pure jump component of X has Blumenthal-Getoor index $\beta < 2$. Let $v^2 = \int x^2 \, F(dx) < \infty$. Then*

$$\lim_{T\to 0} \frac{1}{T}\mathbb{E}\left[\left(X_T^2 - \mathbb{E}\left[X_T^2\right]\right)^+\right] = \sigma^2 P\left(\frac{v^2}{\sigma^2}\right) + v^2 Q\left(\frac{v^2}{\sigma^2}\right),$$

where $P(r)$ resp. $Q(r)$ are strictly decreasing resp. increasing functions on $[0,\infty)$, given by

$$P(r) = \sqrt{\frac{2(1+r)}{\pi\exp(1+r)}}, \quad \text{and} \quad Q(r) = 2\Phi(\sqrt{1+r}) - 1,$$

and $\Phi(\cdot)$ denotes the standard normal distribution function.

Remark 3.4. The assumption that the pure jump component L of X has Blumenthal-Getoor index $\beta < 2$ can be replaced by the weaker requirement that $\lim_{t\to 0} \frac{|L_t|}{\sqrt{t}} = 0$ a.s.

Remark 3.5. For $\sigma = 0$ the theorem should be interpreted in the sense that $v^2/\sigma^2 = \infty$, and that $P(\infty)$ and $Q(\infty)$ are the finite limiting values of $P(r)$ and $Q(r)$ as $r \to \infty$.

Remark 3.6. Virtually all Lévy processes used in Mathematical Finance satisfy the condition that the Blumenthal-Getoor index β of their jump component is strictly smaller than 2. Nevertheless a pure-jump Lévy process with $\beta = 2$ can be constructed from the Lévy measure $F(dx) = \mathbf{1}_{(0,1/2)}(x)x^{-3}\log(x)^{-2}dx$. A simple calculation shows that $F(dx)$ integrates $1 \wedge x^2$ and thus is indeed a Lévy measure. Now choosing any $\epsilon > 0$ it holds that $x^{\epsilon/2}\log(x)^2 \leq 1$ for all x in some interval $(0, x_0)$. Thus we may estimate

$$\int_{-1}^{1} x^{2-\epsilon} F(dx) \geq \int_0^{x_0\wedge 1/2} \frac{1}{x^{(1+\epsilon/2)}x^{\epsilon/2}\log(x)^2}dx \geq \int_0^{x_0\wedge 1/2} \frac{1}{x^{1+\epsilon/2}}dx = \infty,$$

which shows that the Blumenthal-Getoor index of the corresponding Lévy process is 2.

In the boundary case of $v^2 = 0$ we have $P(0) = \sigma^2\sqrt{\frac{2}{\pi e}}$ and recover from Theorem 3.3 the result of Proposition 2.2 for a constant variance process. For $\sigma = 0$ we obtain $Q(\infty) = 1$ and recover the result of Lemma 3.2. The proof of the theorem is lengthy, and therefore deferred to the Appendix.

4. Exact pricing methods for options on realized variance

4.1. **Option pricing using integral transform methods.** To come up with exact valuation schemes, we first recall how to price European-style options using the integral transform approach of [4, 15]. The key assumption is the existence of an integral representation of the option's payoff function f in the following sense:

$$f(x) = \int_{R-i\infty}^{R+i\infty} p(z)e^{-zx}dz,$$

for $p : \mathbb{C} \to \mathbb{C}$ and $R > 0$ such that $v \mapsto p(R + iv)$ is integrable.

Example 4.1. For a put on variance we have

$$f(x) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} \frac{e^{Kz}}{z^2}e^{-zx}dz = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{K(R+iv)}}{(R+iv)^2}e^{-(R+iv)x}\right)dv,$$

for $x \geq 0$ and any $R > 0$ (cf. e.g. [6, Corollary 7.8]).

In view of Fubini's theorem, the valuation of options which can be represented like this boils down to the computation of the Laplace transform of the underlying. E.g. for the put on quadratic variation we have

$$\mathbb{E}\left[(K - [X,X]_T)^+\right] = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{K(R+iv)}}{(R+iv)^2}\mathbb{E}\left[\exp\left(-(R+iv)[X,X]_T\right)\right]\right)dv.$$

Using the put-call parity $(x - K)^+ = x - K + (K - x)^+$, this leads to the analogous formula

$$\begin{aligned}&\mathbb{E}\left[([X,X]_T - K)^+\right] \\ &\quad = \mathbb{E}\left[[X,X]_T\right] - K + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{K(R+iv)}}{(R+iv)^2}\mathbb{E}\left[\exp\left(-(R+iv)[X,X]_T\right)\right]\right)dv\end{aligned}$$

for calls on variance, provided that $[X,X]_T$ is integrable. Evidently, one just has to replace the normalized quadratic variation $\frac{1}{T}[X,X]$ by RV_T to come up with the corresponding formulas for option on *discrete* realized variance. Summing up, it remains to compute the Laplace transforms of the objects of interest.

4.2. **Options on quadratic variation.** For exponential Lévy models, the quadratic variation process $[X,X]$ also follows a Lévy process (cf. [5] for the self-decomposable and [13] for the general case). More specifically, we have the following

Lemma 4.2. *Suppose the* log-*price X follows a Lévy process with Lévy-Khintchine triplet (b, σ^2, F). Then $[X,X]$ also is a Lévy process and its Lévy-Khintchine triplet is given by $(\sigma^2, 0, F^{[X,X]})$ relative to the truncation function $h(x) = 0$, where,*

$$F^{[X,X]}(G) = \int 1_G(x^2)F(dx), \quad \forall G \in \mathcal{B}.$$

Proof. See [13, Lemma 4.1] . □

Combined with the Lévy-Khintchine formula [17, Theorem 8.1], this result immediately yields the required Laplace transform.

Corollary 4.3. *We have $\mathbb{E}\left[e^{-u[X,X]_T}\right] = \exp[T\psi^{[X,X]}(-u)]$ for*

$$\psi^{[X,X]}(-u) = \left(-cu + \int (e^{-ux^2} - 1)F(dx)\right), \quad \mathrm{Re}(u) \geq 0.$$

Consequently, the we obtain

$$\mathbb{E}\left[(K - [X,X]_T)^+\right] = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{K(R+iv)}}{(R+iv)^2}e^{T\psi^{[X,X]}(-(R+iv))}\right)dv \tag{4.1}$$

for puts on variance. Likewise, for calls on variance,

$$\begin{aligned}(4.2)\quad &\mathbb{E}\left[([X,X]_T - K)^+\right] \\ &= T\left(\sigma^2 + \int x^2 F(dx)\right) - K + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{K(R+iv)}}{(R+iv)^2} e^{T\psi^{[X,X]}(-(R+iv))}\right) dv,\end{aligned}$$

provided that $\int x^2 F(dx) < \infty$. Some examples where the Lévy exponent $\psi^{[X,X]}$ of $[X,X]$ can be computed in closed form are summarized in Section 5 below.

4.3. **Options on realized variance.** In this section we develop a corresponding integral transform pricing method for claims on the *exact* quantity (1.1) without any use of approximation. There are several benefits to such a method: First, as shown in Section 3, there are Lévy processes for which the discretization gap between claims on quadratic variation and discrete realized variance does not vanish. For such processes it is clear that an exact valuation method should be preferred over the valuation methods (4.1) and (4.2) based on quadratic variation. Second, while the results of Section 3 show that the discretization gap vanishes for pure-jump models in the limit $T \to 0$, it is a priori not clear whether the gap will also be small enough for practical purposes when $T > 0$. Third, the method we present will in general be of similar computational complexity as the method based on quadratic variation, especially in cases where the Lévy exponent ψ of X is of more tractable form than the Lévy exponent $\psi^{[X,X]}$ of $[X,X]$. Such cases include for example subordination-based processes like the normal inverse Gaussian or generalized hyperbolic process. See Remark 4.9 for more details.

As in the previous section, the crucial quantity is again a Laplace transform, namely

$$\mathbb{E}\left[\exp\left(-u\sum_{k=1}^{dT}(X_{k/d} - X_{(k-1)/d})^2\right)\right] = \left(\mathbb{E}\left[\exp(-uX_{1/d}^2)\right]\right)^{dT},$$

where the equality is due to the independence and stationarity of the increments of the Lévy process X. Consequently, if the Laplace transform of the *squared* process is known, the price of e.g. puts and calls on discrete realized variance can be recovered by an inverse Laplace transform as above.

Our approach is based on the following identity: If Z is a normally distributed random variable, independent of X_t, then using the characteristic function of the normal distribution it holds that

$$\mathbb{E}\left[e^{-uX_t^2}\right] = \mathbb{E}\left[e^{i\sqrt{2u}X_t Z}\right] = \mathbb{E}\left[e^{t\psi(iZ\sqrt{2u})}\right], \tag{4.3}$$

for all $u \in \mathbb{R}_+$. Note that the first expectation is taken with respect to the law of the Levy process X_t, the middle expectation with respect to the product law of X_t and Z, and the final expectation with respect to the law of the normal random variable Z only. The exchange in the order of integration is justified by the Fubini theorem, and the fact that the integrands on the left and right hand side are bounded by 1 in absolute value. The benefit of formula (4.3) is to replace an integration with respect to the law of the Lévy process - which is typically not known explicitly – by an integration with respect to a standard normal distribution. The characteristic exponent ψ which appears in the expectation on the right is in most cases analytically known and of considerably simpler form then the law of the Lévy process.

Let us remark here, that the randomization approach of formula (4.3) can be extended to the Laplace transform of powers $|X_t|^p$ with $p \in (0,2)$, and consequently to the *discrete realized p-variation* $\sum_k |X_{k/d} - X_{(k-1)/d}|^p$ of a Lévy process X. To this end, replace the standard normal variable Z by a symmetric α-stable random variable S_p with parameters $(\alpha, \beta, c, \tau) = (p, 0, 1, 0)$ (cf. [17, Theorem 14.15]). Using that $\mathbb{E}\left[e^{iwS_p}\right] = \exp(-|w|^p)$ we obtain

$$\mathbb{E}\left[e^{-u|X_t|^p}\right] = \mathbb{E}\left[e^{iu^{1/p}X_t S_p}\right] = \mathbb{E}\left[e^{t\psi(iS_p u^{1/p})}\right], \tag{4.4}$$

for all $u \in \mathbb{R}_+$.

Remarks on Laplace Inversion. The integral in formula (4.2) can be considered as inverting a Laplace transform by integration along a contour in the complex plane. There are many alternatives to this inversion method, see e.g. [8] for an overview. Some of these methods only require knowledge of the Laplace transform *on the positive real line*, and thus seem tailor-made for formula (4.3) which holds – unless further conditions are imposed – only on $\mathbb{R}_+$. The best-known such method is probably the *Post-Widder inversion formula*

$$f(x) = \lim_{n\to\infty} \frac{(-1)^n}{n} \left(\frac{n+1}{x}\right)^{n+1} \widehat{f}^{(n)}((n+1)/x), \qquad x > 0, \tag{4.5}$$

where $\widehat{f}$ denotes the Laplace transform of a function f, and $\widehat{f}^{(n)}$ its n-th derivative. The Post-Widder method suffers from slow convergence and cancelation errors, and modifications such as the Gaver-Stehfest algorithm have been introduced to improve its performance. After implementing the Gaver-Stehfest algorithm and performing some numerical tests, we observed however, that small errors in $\widehat{f}$ – which invariably result from the evaluation of (4.3) – are strongly amplified by this method and lead to a huge errors in f, probably due to the use of very high-order derivatives in (4.5). Moreover, as [8] shows, inversion algorithms that evaluate the Laplace transform in the complex half-plane are in general numerically superior to algorithms that evaluate the Laplace transform only on the positive half-plane. For these reasons we decided to concentrate on the contour integration formula (4.2), and as a next step to extend (4.3) to the complex half plane $\mathrm{Re}\,(u) > 0$.

Extension to the complex half plane. Extending (4.3) to the complex half plane will not be possible without imposing some conditions on ψ. The following is sufficient:

Condition 4.4. The characteristic exponent ψ has an analytic extension from the imaginary halfline $i\mathbb{R}_+$ to the sector

$$\Lambda = \left\{ u \in \mathbb{C} : \frac{\pi}{4} < \arg(u) < \frac{3\pi}{4} \right\}.$$

Moreover, the extended function ψ satisfies the growth bound

$$\limsup_{r\to\infty} \frac{\mathrm{Re}\,(\psi(re^{i\theta}))}{r^2} \le 0 \quad \text{for all} \quad \theta \in \left(\tfrac{\pi}{4}, \tfrac{3\pi}{4}\right). \tag{4.6}$$

Remark 4.5. An analytic function satisfying the growth bound (4.6) is often said to be of *order* 2 *and type* 0 *in the sector* Λ.

An elementary symmetry argument shows that given the above condition, ψ can also be analytically extended from the negative imaginary halfline $-i\mathbb{R}_+$ to the conjugate sector $\overline{\Lambda}$.

Lemma 4.6. *Suppose ψ satisfies Condition 4.4. Then it can also be analytically extended from the halfline $-i\mathbb{R}_+$ to the conjugate sector $\overline{\Lambda}$. Overall, ψ has a unique extension to the hourglass shaped region $\Lambda^{\bowtie} = \Lambda \cup \{0\} \cup \overline{\Lambda}$, which is analytic on both Λ and $\overline{\Lambda}$ and satisfies the growth bound*

$$\limsup_{r\to\pm\infty} \frac{\mathrm{Re}\,(\psi(re^{i\theta}))}{r^2} \le 0 \quad \textit{for all} \quad \theta \in \left(\tfrac{\pi}{4}, \tfrac{3\pi}{4}\right). \tag{4.7}$$

Proof. Suppose that ψ satisfies Condition 4.4, i.e. it is an analytic function defined on Λ. For $u \in \overline{\Lambda}$ define $\psi(u) = \overline{\psi(\overline{u})}$. On the imaginary axis, this definition agrees with the Lévy-Khintchine representation of ψ. The analyticity of ψ on $\overline{\Lambda}$ follows directly, e.g. by verifying the Cauchy-Riemann differential equations. The growth bound on $\Lambda^{\bowtie}$ is an immediate consequence of the construction of the extension. □

We can now establish the central result of this section:

Theorem 4.7. *Let X_t be a Lévy process with characteristic exponent ψ and let Z be an independent standard normal random variable. Then*

$$\mathbb{E}\left[e^{-uX_t^2}\right] = \mathbb{E}\left[e^{t\psi(iZ\sqrt{2u})}\right] \tag{4.8}$$

holds for all u on the positive real line. *If X_t satisfies Condition 4.4, then* (4.8) *holds for all u in the* positive half-plane $\{u \in \mathbb{C} : \mathrm{Re}\,(u) > 0\}$, *with ψ denoting the unique analytic extension described in Lemma 4.6.*

Remark 4.8. The square root denotes the principal branch of the complex square root function with branch cut along the negative real line.

Remark 4.9. For most Lévy processes proposed in the literature, the Lévy exponent ψ can be computed in closed form. Hence, the evaluation of the Laplace transform of X_T^2 typically requires one numerical integration. The corresponding formula for the Laplace transform of $[X, X]_T$ in Corollary 4.3 is therefore simpler, if the integral $\int(e^{-ux^2} - 1)F(dx)$ can be computed in closed form. However, even if this is possible as e.g. for CGMY processes and the models of Merton and Kou, one usually has to employ special functions (cf. Section 5) such that the numerical advantage is not too big. On the other hand, e.g. for NIG or generalized hyperbolic Lévy processes, $\int(e^{-ux^2} - 1)F(dx)$ has to be evaluated using numerical quadrature, such that both formulas turn out to be of a similar complexity.

Proof. Let $u \in \mathcal{H}_+ := \{u \in \mathbb{C} : \mathrm{Re}\,(u) > 0\}$. The function $u \mapsto i\sqrt{2u}$ (using the principal branch of the square root) is a single-valued analytic function on $\mathcal{H}_+$, mapping u to $\sqrt{2|u|}\exp\left(\frac{i}{2}(\arg(u) + \pi)\right)$, and thus $\mathcal{H}_+$ to Λ. With the normal random variable Z taking values in $\mathbb{R}$ it follows that $iZ\sqrt{2u} \in \Lambda^{\bowtie}$. Let $\epsilon > 0$. Then (4.7) implies that there exists $M_\theta > 0$ such that

$$\mathrm{Re}\,(\psi(re^{i\theta})) \le \epsilon r^2 + M_\theta, \quad \text{for} \quad r \in \mathbb{R}, \quad \theta \in \left(\tfrac{\pi}{4}, \tfrac{3\pi}{4}\right). \tag{4.9}$$

Thus

$$\left|e^{\psi(iZ\sqrt{2u})}\right| = \exp\left[\mathrm{Re}\left(\psi\left(Z\sqrt{2|u|}e^{i(\arg u + \pi)/2}\right)\right)\right] \le \exp\left(\epsilon 2|u|Z^2 + M_\theta\right).$$

Note that Z^2 has chi-squared distribution with one degree of freedom. The right hand side thus has a finite expectation of value $(1 - 4\epsilon|u|)^{-1/2}e^{M_\theta}$, whenever $|u| < 1/(4\epsilon)$. Since ϵ was arbitrary, it can be chosen small enough to satisfy this condition. We have shown that

$$f(u) = \mathbb{E}\left[e^{\psi(iZ\sqrt{2u})}\right]$$

exists for all $u \in \mathcal{H}_+$. Next we show that it is also analytic. Let $Z_n = Z\mathbf{1}_{\{|Z| \le n\}}$ be a sequence of truncations of Z and define

$$f_n(u) = \mathbb{E}\left[e^{\psi(iZ_n\sqrt{2u})}\right].$$

Since ψ is continuous on $\Lambda^{\bowtie}$, $f_n \to f$ pointwise in $\mathcal{H}_+$. Moreover, since the integrand is absolutely bounded for u in compacts, each f_n is analytic in $\mathcal{H}_+$ (cf. [16, Chapter 10, Exercise 15]). Let $\mathcal{K}$ be a compact subset of $\mathcal{H}_+$. On $\mathcal{K}$ the bound (4.9) can be turned into a *uniform* bound

$$\mathrm{Re}\,(\psi(u)) \le \epsilon R^2 + M, \qquad u \in \mathcal{K},$$

where R and M depend only on $\mathcal{K}$, and we again use the continuity of ψ on $\Lambda^{\bowtie}$. By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|f(u) - f_n(u)|^2 &= \mathbb{E}\left[\exp\left(2\mathrm{Re}\left(\psi(iZ\sqrt{2u})\right)\right)\mathbf{1}_{\{|Z|>n\}}\right]^2 \\ &\le \mathbb{E}\left[\exp\left(\epsilon 4RZ^2 + 2M\right)\right]\mathbb{P}(|Z|>n) \\ &= (1-8R\epsilon)^{-1/2}e^{2M}\mathbb{P}(|Z|>n),\end{aligned}$$

for all $u \in \mathcal{K}$. This shows that the convergence of f_n to f is uniform on compact subsets of $\mathcal{H}_+$. But analyticity is preserved by uniform convergence on compacts (cf. [16, Theorem 10.27]), such that f is analytic. We have now shown that both sides of (4.8) are well-defined analytic functions on $\mathcal{H}_+$. Since they coincide on the positive real line, they must coincide on all of $\mathcal{H}_+$, and the proof is complete. □

The following example shows that Condition 4.4 can not be reduced to analyticity in the sector Λ alone:

Example 4.10. Let $X_t = N_t - t\gamma$, where N_t is a Poisson process with intensity 1, and $\gamma = e - 1$, such that e^X is a martingale. The Lévy exponent of this process is given by $\psi(u) = e^u - 1 - u\gamma$. Clearly ψ has an analytic extension to the whole complex plane, and in particular to the sector Λ. But $\mathrm{Re}\left(\psi(re^{i\theta})\right) = e^{r\cos(\theta)}\cos(r\sin(\theta)) - 1 - r\gamma\cos(\theta)$, such that the growth condition (4.6) is not satisfied e.g. in the direction $\theta = 3\pi/8$. Finally the formula (4.8) is *not* well-defined on the whole complex half-plane $\{u \in \mathbb{C} : \mathrm{Re}\,(u) \ge 0\}$. Indeed, a tedious calculation shows that the expectation of $\mathrm{Re}\left(\mathbb{E}\left[\left(e^{t\psi(iZ\sqrt{2u})}\right)^+\right]\right)$ is infinite for e.g. $t = 1$ and $u = 3/8 - i/2$, and thus that $\mathbb{E}\left[e^{t\psi(iZ\sqrt{2u})}\right]$ does not exist.

Even though Theorem 4.7 fails in this simple case, Condition 4.4 holds for most Lévy processes used in applications.

Example 4.11. Condition 4.4 is satisfied for the following Lévy processes.

(1) *Brownian motion*: In this case, $\psi(u) = \frac{\sigma^2}{2}(u^2 - u)$ is an entire function. Moreover, $\limsup_{r\to\infty}\mathrm{Re}(\psi(re^{i\theta}))/r^2 = \frac{\sigma^2}{2}\mathrm{Re}(e^{2i\theta}) \le 0$ for all $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$.

(2) The *Kou* model: This jump-diffusion process corresponds to

$$\psi(u) = \mu u + \frac{1}{2}\sigma^2u^2 + \frac{\lambda_+ u}{\nu_+ - u} - \frac{\lambda_- u}{\nu_- + u},$$

for $\lambda_+, \lambda_-, \nu_+, \nu_- \ge 0$ and $\mu \in \mathbb{R}$ determined by the martingale condition $\psi(1) = 0$. Again, ψ obviously admits an analytic extension to Λ and, in addition, $\limsup_{r\to\infty}\mathrm{Re}(\psi(re^{i\theta}))/r^2 = \frac{\sigma^2}{2}\mathrm{Re}(e^{2i\theta}) \le 0$ for all $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$.

(3) The *Merton* model: For this jump-diffusion process, we have

$$\psi(u) = \mu u + \frac{\sigma^2}{2}u^2 + \lambda\left[\exp\left(\gamma u + \frac{\delta^2}{2}u^2\right) - 1\right],$$

for $\sigma \ge 0$, $\lambda, \delta > 0$, $\gamma \in \mathbb{R}$ and $\mu \in \mathbb{R}$ determined by the martingale condition $\psi(1) = 0$. Consequently, ψ can be analytically extended to Λ. Furthermore, since $\mathrm{Re}(\gamma re^{i\theta} + \frac{\delta^2}{2}r^2e^{2i\theta}) \le 0$ for sufficiently large r, it follows that $\limsup_{r\to\infty}\mathrm{Re}(\psi(re^{i\theta}))/r^2 = \frac{\sigma^2}{2}\mathrm{Re}(e^{2i\theta}) \le 0$ for all $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$.

(4) *NIG* processes: In this pure jump specification,

$$\psi(u) = \mu u + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2}),$$

where $\delta, \alpha > 0$, $\beta \in (-\alpha, \alpha)$ and μ is determined by the martingale condition $\psi(1) = 0$. Once more, ψ admits an analytic extension to Λ. Moreover, $\limsup_{r\to\infty}\mathrm{Re}(\psi(re^{i\theta}))/r^2 = 0$ for all $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$.

(5) *CGMY* processes: These generalizations of the VG process correspond to

$$\psi(u) = C\Gamma(-Y)((M-u)^Y - M^Y + (G+u)^Y - G^Y),$$

for parameters $C, G, M > 0$ and $Y < 2$. In particular, ψ can be analytically extended to Λ and $\limsup_{r\to\infty} \mathrm{Re}(\psi(re^{i\theta}))/r^2 = 0$ for all $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$.

Based on these examples and the above counterexample we conjecture that Condition 4.4 is related to the absolute continuity or smoothness of the Lévy measure.

5. Numerical Illustration

We now consider three numerical examples. First, we take a look at the Black-Scholes model. For $\sigma = 0.3$, the call prices on realized variance resp. quadratic variation are depicted in Figure 1 for maturities up to 50 days.

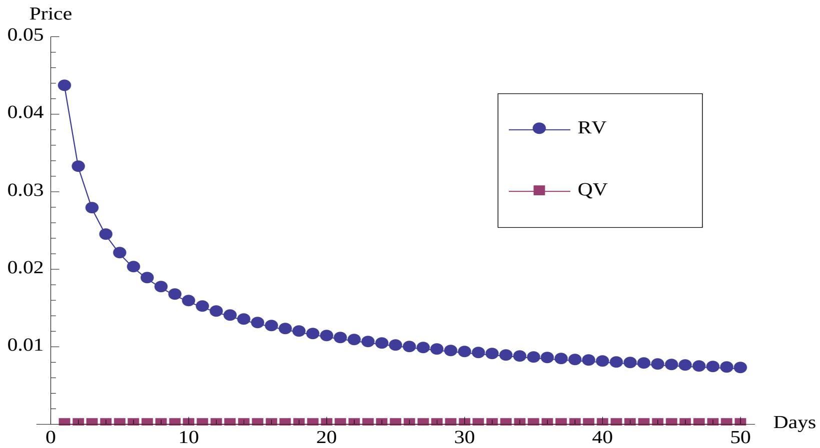

Figure 1. ATM call prices on normalized quadratic variation resp. realized variance in the Black-Scholes model. The analytical short-time limit for realized variance from Theorem 2.2 is 0.04356.

Evidently, the prices of calls on realized variance converge to the prices of calls on quadratic variation (which are zero) even slower than in the results for the Heston model reported in [3]. Moreover, for maturity 1 day, we obtain a call price of 0.04356, which virtually coincides with the short-time limit from Theorem 2.2.

Next, we turn to the pure-jump CGMY process. By [5, Section 4],

$$\psi^{[X,X]}(u) = C\Bigg(\left(\frac{2u}{Y} - \frac{M^2}{Y(1-Y)}\right) I(2-Y, M, -u) + \left(\frac{2u}{Y} - \frac{G^2}{Y(1-Y)}\right) I(2-Y, G, -u) + \frac{2uM}{Y(1-Y)} I(3-Y, M, -u) + \frac{2uG}{Y(1-Y)} I(3-Y, G, -u) + \frac{M^Y + G^Y}{Y(1-Y)}\Gamma(2-Y)\Bigg),$$

where

$$I(\kappa, \nu, \tau) := 2^{-\kappa}\tau^{-\kappa/2}\Gamma(\kappa) U\left(\frac{\kappa}{2}, \frac{1}{2}, \frac{\nu^2}{4\tau}\right) \tag{5.1}$$

for the *confluent hypergeometric U-function U*. Using the calibrated (yearly) parameters

$$C = 0.3251, \quad G = 3.7103, \quad M = 18.4460, \quad Y = 0.6029,$$

from [5, Table 1], Lemma 4.3 and Theorem 4.7 lead to the results in Figure 2.

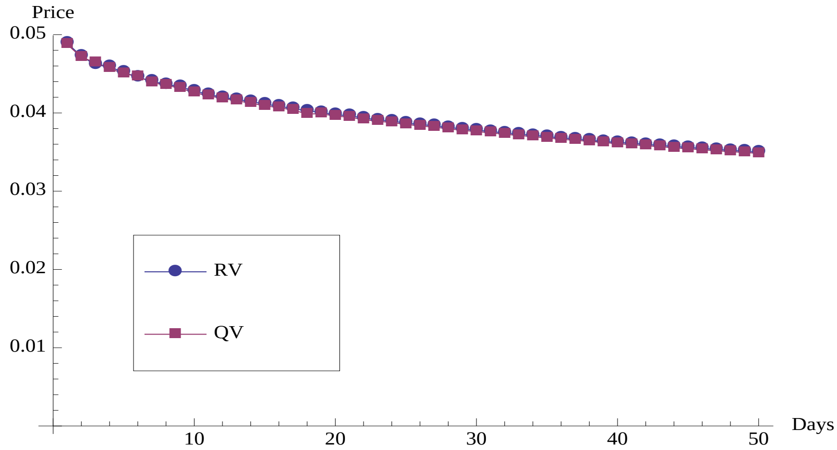

FIGURE 2. ATM call prices on normalized quadratic variation resp. realized variance in the CGMY model. The analytical short-time limits from Lemma 3.1 and 3.2 coincide at a value of 0.0511.

Evidently, the respective prices can barely be distinguished by eye. This drastically differs from the results reported for the Heston model in [3]. Also notice that the short-time limits from Lemma 3.1 and 3.2 coincide here for both prices at 0.0511, which is quite close to the values 0.0488 resp. 0.0489 for calls with maturity 1 day on quadratic variation resp. realized variance.

As a third example we consider the model of Kou, which includes both jumps and a Brownian component. Similarly as above, the Lévy exponent $\psi^{[X,X]}$ of $[X,X]$ can again be expressed in terms of the confluent hypergeometric U-function,

$$\psi^{[X,X]}(u) = \sigma^2 u + \lambda_+(\nu_+ I(1,\nu_+,-u) - 1) - \lambda_-(\nu_- I(1,\nu_-,-u) - 1),$$

with the function I from (5.1). Using the calibrated yearly parameters

$$\sigma = 0.3, \quad \lambda_+ = 0.5955, \quad \nu_+ = 16.6667, \quad \lambda_- = 3.3745, \quad \nu_- = 10,$$

from [19, Section 7.3], we obtain the results depicted in Figure 3. In line with the short-time limit results obtained in Lemmas 3.1 and 3.2, the difference is less pronounced than in the purely continuous models of Black-Scholes and Heston, but significant for maturities shorter than 10 days. For maturity 1 day, we obtain prices of 0.0706 resp. 0.0972 for calls on quadratic variation resp. realized variance. The corresponding limiting values from Lemma 3.1 resp. Theorem 3.3 are given by 0.0718 resp. 0.0980. Also note that the upper bound from Lemma 3.2 is considerably less accurate at 0.1618.

Finally, Figure 4 shows the results for the Kou model if σ is changed to 0.2 while all other parameters are kept the same. Evidently, the prices of calls on the quadratic variation are not affected. On the other hand, the discretization gap becomes much smaller in line with Theorem 3.3. More specifically, for maturity 1 day we obtain prices of 0.0706 resp. 0.0773 for calls on quadratic variation resp. realized variance. For the corresponding limiting values, we have 0.0718 resp. 0.0782, whereas the upper bound from Lemma 3.2 is given by 0.1118.

6. OUTLOOK

Both from a theoretical and a practical point of view it would be interesting to extend the results of this paper to stochastic volatility models, for example to the

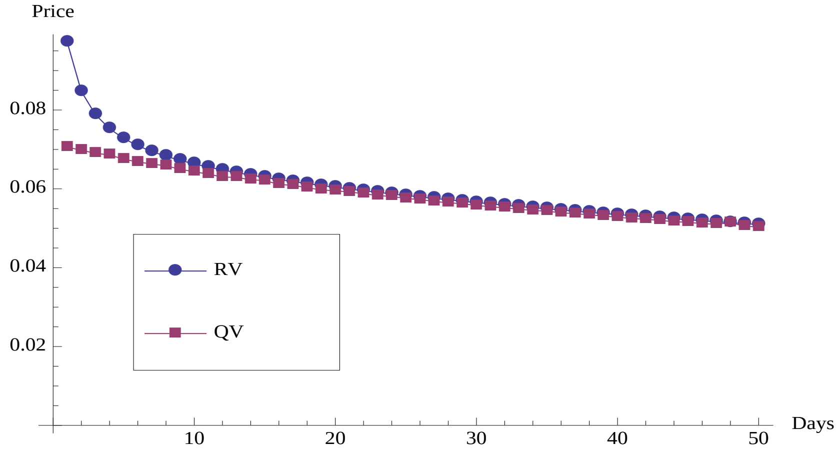

FIGURE 3. ATM call prices on normalized quadratic variation resp. realized variance in the Kou model for $\sigma = 0.3$. The analytical short-time limits from Lemma 3.1 resp. Theorem 3.3 are 0.0718 resp. 0.0980.

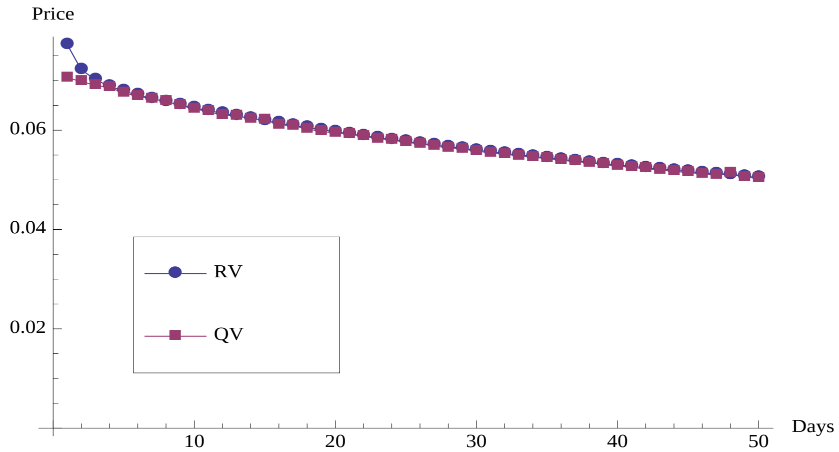

FIGURE 4. ATM call prices on normalized quadratic variation resp. realized variance in the Kou model for $\sigma = 0.2$. The analytical short-time limits from Lemma 3.1 resp. Theorem 3.3 are 0.0718 resp. 0.0782.

class of affine stochastic volatility models (see for example [13, 14]), which includes the Heston model, the SVJ and SVJJ models of [10] and most time-change based stochastic volatility models. In such an affine stochastic volatility model, the log-price X and the stochastic variance process V have a joint conditional characteristic function of the form

$$\mathbb{E}\left[e^{uX_t+wV_t}\middle|\mathcal{F}_h\right] = \exp\Big(\phi(t-h,u,w) + V_h\psi(t-h,u,w) + X_h u\Big), \tag{6.1}$$

for $u, w \in i\mathbb{R}$. Contrary to the Lévy case, independence or stationarity of increments can not be used to reduce the computation of the Laplace transform of realized variance to the Laplace transform $\mathbb{E}\left[e^{-uX_T^2}\right]$ of the squared process. However it seems possible to use a *conditional* version of the identity (4.8) in each time-step between business days, and to use the special form (6.1) of the characteristic function to convert this conditional identity into a recursive algorithm for the computation of the Laplace transform of realized variance. The delicate point is to find analyticity conditions analogous to Condition 4.4 that allow to extend the identity to the positive half-plane $\{u \in \mathbb{C} : \mathrm{Re}\,(u) \geq 0\}$. A rigorous analysis of the necessary technical conditions as well as an efficient numerical implementation for this case is therefore deferred to future research.

Appendix A. Proof of Theorem 3.3

If $\sigma^2 = 0$, the claim follows from Lemma 3.2. Now let $\sigma^2 > 0$. By the Lévy-Itô decomposition, $X_T = -\frac{\sigma^2}{2}T + \sigma W_T + L_T$ for a Brownian motion W and an independent jump process L. For

$$D_T := 1 + \frac{\sigma^2 T}{4} - \mathbb{E}\left[L_T\right] + \frac{\mathbb{E}\left[L_T^2\right]}{\sigma^2 T},$$

we have

$$\lim_{T\to 0} \frac{\mathbb{E}\left[L_T^2\right]}{\sigma^2 T} = \lim_{T\to 0} \frac{\mathrm{Var}(L_T)}{\sigma^2 T} = \frac{1}{\sigma^2}\int x^2 F(dx) =: r \geq 0\,,$$

by [7, Proposition 3.13] and thus $D_T \to (1+r) \geq 0$. Hence $D_T \geq 0$ for sufficiently small T and we can rewrite the expectation in Theorem 3.3 as

$$\mathbb{E}\left[\left(X_T^2 - \mathbb{E}\left[X_T^2\right]\right)^+\right] = \sigma^2 T\, \mathbb{E}\left[\left(\left(\frac{W_T}{\sqrt{T}} - d_+\right)\left(\frac{W_T}{\sqrt{T}} - d_-\right)\right)^+\right], \tag{A.1}$$

where

$$d_\pm := \frac{\sigma\sqrt{T}}{2} - \frac{L_T}{\sigma\sqrt{T}} \pm \sqrt{D_T}\,.$$

Writing

$$M(L_T) = \frac{L_T^2 - \mathbb{E}\left[L_T^2\right]}{\sigma^2 T} - (L_T - \mathbb{E}\left[L_T\right]),$$

we can evaluate (A.1) conditional on L and obtain

$$\begin{aligned} &\mathbb{E}\left[\left(X_T^2 - \mathbb{E}\left[X_T^2\right]\right)^+ \Big| L_T\right] \\ &\quad = \frac{\sigma^2 T}{\sqrt{2\pi}}\left(d_+ e^{-d_-^2/2} - d_- e^{-d_+^2/2}\right) + \sigma^2 T M(L_T)\left(\Phi(d_-) + \overline{\Phi}(d_+)\right), \end{aligned} \tag{A.2}$$

for sufficiently small T, where $\Phi(\cdot)$ denotes the standard normal distribution function, and $\overline{\Phi}(\cdot) = 1 - \Phi(\cdot)$ its complementary function. Thus

$$\begin{aligned} &\lim_{T\to 0}\frac{1}{T}\mathbb{E}\left[\left(X_T^2 - \mathbb{E}\left[X_T^2\right]\right)^+\right] \\ &= \frac{\sigma^2}{\sqrt{2\pi}}\lim_{T\to 0}\mathbb{E}\left[d_+ e^{-d_-^2/2} - d_- e^{-d_+^2/2}\right] + \sigma^2 \lim_{T\to 0}\mathbb{E}\left[M(L_T)\left(\Phi(d_-) + \overline{\Phi}(d_+)\right)\right]. \end{aligned} \tag{A.3}$$

Our assumption that L has Blumenthal-Getoor index $\beta < 2$ implies by [1, Theorem 3.1] that $\lim_{T\to 0}\frac{|L_T|}{\sqrt{T}} = 0$ almost surely. The first term on the right hand side of (A.3) can be dominated in absolute value by

$$C(|d_+| + |d_-|) \leq C\left(\sigma\sqrt{T} + 2\frac{|L_T|}{\sigma\sqrt{T}} + 2\sqrt{D_T}\right),$$

for some constant $C > 0$ independent of T. The upper bound converges to $2C\sqrt{1+r}$ almost surely, and, since $T \mapsto |L_T|/\sqrt{T}$ is bounded in L^2 on any compact interval, also in expectation. Thus we may apply an extension of Lebesgue's dominated convergence theorem (cf. [12, Theorem 1.21]) and see that

$$\frac{\sigma^2}{\sqrt{2\pi}} \lim_{T\to 0} \mathbb{E}\left[d_+ e^{-d_-^2/2} - d_- e^{-d_+^2/2}\right] = \sigma^2 \sqrt{\frac{2(1+r)}{\pi \exp(1+r)}} =: \sigma^2 P(r).$$

It remains to analyze the second term on the right hand side of (A.3),

$$\text{(A.4)} \quad \mathbb{E}\left[M(L_T)\left(\Phi(d_-) + \overline{\Phi}(d_+)\right)\right]$$
$$= \mathbb{E}\left[\frac{L_T^2}{\sigma^2 T}\left(\Phi(d_-) + \overline{\Phi}(d_+)\right)\right] - \mathbb{E}\left[\left(\frac{\mathbb{E}\left[L_T^2\right]}{\sigma^2 T} + L_T - \mathbb{E}\left[L_T\right]\right)\left(\Phi(d_-) + \overline{\Phi}(d_+)\right)\right]$$
$$=: A_T - B_T.$$

The term B_T can be bounded in absolute value by

$$|B_T| \le \left(\frac{\mathbb{E}\left[L_T^2\right]}{\sigma^2 T} + |L_T| + \mathbb{E}\left[|L_T|\right]\right),$$

which converges a.s. and, since $T \mapsto |L_T|$ is bounded in L^2 on any compact interval, also in expectation to r. Therefore we can use dominated convergence to obtain

$$\text{(A.5)} \qquad \lim_{T\to 0} B_T = r\left(\Phi(-\sqrt{1+r}) + \overline{\Phi}(\sqrt{1+r})\right) = 2r\Phi(-\sqrt{1+r}).$$

To evaluate A_T, let $N > 0$, $\epsilon > 0$ and let T be small enough such that $\sqrt{T} \le \epsilon/N$. Write

$$\begin{aligned} A_T &= \mathbb{E}\left[\mathbf{1}_{\{L_T \le -N\sqrt{T}\}} \frac{L_T^2}{\sigma^2 T}\left(\Phi(d_-) + \overline{\Phi}(d_+)\right)\right] \\ &\quad + \mathbb{E}\left[\mathbf{1}_{\{|L_T| < N\sqrt{T}\}} \frac{L_T^2}{\sigma^2 T}\left(\Phi(d_-) + \overline{\Phi}(d_+)\right)\right] \\ &\quad + \mathbb{E}\left[\mathbf{1}_{\{L_T \ge N\sqrt{T}\}} \frac{L_T^2}{\sigma^2 T}\left(\Phi(d_-) + \overline{\Phi}(d_+)\right)\right] \\ &=: A_T^1 + A_T^2 + A_T^3. \end{aligned}$$

The middle term can be estimated by

$$\text{(A.6)} \qquad |A_T^2| \le \frac{1}{\sigma^2 T}\mathbb{E}\left[\mathbf{1}_{\{|L_T| < \epsilon\}} L_T^2\right] \to \frac{1}{\sigma^2}\int_{-\epsilon}^{\epsilon} x^2 \, F(dx),$$

where the convergence follows from [9, Theorem 1.1]. For $A_T^1 + A_T^3$ we first derive the upper bound

$$\text{(A.7)} \qquad A_T^1 + A_T^3 \le \frac{\mathbb{E}\left[L_T^2\right]}{\sigma^2 T} \to r,$$

Next, we obtain the lower bound

$$A_T^1 \ge \mathbb{E}\left[\mathbf{1}_{\{L_T \le -N\sqrt{T}\}} \frac{L_T^2}{\sigma^2 T}\Phi(N/\sigma + \delta_T - \sqrt{1+r})\right],$$

by estimating $d_- \ge N/\sigma + \delta_T - \sqrt{1+r}$ on $\{L_T \le -N\sqrt{T}\}$, where δ_T is a deterministic quantity converging to 0 as $T \to 0$. For A_T^3 we obtain the analogous bound

$$A_T^3 \ge \mathbb{E}\left[\mathbf{1}_{\{L_T \ge N\sqrt{T}\}} \frac{L_T^2}{\sigma^2 T}\overline{\Phi}(-N/\sigma + \delta_T' + \sqrt{1+r})\right],$$

which in combination yields

$$\begin{aligned}
A_T^1 + A_T^3 \geq \frac{1}{\sigma^2 T}\mathbb{E}\left[\mathbf{1}_{\{|L_T|\geq N\sqrt{T}\}} L_T^2\right] \\
\times \min\left(\Phi(N/\sigma + \delta_T - \sqrt{1+r}), \overline{\Phi}(-N/\sigma + \delta_T' + \sqrt{1+r}\right) \\
\to r\,\Phi(N/\sigma - \sqrt{1+r}) \qquad \text{as } T \to 0.
\end{aligned} \tag{A.8}$$

Note that N can be chosen arbitrarily big, and ϵ arbitrarily small. Thus combining this estimate with (A.6) and (A.7) finally shows that $A_T \to r$. Together with (A.4) and (A.5) this proves the claim.

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