

Overview of utility-based valuation

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Abstract

We review the utility-based valuation method for pricing derivative securities in incomplete markets. In particular, we review the practical approach to the utility-based pricing by the means of computing the first order expansion of marginal utility-based prices with respect to a small number of random endowments.

Key words: utility-based prices, price corrections, risk-tolerance

1 Introduction

The valuation of derivative securities by an economic agent represents a basic problem of financial theory and practice. It is also one of the most studied problems within various models. In the framework of a complete financial model each contingent claim can be replicated by a portfolio of traded securities. Therefore, it admits a uniquely defined arbitrage-free price given as the initial wealth of such a portfolio. While complete financial models have many computational advantages, they are still only an idealistic representation (or approximation) of real financial markets, as the exact replication of options is usually not possible. Hence, the resulting arbitrage-free prices computed in these models should be used in practice rather cautiously. Indeed, assume for a moment that the illiquid contingent claims can suddenly be bought or sold at a price p^{trade} which only slightly differs from the price p computed

in a complete financial model. The naive interpretation of the price p leads the investor to take an infinite position in the contingent claims, which is, clearly, nonsense from a practical point of view.

Due to inability to replicate non-replicable derivative securities perfectly, the ownership of these derivatives bears some risk. Therefore pricing in incomplete financial markets becomes a non-trivial task. A classical approach in the economic theory is to view the valuation of derivatives as a part of the problem of optimal investment. Of course, in this case the resulting prices will depend not only on the financial market of traded assets (as in arbitrage-free valuation approach) but also on “subjective” characteristics of an economic agent such as

- His or her risk preferences, which in the classical framework of Von Neumann-Morgenstern are specified by the reference probability measure \mathbb{P} and the utility function U for consumption at maturity.
- The current portfolio (x, q) of the investor, where x is the wealth invested into liquid securities, and $q = (q_i)_{1 \leq i \leq m}$ is the vector of his holdings in the non-traded contingent claims.
- Investor’s trading volume in the derivative securities.

The first item is required, since in our framework pricing is similar to investment. The risk preferences specify the trade-off between risk and return. Regarding the last two items note that in our utility-based valuation framework the prices of non-replicable derivatives have non-linear dependence on trading quantities, that is, the price of q such securities each valued at p dollars is different from qp dollars. In contrast to arbitrage-free prices the utility based prices are not given by a single number. They represent a function depending on the trading volume and the current position of the investor.

In this paper we will give an overview of the utility-based valuation method and important properties of a risk-tolerance wealth process. Particular examples of computations of the prices in various incomplete market models can be found in the companion paper [Ger].

2 Model of financial market

Let us consider a model of a security market which consists of $d + 1$ traded or liquid assets: one zero coupon bond or a savings account with zero interest rate, and d stocks. We work in discounted terms, i.e. we suppose that the price of the bond is constant, and denote by $S = (S^i)_{1 \leq i \leq d}$ the price process

of the d stocks. Stock prices are assumed to be semimartingales on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < T}, \mathbb{P})$. Here T is a finite time horizon, and $\mathcal{F} = \mathcal{F}_T$.

A (self-financing) portfolio is defined as a pair (x, H) , where the constant x represents the initial capital and $H = (H^i)_{1 \leq i \leq d}$ is a predictable S -integrable process, where H_t^i specifies how many units of asset i are held in the portfolio at time t . The wealth process $X = (X_t)_{0 \leq t \leq T}$ of the portfolio evolves in time as the stochastic integral of H with respect to S :

$$X_t = x + \int_0^t H_u dS_u, \quad 0 \leq t \leq T. \quad (1)$$

We denote by $\mathcal{X}(x)$ the family of wealth processes with non-negative capital at any instant and with initial value equal to x :

$$\mathcal{X}(x) \triangleq \{X \geq 0 : X \text{ is defined by (1)}\}. \quad (2)$$

A non-negative wealth process is said to be *maximal* if its terminal value cannot be dominated by that of any other non-negative wealth process with the same initial value. In general, a wealth process X is called *maximal* if it admits a representation of the form

$$X = X' - X'',$$

where both X' and X'' are non-negative maximal wealth processes. A wealth process X is said to be *acceptable* if it admits a representation as above, where both X' and X'' are non-negative wealth processes and, in addition, X'' is maximal. A paper [DS97] by Delbaen and Schachermayer contains many deep results on maximal and acceptable wealth processes.

A probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an *equivalent local martingale measure* if any $X \in \mathcal{X}(1)$ is a local martingale under \mathbb{Q} . The family of equivalent local martingale measures is denoted by \mathcal{Q} . We assume throughout that

$$\mathcal{Q} \neq \emptyset. \quad (3)$$

By the First Fundamental Theorem of Asset Pricing this condition is equivalent to the absence of arbitrage opportunities in the model. A precise statement of this important result is given in the seminal papers [DS94] and [DS98] by Delbaen and Schachermayer. In particular, (3) implies that a constant positive process is maximal.

In addition to the set of traded securities we consider a family of m non-traded European contingent claims with payment functions $f = (f_i)_{1 \leq i \leq m}$, which are \mathcal{F} -measurable random variables, with maturity T . We assume that

this family is dominated by the terminal value of some non-negative wealth process X , that is

$$\|f\| \triangleq \sqrt{\sum_{i=1}^m f_i^2} \leq X_T,$$

which is also equivalent (see [DS94], Theorem 5.7) to the following integrability condition

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\|f\|] < \infty.$$

We are interested in the case when it is not possible to replicate (in some appropriate way) these securities, and therefore the following assumption is required.

Assumption 2.1 ([KS06b]). *For any $q \in \mathcal{R}^m$ such that $q \neq 0$, the random variable $\langle q, f \rangle \triangleq \sum_{i=1}^m q_i f_i$ is not replicable.*

This assumption is made only for simplicity of notation. It does not restrict generality.

Now let us consider an investor or an economic agent. The agent's preferences are specified by the utility function U for consumption at maturity. The utility function

$$U : (0, \infty) \rightarrow (-\infty, \infty),$$

is assumed to be strictly concave, strictly increasing and continuously differentiable, and to satisfy the Inada conditions:

$$U'(0) = \lim_{x \rightarrow 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0. \quad (4)$$

In addition to these standard conditions, following [KS99], it is assumed that the asymptotic elasticity of U is strictly less than 1, that is

$$\limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1. \quad (5)$$

In order to satisfy conditions of the theorems in [KS06b] the following assumption is made.

Assumption 2.2 ([KS06b]). *The utility function U is two times continuously differentiable on $(0, \infty)$ and its relative risk-aversion coefficient*

$$A(x) \triangleq -\frac{xU''(x)}{U'(x)}, \quad x > 0, \quad (6)$$

is uniformly bounded away from zero and infinity, that is, there are constants $c_1 > 0$ and $c_2 < \infty$ such that

$$c_1 < A(x) < c_2, \quad x > 0. \quad (7)$$

In fact, this assumption implies both the Inada conditions (4) and the condition on asymptotic elasticity (5). For further discussion regarding this assumption and two times differentiability of U in general, see [KS06a].

Assume that the initial portfolio of the investor has the form (x, q) , where x is the liquid capital invested in the savings account and *liquid* stocks, and the vector q represents the quantities of the *illiquid* contingent claims in the portfolio. The liquid part of the portfolio will be changing over time, while the illiquid part is fixed until maturity.

The goal of the investor is to maximize the expected utility of terminal wealth. Given the portfolio (x, q) , the quantity $u(x, q)$ that allows to distinguish between different portfolio configurations and trading strategies is called an indirect utility and is defined as

$$u(x, q) = \sup_{X \in \mathcal{X}(x, q)} \mathbb{E}[U(X_T + \langle q, f \rangle)], \quad (x, q) \in \mathcal{K}, \quad (8)$$

where $\mathcal{X}(x, q)$ is the set of acceptable processes with initial capital x whose terminal values dominate $-\langle q, f \rangle$, that is

$$\mathcal{X}(x, q) \triangleq \{X : X \text{ is acceptable, } X_0 = x \text{ and } X_T + \langle q, f \rangle \geq 0\}$$

and \mathcal{K} is the interior of the cone of points (x, q) such that the set $\mathcal{X}(x, q)$ is not empty, that is

$$\mathcal{K} \triangleq \text{int}\{(x, q) \in \mathcal{R}^{m+1} : \mathcal{X}(x, q) \neq \emptyset\}.$$

The problem of optimal investment with random endowment (8) has been carefully studied by Hugonnier and Kramkov in [HK04]. It was shown there that under the conditions of no-arbitrage (3) and the asymptotic elasticity (5) the upper bound in (8) is attained provided that $u(x, q) < \infty$.

3 Marginal-utility based prices

We are interested in the problem of evaluation of non-traded contingent claims $f = (f_i)_{1 \leq i \leq m}$. We need to attach some meaning to the price of a security that is not traded. We begin with an intuitive explanation, and

then will make it precise. Intuitively, a “price” is defined as a “threshold” $p = (p_i)_{1 \leq i \leq m}$ such that the economic agent is willing to buy the i th contingent claim at a price less than p_i , sell it at a price greater than p_i , and do nothing at p_i .

To make the above description precise we need to introduce an order relation in the space of portfolio configurations involving random endowments. In other words, given two arbitrary portfolios (x_i, q_i) , $i = 1, 2$, the investor should be able to say that (x_1, q_1) is “better” than (“worse” than, “equal” to) (x_2, q_2) . The classical approach of Financial Economics is to define the preferences of the investor with respect to the future random payoffs in terms of their expected utilities, i.e. using (8). In this case, the “quality” of a portfolio (x, q) is expressed as the maximal expected utility $u(x, q)$, which can be achieved by investing the liquid amount x in the financial market according to the optimal trading strategy. This leads us to the following definition.

Definition 3.1 ([KS06b]). *Let $(x, q) \in \mathcal{K}$ be the initial portfolio of the agent. A vector $p \in \mathcal{R}^m$ is called a marginal utility based price (for the contingent claims f) at (x, q) if*

$$u(x, q) \geq u(x', q')$$

for any $(x', q') \in \mathcal{K}$ such that

$$x + \langle q, p \rangle = x' + \langle q', p \rangle.$$

The interpretation of this definition is that the agent’s holdings q in f are optimal in the model where the contingent claims can be traded at time zero at the marginal utility based price p . In other words, given the portfolio (x, q) the investor will not trade the options at the price $p(x, q)$.

4 Davis price

When the initial portfolio of the investor does not contain contingent claims, i.e. the portfolio consists of liquid wealth x only, and $q = 0$, the marginal utility based price

$$p(x) \triangleq p(x, 0)$$

can often be computed explicitly. Note that such a price $p(x)$ specifies the direction of trade (but not the optimal trading volume!). This case was extensively studied in the literature. The common references include the paper [Rub76] by Rubinstein (in economic literature) and the papers [Dav97] by Davis and [HKS05] by Hugonnier, Kramkov and Schachermayer (in mathematical finance literature). The latter paper contains precise mathematical

conditions for the price $p(x)$ to be defined uniquely and to satisfy the key formula (14) below.

Let $u(x)$ be a short notation for the value function in the case without random endowments, that is

$$u(x) \triangleq u(x, 0) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0. \quad (9)$$

We assume that

$$u(x) < \infty \text{ for some } x > 0. \quad (10)$$

An important role in the future analysis will be played by the *marginal utility of the terminal wealth of the optimal investment strategy*, that is by the random variable $U'(\widehat{X}_T(x))$, where $\widehat{X}(x)$ is the solution to (9). Note that it is often easier to compute $U'(\widehat{X}_T(x))$, rather than the terminal wealth $\widehat{X}_T(x)$ itself.

Let $V(y)$ be the Legendre transform (the conjugate function) of the investor's utility function $U(x)$ defined as

$$V(y) \triangleq \sup_{x > 0} \{U(x) - xy\}, \quad y > 0. \quad (11)$$

Consider the following dual optimization problem:

$$v(y) \triangleq \inf_{Y \in \mathcal{Y}(y)} [V(Y_T)], \quad y > 0, \quad (12)$$

where $\mathcal{Y}(y)$ is the family of non-negative supermartingales Y such that $Y_0 = y$ and XY is a supermartingale for all $X \in \mathcal{X}(1)$. Note that $\mathcal{Y}(1)$ contains the density processes of all $\mathbb{Q} \in \mathcal{Q}$.

We remind the reader that $v(y)$ is the dual function in the case that the investor initially has only liquid wealth x (as defined in (9)). If the lower bound in (12) is attained, then the process $\widehat{Y}(y)$ attaining the infimum is called the dual minimizer. Note that in general the process $\widehat{Y}(y)/y$ is not the density process of some martingale measure $\mathbb{Q} \in \mathcal{Q}$. It was shown in [KS99] that under the conditions of no-arbitrage (3), asymptotic elasticity (5) and boundedness of value function (10) the dual minimizer exists for any $y > 0$.

Assume that the Lagrange multiplier y is dual to the initial wealth x in the sense that

$$y = u'(x) \quad (\text{or, equivalently, } x = -v'(y))$$

and that

$$\frac{\widehat{Y}_T(y)}{y} = \frac{d\widehat{\mathbb{Q}}(y)}{d\mathbb{P}} \quad (13)$$

for some $\widehat{\mathbb{Q}}(y) \in \mathcal{Q}$. In this case the marginal utility price $p(x)$ is given by the following risk neutral evaluation formula

$$p(x) = \mathbb{E}_{\widehat{\mathbb{Q}}(y)}[f]. \quad (14)$$

It was shown in [HKS05] (under the additional condition that S is locally bounded) that the condition (13) is necessary and sufficient for the marginal utility based price $p(x)$ to be uniquely defined for *any bounded* contingent claim f .

5 Sensitivity analysis and risk-tolerance wealth processes

Assume now that the investor can trade contingent claims at the initial time at a price p^{trade} . A very important question from the practical point of view is what quantity $q = q(p^{trade})$ the investor should trade at the price p^{trade} . If the initial portfolio of the economic agent consists exclusively of liquid wealth x , the optimal (static) position $q(p^{trade})$ in the illiquid contingent claims can be computed (at least intuitively) using marginal utility based prices $p(x, q)$ from the following “equilibrium” condition:

$$p^{trade} = p(x - \langle p^{trade}, q(p^{trade}) \rangle, q(p^{trade})). \quad (15)$$

This equation has a natural economic interpretation:

1. In order to acquire q stocks at the price p the investor needs to spend the cash amount

$$\langle p, q \rangle \triangleq \sum_{i=1}^m p_i q_i.$$

2. The position (x, q) is optimal given an opportunity to trade derivatives at p if and only if $p = p(x, q)$.

The practical use of (15) is rather limited. In the literature there are almost no explicit computations of $p(x, q)$ with q different from zero. As an exception we refer to the papers [MZ04] by Musiela and Zariphopoulou and [Hen02] by Henderson where some explicit computations are done for the case of exponential utilities.

If it is not possible to compute the price $p(x, q)$ explicitly, one may try to compute a linear approximation of the price. That is a linear expansion of the first order for “small” values of Δx and q .

$$p(x + \Delta x, q) = p(x) + p'(x)\Delta x + D(x)q + o(|\Delta x| + \|q\|). \quad (16)$$

Here $p'(x) = (p'_i(x))_{1 \leq i \leq m}$ is an m -dimensional vector and $D_{ij}(x) = \frac{\partial p_i}{\partial q_j}(x, 0)$, is an $m \times m$ matrix with $1 \leq i, j \leq m$.

The detailed analysis of the linear approximation (16) is given in the paper [KS06b] by Kramkov and Sîrbu. They compute the sensitivity parameters $p'(x)$ and $D(x)$ under very general (essentially minimal) assumptions. In addition to the natural *quantitative* problem of the computation of the vector $p'(x)$ and the matrix $D(x)$ for any family of contingent claims f , Kramkov and Sîrbu also study the important questions of the *qualitative nature* such as

1. Are the marginal utility based prices computed at $q = 0$ locally independent of the initial capital, that is, does

$$p'(x) = 0$$

hold true?

This is an important property since, for example, the price in Black and Scholes model does not depend on the initial wealth of an economic agent.

2. Does the sensitivity matrix $D(x)$ have full rank, that is, does

$$D(x)q = 0 \text{ if and only if } q = 0, \quad q \in \mathcal{R}^m,$$

hold true?

To illustrate this property consider a model with only one contingent claim f and $D(x) = 0$. Then the linear expansion of first order does not show any dependence of price on quantity q . At least second order expansion is required to see the dependence.

3. Is the sensitivity matrix $D(x)$ symmetric?

Consider two claims f_1 and f_2 with the corresponding Davis prices $p_1(x)$ and $p_2(x)$ respectively. Suppose it is possible to trade *only* in the claim f_1 at the price p_1^{trade} , such that $p_1(x) > p_1^{trade}$. Then the agent would buy f_1 . Now, suppose that it is possible to trade *only* in the claim f_2 at the price p_2^{trade} , such that $p_2(x) > p_2^{trade}$. Similarly, the agent would buy f_2 . Assume now that it is possible to trade both securities *simultaneously*. In this case it is not necessarily true that the agent should buy f_1 and buy f_2 . Here is an example. Suppose $f_1 = c + f_2$ for some constant c . If at some moment $p_1^{trade} - p_2^{trade} < c$, the agent would buy f_1 and sell f_2 .

It is very desirable, therefore, to be able to work with different groups of financial markets independently, i.e., to decompose a multidimensional problem into a sequence of one-dimensional problems. In other words, it is important to be able to find a family of contingent claims $h = (h)_{1 \leq i \leq m}$, spanning the same space as the contingent claims $f = (f)_{1 \leq i \leq m}$ and such that, for the contingent claims h , a change in the traded price of h_i will only determine the agent to take a position in the i -th claim alone. If the sensitivity matrix for h is diagonal, then it is possible to decompose a multidimensional problem into a sequence of one-dimensional problems. Therefore we can work with different groups of financial markets independently in the first order if and only if the matrix $D(x)$ is symmetric (and therefore can be diagonalized.)

4. Is the sensitivity matrix $D(x)$ negative semi-definite, that is, does

$$\langle q, D(x)q \rangle \leq 0, \quad q \in \mathcal{R}^m,$$

hold true?

In the case of one contingent claim, this property simply means that $q(x)$ has to have the same sign as $p(x) - p^{trade}$, which is again related to the correct direction of trade.

5. Is the linear approximation stable? Is it true that for any p^{trade} the linear approximation

$$p^{trade} \approx p(x) - \langle p^{trade}, q(x) \rangle p'(x) + D(x)q$$

of the “equilibrium” equation

$$p^{trade} = p(x - \langle p^{trade}, q(x) \rangle, q(x))$$

has the “correct” solution? For example, in one-dimensional case will we have the property that the solution $q(x)$ of the approximation equation is positive if and only if p^{trade} is greater than $p(x)$?

It is very interesting that there is one single property of financial markets that is responsible for the positive answer to all of the above questions.

Definition 5.1 ([KS06b]). *Let $x > 0$ and denote by $\widehat{X}(x)$ the solution to (9). The process $R(x)$ is called the risk-tolerance wealth process if it is maximal and*

$$R_T(x) = -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))}. \quad (17)$$

In other words, $R(x)$ is the replication process for the random payoff defined in the right hand side of (17). Since we are in the framework of incomplete markets, it is either possible to replicate this random payoff, or not. The key message of [KS06b], see Theorems 8 and 9, is that the financial models where the risk-tolerance wealth process exist are exactly the models with “good” qualitative properties of the first order expansion (16) for an *arbitrary* family of contingent claims f .

6 Properties of risk-tolerance wealth processes

Provided that the risk-tolerance wealth process exists, its initial value is

$$R_0(x) = -\frac{u'(x)}{u''(x)}. \quad (18)$$

Note that the expression on the right-hand side is the risk-tolerance of the economic agent at initial time. This quantity can be extracted in practice from the current mean-variance preferences of the investor.

Regarding the evolution of the risk-tolerance wealth process over time we note that by Theorem 4 in [KS06b]

$$\frac{R(x)}{R_0(x)} = \widehat{X}'(x) \triangleq \lim_{\Delta x \rightarrow 0} \frac{\widehat{X}(x + \Delta x) - \widehat{X}(x)}{\Delta x}. \quad (19)$$

The intuitive understanding of the above formula is the following. Our economic agent has initial wealth x and invests it according to optimal investment strategy $\widehat{X}(x)$ (optimal in the sense that $\widehat{X}(x)$ is the solution of (9)). If this investor is given a small additional cash amount Δx , then the investor will use it according to the investment strategy $\widehat{X}(x + \Delta x) - \widehat{X}(x)$, which for small Δx is proportional to $R(x)$ by (19). That is, $R(x)$ is the answer to a simple question: ”If you have an extra dollar, how would you invest it?” In practice, every investor/bank/mutual fund should be able to answer this question.

The following heuristic argument explains the formulas (18) and (19). Assume that the investor receives a small additional cash amount Δx . Denote by Φ the terminal wealth of the strategy used for the investment per dollar of Δx . As the initial capital of the strategy is 1 we have that

$$\mathbb{E}_{\widehat{\mathbb{Q}}(y)}[\Phi] = 1, \quad (20)$$

where $\widehat{\mathbb{Q}}(y)$ is the martingale measure defined in (13) and $y = u'(x)$. We want to choose the asset Φ independently of Δx so that

$$\mathbb{E}[U(\widehat{X}_T(x) + \Delta x \Phi)] = u(x + \Delta x) + o((\Delta x)^2). \quad (21)$$

The Taylor expansion of the left hand side of (21) gives

$$\begin{aligned} & \mathbb{E}[U(\widehat{X}_T(x) + \Delta x \Phi)] \\ &= \mathbb{E}[U(\widehat{X}_T(x))] + \mathbb{E}[U'(\widehat{X}_T(x))\Delta x \Phi] + \frac{1}{2}\mathbb{E}[U''(\widehat{X}_T(x))(\Delta x)^2 \Phi^2] + o((\Delta x)^2) \\ &= u(x) + \Delta x \mathbb{E}\left[y \frac{d\widehat{\mathbb{Q}}(y)}{d\mathbb{P}} \Phi\right] \\ &\quad + \frac{1}{2}(\Delta x)^2 \mathbb{E}\left[\frac{U''(\widehat{X}_T(x))}{U'(\widehat{X}_T(x))} y \frac{d\widehat{\mathbb{Q}}(y)}{d\mathbb{P}} \Phi^2\right] + o((\Delta x)^2) \\ &= u(x) + u'(x)\Delta x - \frac{1}{2}(\Delta x)^2 u'(x) \mathbb{E}_{\widehat{\mathbb{Q}}(y)}\left[\frac{\Phi^2}{R_T(x)}\right] + o((\Delta x)^2). \end{aligned}$$

Hence, we need to select Φ so that

$$\mathbb{E}_{\widehat{\mathbb{Q}}(y)}\left[\frac{\Phi^2}{R_T(x)}\right] \rightarrow \min.$$

A quick way to show that the lower bound above is attained at

$$\Phi = \frac{R_T(x)}{R_0(x)} \quad (22)$$

is by using the change of numéraire technique. We select the risk-tolerance wealth process $R(x)$ as the new numéraire and denote by $\mathbb{Q}^R(x)$ the corresponding martingale measure:

$$\frac{d\mathbb{Q}^R(x)}{d\widehat{\mathbb{Q}}(y)} = \frac{R_T(x)}{R_0(x)}. \quad (23)$$

We have

$$\mathbb{E}_{\widehat{\mathbb{Q}}(y)}\left[\frac{\Phi^2}{R_T(x)}\right] = \frac{1}{R_0(x)} \mathbb{E}_{\mathbb{Q}^R(x)}\left[\left(\frac{\Phi R_0(x)}{R_T(x)}\right)^2\right] \leq \frac{1}{R_0(x)}, \quad (24)$$

where in the last step we used the Cauchy inequality. On the other hand,

$$\mathbb{E}_{\widehat{\mathbb{Q}}(y)}\left[\frac{\left(\frac{R_T(x)}{R_0(x)}\right)^2}{R_T(x)}\right] = \frac{1}{R_0(x)} \mathbb{E}_{\widehat{\mathbb{Q}}(y)}\left[\frac{R_T(x)}{R_0(x)}\right] = \frac{1}{R_0(x)}. \quad (25)$$

Therefore by comparing (24) and (25) we obtain

$$\mathbb{E}_{\hat{\mathbb{Q}}(y)} \left[\frac{\Phi^2}{R_T(x)} \right] \leq \mathbb{E}_{\hat{\mathbb{Q}}(y)} \left[\frac{\left(\frac{R_T(x)}{R_0(x)} \right)^2}{R_T(x)} \right]$$

for any Φ satisfying (20). This proves the optimality of the choice (22) for Φ and, hence, the formula (19) for $R(x)$.

Once the optimality of (22) has been established we can compare the second order terms in the quadratic expansions of the left and right parts of (21). Direct computations show that this leads to (18).

7 Existence of risk-tolerance wealth process

The incomplete markets that we consider consist of two main ingredients: the investor's utility function, and a set of traded securities on a filtered probability space. Theorem 8 in [KS06b] gives the equivalence conditions for existence of $R(x)$ and the required qualitative properties of the sensitivity matrix $D(x)$. It is also interesting to know whether it is possible to relax the market model and still get the required properties of the matrix $D(x)$. Theorems 6 and 7 in [KS06b] give the answer to this question.

Let us recall that a market model is called complete if it is arbitrage-free and every bounded non-negative contingent claim is replicable. The market is complete if and only if the family of equivalent probability measures contains only one element $\mathbb{Q} \sim \mathbb{P}$. It can be shown that if the financial model is complete, then the risk-tolerance wealth process is well-defined for any utility function U such that (10) and Assumption 2.2 hold true (in a complete market every claim is replicable).

If the model is arbitrage-free but is incomplete then the family of equivalent martingale measures \mathcal{Q} contains an infinite number of elements.

Definition 7.1. *Let f and g be non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. f is said to second order stochastically dominate g ($f \succeq_2 g$) if*

$$\int_0^t \mathbb{P}(f \geq x) dx \geq \int_0^t \mathbb{P}(g \geq x) dx, \quad t \geq 0.$$

Remark 7.1. It is well-known that $f \succeq_2 g$ if and only if

$$\mathbb{E}[\phi(f)] \leq \mathbb{E}[\phi(g)]$$

for any function $\phi = \phi(x)$ on $[0, \infty)$ that is convex, decreasing, and such that the expected values above are well-defined.

Definition 7.2. Let \mathcal{Q} be the family of measures equivalent to \mathbb{P} . The probability measure $\widehat{\mathbb{Q}} \in \mathcal{Q}$ is called the universal minimal martingale measure if its Radon-Nikodym derivative dominates the Radon-Nikodym derivatives of other elements of \mathcal{Q} in the sense of the second order stochastic dominance, that is,

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \succeq_2 \frac{d\mathbb{Q}}{d\mathbb{P}} \quad \text{for all } \mathbb{Q} \in \mathcal{Q}. \quad (26)$$

Theorem 6 in [KS06b] asserts that if there exists the universal minimal martingale measure $\widehat{\mathbb{Q}} \in \mathcal{Q}$, then for *any* utility function U satisfying (10) and Assumption 2.2 the process $R(x)$ is well-defined (qualitative properties of $p(x)$ hold true) and vice versa. This also explains the wording of the Definition 7.2, since $\widehat{\mathbb{Q}}$ solves the dual problem of optimal investment (12) for any utility function U and any initial wealth x .

A complimentary result is stated in Theorem 7 in [KS06b]. If an arbitrary financial model is considered, then the *only* utility functions allowing the existence of the risk-tolerance wealth process are power utilities and exponential utilities.

8 Computation of $D(x)$

Remember, that the described theory was developed in order to compute the first order correction to the price $p(x)$ due to the presence of non-replicable assets in the portfolio. In addition to establishing an equivalence between the existence of the process $R(x)$ and the desired properties of the matrix $D(x)$, Theorem 8 in [KS06b] provides the necessary machinery required to compute the first order correction to the price $p(x)$.

Hereafter we assume that the risk-tolerance wealth process $R(x)$ is well-defined. We choose $\frac{R(x)}{R_0(x)} = \widehat{X}'(x)$ to be a new numéraire. Let

$$f^R = \frac{f R_0(x)}{R_T(x)}$$

be the discounted payoffs of the contingent claims and for any wealth process X

$$X^R = \frac{X R_0(x)}{R(x)}.$$

be its wealth expressed in terms of the numéraire $\frac{R(x)}{R_0(x)}$.

Let $\mathbb{Q}^R(x)$ be the probability measure on (Ω, \mathcal{F}) whose Radon-Nikodym derivative under \mathbb{P} is given by

$$\frac{d\mathbb{Q}^R(x)}{d\mathbb{P}} = \frac{R_T(x)\widehat{Y}_T(y)}{R_0(x)y}, \quad (27)$$

where $y = u'(x)$, $\widehat{X}(x)$ is the solution to (9) and $\widehat{Y}(y)$ is the solution to (12). Note that the process $\frac{R_t(x)\widehat{Y}_t(y)}{R_0(x)y}$ is a positive uniformly integrable martingale starting at 1, and therefore it defines a density process for the probability measure $\mathbb{Q}^R(x)$. See Theorem 2.2 in [KS99] for details. Note that X^R is a supermartingale under this measure. Under this measure the price process of the contingent claim f^R expressed in the number of units R/R_0 is

$$P_t^R = \mathbb{E}_{\mathbb{Q}^R(x)}[f^R \mid \mathcal{F}_t].$$

Consider now the Kunita-Watanabe orthogonal decomposition of the price process P^R under $\mathbb{Q}^R(x)$

$$P_t^R = M_t + N_t, \quad N_0 = 0. \quad (28)$$

One can think of this decomposition in the following way. The process M is an $R(x)/R_0(x)$ -discounted wealth process, which represents the hedging process. The process N is a martingale under $\mathbb{Q}^R(x)$, which is orthogonal to all $R(x)/R_0(x)$ -discounted wealth processes. Therefore N is the “risk process” – the part of the price process P^R that cannot be hedged due to the incompleteness of the market. Now, Theorem 8 in [KS06b] gives the explicit form of the price correction:

$$D^{ij}(x) = \frac{u''(x)}{u'(x)} \mathbb{E}_{\mathbb{Q}^R(x)}[N_T^i N_T^j], \quad 1 \leq i, j \leq m, \quad (29)$$

where N is defined in (28).

9 Further assumptions

For completeness we have to mention that Theorem 8 of [KS06b] holds true under the following technical assumptions.

Following [KS06a] we call a d -dimensional semimartingale R *sigma-bounded* if there is a strictly positive predictable (one-dimensional) process h such that the stochastic integral $\int h dR$ is well-defined and is locally bounded.

Assumption 9.1 ([KS06b]). *The price process of the traded securities discounted by the solution $\widehat{X}(x)$ to (9), that is the $d+1$ -dimensional semimartingale*

$$S^{\widehat{X}(x)} \triangleq \left(\frac{1}{\widehat{X}(x)}, \frac{S}{\widehat{X}(x)} \right), \quad (30)$$

is sigma-bounded.

We refer to [KS06a], Theorem 3 for sufficient conditions that ensure the validity of this assumption. In particular, this assumption is satisfied if S is a continuous process, or if the original (incomplete) model can be extended to a complete one by adding a finite number of securities.

To facilitate the formulation of the assumptions on the random endowments f , we change the numéraire from the bond to the normalized optimal wealth process $\widehat{X}(x)/x$ and denote by

$$g_i(x) \triangleq x \frac{f_i}{\widehat{X}_T(x)}, \quad 1 \leq i \leq m,$$

the payoffs of the European options discounted by $\widehat{X}(x)$.

Let $\widetilde{\mathbb{Q}}(x)$ be a probability measure equivalent to \mathbb{P} such that

$$\frac{d\widetilde{\mathbb{Q}}(x)}{d\mathbb{P}} \triangleq \frac{\widehat{X}(x)\widehat{Y}(x)}{xy}, \quad y = u'(x), \quad (31)$$

and let $\mathbf{H}_0^2(\widetilde{\mathbb{Q}}(x))$ be the space of square integrable martingales with initial value 0 under the measure $\widetilde{\mathbb{Q}}(x)$ defined in (31). Denote

$$\mathcal{M}^2(x) \triangleq \left\{ M \in \mathbf{H}_0^2(\widetilde{\mathbb{Q}}(x)) : M = \int H dS^{\widehat{X}(x)} \right\},$$

where $S^{\widehat{X}(x)}$ was defined in (30).

Assumption 9.2 ([KS06b]). *There is a constant $c > 0$ and a process $M \in \mathcal{M}^2(x)$ such that*

$$\sum_{i=1}^m |g_i(x)| \leq c + M_T.$$

10 Remark on implementation in practice

Let us consider an investor who is already trading on the market and uses his proprietary model to find the risk-neutral pricing probability measure $\widehat{\mathbb{Q}}$ and as the result he can compute the price $p(x)$. However, the investor's pricing model is linear in the size of trade. The advantage of the previously described theory is that it makes it possible to compute the price correction to the *already computed* linear price $p(x)$ without changing anything and with a rather minimal effort. The ingredients that are required are:

- The investor's risk-neutral probability measure $\widehat{\mathbb{Q}}$, which is already implemented by the investor.
- The investor's relative risk aversion coefficient $\alpha(x) \triangleq -x \frac{u''(x)}{u'(x)}$, which can be deduced from the mean-variance preferences.
- $R(x)/R_0(x)$, the investor's decision how to spend an "extra dollar".

Once we have these three ingredients, we can compute the matrix $D(x)$ using formula (29). Examples of such computations can be found in [Ger].

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