A proof of a conjecture in the Cramér-Lundberg model with investments

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Abstract

In this paper, we discuss the Cramér-Lundberg model with investments, where the price of the invested risk asset follows a geometric Brownian motion with drift a and volatility $\sigma > 0$. By assuming there is a cap on the claim sizes, we prove that the probability of ruin has at least an algebraic decay rate if $2a/\sigma^2 > 1$. More importantly, without this assumption, we show that the probability of ruin is certain for all initial capital u, if $2a/\sigma^2 \leq 1$.

1 Introduction

In the classical Cramér-Lundberg model, if the claim sizes have finite exponential moments, then it is well-known that the ruin probability decays exponentially as the initial surplus increases; see for instance the books by Asmussen [1] and Embrechts et. al [2]. For the case of heavy-tailed claims there also exists numerous results in the literature. However, when the insurance company invests in a risky asset, for example a stock, whose price is described by a geometric Brownian motion with drift $a > 0$ and volatility $\sigma > 0$, then the probability of ruin either decays algebraically as the initial surplus increases or the ruin is certain, provided the claim size is exponentially distributed. This result was shown by Frolova et. al [3]. Under the assumption that the claim size distributions have moment generating functions defined on a neighborhood of the origin, Constantinescu and Thommann [4] proved that if the probability of ruin decays as the initial capital $u \to \infty$, then $\rho = \frac{2a}{\sigma^2} > 1$, and that if $1 < \rho < 2$, then the probability of ruin decays algebraically as the initial capital $u \to \infty$. Furthermore, they conjectured that if $\rho \leq 1$, then the ruin probability $\psi(u) = 1$ for all $u \geq 0$.

In this paper, our main goal is to prove that the conjecture is true. This work was motivated by a paradox of risk without the possibility of reward discussed by Steele [5]. In the setting of this paradox of risk, the price of a risky asset is modeled by a geometric Brownian motion with an expected return rate a. Steele pointed out that if $\rho < 1$, the price of the risky asset approaches to zero with probability one, despite the fact that the expected value goes to positive infinity at an exponential rate. We observe that if the price of our risky asset is very close to zero, then even a small jump will trigger the ruin. Similarly, if the price of the risky asset drops below a threshold with probability one and if there is a positive probability that the price of the risky asset may have jumps larger than the threshold, then the ruin occurs almost surely. If the jump is modeled by a compound Poisson process, then this leads to the conjecture that is discussed in this paper.

We first recall the Cramér-Lundberg model with investments. The risk process is given by

$$X_t = X_0 + \int_0^t aX_s dt + \int_0^t \sigma X_s dW_s + ct - \sum_{j=1}^{N(t)} \xi_j, \tag{1.1}$$

or

$$dX_t = (aX_t + c)dt + \sigma X_t dW_t - dP_t, \tag{1.2}$$

where W_t is the Wiener process (standard Brownian motion), $N(t)$ is a Poisson process with parameter λ, and the claim sizes ξ_i; $i = 1, 2, 3, ...$, are independent, identically distributed random variables, having the density function $p(x)$, with positive mean μ and finite variance. c is the fixed rate of premium and X_0 is the initial capital. $P_t = \sum_{j=1}^{N(t)} \xi_j$. The capital X_t is continuously invested in a risky asset, with relative price increments $dX_t = aX_t dt + \sigma X_t dW_t$, where $a > 0$ and $\sigma > 0$ are the drift and volatility of the returns of the asset.

Our paper is organized as follows. By assuming there is a cap on the claim size, in Section 2, we prove two important results that (1) the probability of ruin has at least an algebraic decay rate if $2a/\sigma^2 > 1$ and (2) the price of the risky asset will drop below a threshold with probability one for all initial capital $X_0 = u$, if $2a/\sigma^2 \leq 1$. In Section 3, we prove that the conjecture is true by coupling the stochastic processes with and without the assumption on the claim sizes.

2 Ruin Probability With A Cap On the Claim Size

We will assume the claim size is bounded by a constant $M > 0$ through the entire section. In insurance, M can be understood as the limit or cap of a policy. Let $T_{u^*} = \inf\{t > 0;\ X_t < u^*\}$ be the first time that $X_t < u^*$, and let

$$\psi_{u^*}(u) = P(T_{u^*} < \infty\, |X_0 = u)$$

be the probability of ruin, where $0 \leq u^* < u$. If $u^* = 0$, we denote the probability of ruin by $\psi(u)$. We will discuss the probability of ruin on the Cramér-Lundberg model with investments based on (1) $\rho > 1$, (2) $\rho = 1$ and (3) $\rho < 1$. We first prove the following

Lemma 2.1. *Let X_t be a stochastic process that satisfies (1.2), if $0 \leq v \leq u$. then*

$$\psi(v) \geq \psi(u).$$

Proof. We first derive a strong solution for (1.2). Let $Y_t = \exp\{(\frac{\sigma^2}{2} - a)t - \sigma W_t\}$. By Itô's formula [6], $dX_tY_t = X_t dY_t + Y_t dX_t + dX_t dY_t$, and simple calculation yields $dX_tY_t = dV_t^u$, where $V_t^u = u + c\int_0^t Y_s\, ds - \int_0^t Y_s\, dP_s$. Integrating both sides, we have $X_tY_t = V_t^u$. Hence

$$X_t = Y_t^{-1} V_t^u \tag{2.3}$$

is a strong solution of (1.1) and (1.2) with initial condition $X_0 = u$.

Next we define $Z_t = Y_t^{-1} V_t^v$, then $Z_t \leq X_t$, $\forall\, t \geq 0$, since $0 \leq v \leq u$. Hence

$$\begin{aligned}\psi(u) &= P(X_t < 0,\ \text{for some}\ \ 0 < t < \infty\, |X_0 = u)\\ &\leq P(Z_t < 0,\ \text{for some}\ \ 0 < t < \infty\, |Z_0 = v).\end{aligned}$$

Note that Z_t also satisfies (1.2) with initial condition $Z_0 = v$. Hence

$$P(Z_t < 0, \text{ for some } 0 < t < \infty \,|Z_0 = v) = \psi(v).$$

Therefore

$$\psi(v) \geq \psi(u).$$

Our main tool is Itô's formula for semimartingales with a jump part. Let $t_1 < t_2 < t_3 < ...$ be the times where the Poisson process $N(t)$ has a jump discontinuity. Then the jump discontinuities for P_t are also at t_i with jump size ξ_i. Following the notations on P. 43 [6], for $t > 0$, and a Borel set U in R, we let

$$N_p((0,t] \times U) = \sharp\{i; t_i \leq t, \xi_i \in U\}.$$

Then $N_p((0,t] \times U)$ defines a random measure $N_p(dtdx)$ on the Borel σ-algebra on $[0,\infty) \times R$. Note that

$$N_p(dtdx) = \sum_{i=1}^{\infty} \delta_{t_i}(dt)\delta_{\xi_i}(dx), \tag{2.4}$$

where δ_{t_i} is the Dirac δ-function centered at t_i (probability measure concentrated at one point t_i). It follows that

$$\int_0^t \int_0^\infty f(s,x) N_p(dsdx) = \sum_{i;t_i \leq t} f(t_i, \xi_i), \tag{2.5}$$

and therefore

$$\int_0^t \int_0^\infty x N_p(dsdx) = \sum_{i;t_i \leq t} \xi_i = P_t. \tag{2.6}$$

It is well-known, see e.g. P. 60 and P. 65 [6], that there exists a continuous process $\hat{N}_p((0,t] \times U)$ such that

$$\tilde{N}_p((0,t] \times U) = N_p((0,t] \times U) - \hat{N}_p((0,t] \times U), \tag{2.7}$$

is a martingale. In our case

$$\hat{N}_p((0,t] \times U) = E[N_p((0,t] \times U)].$$

$E[N_p((0,t] \times U)]$ defines a measure, $n_p(dtdx)$, called the mean (intensity) measure of $N_p(dtdx)$ and it is given by $n_p(dtdx) = \lambda p(x) dtdx$.

The equation (1.1) can be rewritten as

$$X_t = X_0 + \int_0^t aX_s dt + \int_0^t \sigma X_s dW_s + ct - \int_0^t \int_0^\infty x N_p(dsdx). \tag{2.8}$$

By (2.3), the equation (2.8) has a strong solution for each fixed initial condition and it is a semimartingale by Definition 4.1, P. 64 [6].

By (2.3) and direct calculation, we have

$$X_{t+s} = \bar{Y}_t^{-1} X_s + \bar{Y}_t^{-1} \int_0^t c\bar{Y}_u du - \bar{Y}_t^{-1} \int_0^t \bar{Y}_u d\bar{P}_u, \tag{2.9}$$

where

$$\bar{Y}_t = e^{-(a-\frac{\sigma^2}{2})t - \sigma \bar{W}_t}, \tag{2.10}$$
$$\bar{W}_t = W_{t+s} - W_s, \tag{2.11}$$
$$\bar{P}_t = P_{t+s} - P_s. \tag{2.12}$$

Note that $\bar{W}_t$ and $\bar{P}_t$ are independent of $\{X_v; 0 \leq v \leq s\}$ and therefore given $\{X_v; 0 \leq v \leq s\}$, X_{t+s} depends on X_s only. This implies that X_t is a Markov process. Moreover, since $\bar{W}_t = W_{t+s} - W_s$ and W_t have the same distribution, and $\bar{P}_t = P_{t+s} - P_s$ and P_t have the same distribution, we have

$$P(X_{t+s} \in U | X_s = x) = P(X_t \in U | X_0 = x), \tag{2.13}$$

for all $t > 0$, and all Borel sets U. Therefore, $X_t, t \geq 0$ is a Markov process with a stationary transition function. Since the sample paths of X_t are right continuous with left limits, $X_t, t \geq 0$ is a strong Markov process.

Theorem 2.1. *Consider the model given by (1.1) and assume that $\sigma > 0$, $\rho > 1$ and $c > \lambda\mu$. Then*

$$\psi(u) \leq \left(\frac{M}{u}\right)^{\rho-1} \quad \forall\, u \geq M.$$

Remark 2.1. *This theorem shows that the probability of ruin has at least an algebraic decay rate if $2a/\sigma^2 > 1$. In fact, we obtain a slightly stronger result in the proof below:*

$$\psi_M(u) \leq \left(\frac{M}{u}\right)^{\rho-1} \quad \forall\, u \geq M.$$

Proof. Let $F(x) = x^{1-\rho}, x > 0$. Applying Itô's formula [6], we have

$$\begin{aligned} F(X_t) - F(X_0) = &\int_0^t (1-\rho)(X_s)^{-\rho}(aX_s + c)\, ds + \int_0^t (1-\rho)(X_s)^{-\rho}\sigma X_s dW_s \\ &+ \frac{1}{2}\int_0^t (1-\rho)(-\rho)(X_s)^{-\rho-1}\sigma^2 {X_s}^2\, ds \\ &+ \int_0^{t^+}\int_0^M (X_{s^-} - x)^{1-\rho} - (X_{s^-})^{1-\rho}\, N_p(dsdx), \end{aligned}$$

Hence

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t (1-\rho)(X_s)^{-\rho}(aX_s + c)\, ds + \text{ mart.} \\ &+ \frac{1}{2}\int_0^t (1-\rho)(-\rho)(X_s)^{-\rho-1}\sigma^2 {X_s}^2\, ds \\ &+ \int_0^{t^+}\int_0^M (X_{s^-} - x)^{1-\rho} - (X_{s^-})^{1-\rho}\, \tilde{N}_p(dsdx) \\ &\leq F(X_0) + \text{ mart. } + c(1-\rho)\int_0^t (X_s)^{-\rho} ds \\ &+ \int_0^{t^+}\int_0^M (1-\rho)(X_{s^-})^{-\rho}(-x)\lambda p(x) dx ds, \qquad (2.14) \\ &= F(X_0) + \text{ mart. } + (1-\rho)(c - \lambda\mu)\int_0^t (X_s)^{-\rho} ds, \qquad (2.15) \end{aligned}$$

here, and through-out this paper, mart. denotes a martingale at time t. The inequality (2.14) holds because

$$(X_{s^-} - x)^{1-\rho} - (X_{s^-})^{1-\rho} \leq (1-\rho)(X_{s^-})^{-\rho}(-x),\ \forall X_{s^-} \geq M.$$

Now we consider the process X_t on $[M, n)$, where n is an integer ($> M$). Let

$$\tau_n = \inf\{t > 0 :\ X_t \notin [M, n)\}$$

be the first exit time from the interval $[M, n)$. By the Optional Stopping Theorem, it follows that

$$E[F(X_{\tau_n})] \leq E[F(X_0)]. \qquad (2.16)$$

Since $\xi_j > 0$ for all $j = 1, 2, \ldots$, we have $X_{\tau_n} = n$ or $X_{\tau_n} < M$. Moreover, since $F(x)$ is decreasing, we have

$$E[F(X_{\tau_n})] \geq \frac{1}{M^{\rho-1}} P(X_{\tau_n} < M \,|X_0 = u) + \frac{1}{n^{\rho-1}} P(X_{\tau_n} = n \,|X_0 = u).$$

Hence

$$\frac{1}{M^{\rho-1}} P(X_{\tau_n} < M \,|X_0 = u) + \frac{1}{n^{\rho-1}} P(X_{\tau_n} = n \,|X_0 = u) \leq \frac{1}{u^{\rho-1}} \,.$$

Therefore

$$P(X_{\tau_n} < M \,|X_0 = u) \leq \left(\frac{M}{u}\right)^{\rho-1}.$$

Let n go to infinity, we have

$$\psi_M(u) \leq \left(\frac{M}{u}\right)^{\rho-1}.$$

Since $\psi(u) \leq \psi_M(u)$, we have

$$\psi(u) \leq \left(\frac{M}{u}\right)^{\rho-1} \quad \forall\, u \geq M.$$

The cases for $\rho < 1$ and $\rho = 1$ follow from the next two lemmas.

Lemma 2.2. *Consider the model given by (1.1) and assume that $\sigma > 0$ and $\rho < 1$. Then there exists $u^* > M$, such that*

$$\psi_{u^*}(u) = 1, \;\; \forall\, u \geq u^*.$$

Proof. Let $F(x) = x^\alpha, x > M$, where $0 < \alpha < 1 - \rho$. Applying Itô's formula, we have

$$\begin{aligned} F(X_t) - F(X_0) &= \int_0^t \alpha(X_s)^{\alpha-1}(aX_s + c)\, ds + \int_0^t \alpha(X_s)^{\alpha-1}\sigma X_s dW_s \\ &+ \frac{1}{2}\int_0^t \alpha(\alpha-1)(X_s)^{\alpha-2}\sigma^2 {X_s}^2\, ds \\ &+ \int_0^{t^+}\int_0^M (X_{s^-} - x)^\alpha - (X_{s^-})^\alpha\, N_p(dsdx). \end{aligned}$$

Hence

$$
\begin{aligned}
F(X_t) = F(X_0) + \text{ mart. } &+ \int_0^t \alpha (X_s)^{\alpha-1}(aX_s + c)\, ds \\
&+ \frac{1}{2}\int_0^t \alpha(\alpha-1)(X_s)^{\alpha-2}\sigma^2 {X_s}^2\, ds \\
&+ \int_0^{t+}\int_0^M (X_{s^-}-x)^\alpha - (X_{s^-})^\alpha\ \tilde{N}_p(dsdx) \\
&\leq F(X_0) + \text{ mart. } + \alpha \int_0^t (X_s)^\alpha \left(\frac{\sigma^2}{2}(\rho+\alpha-1) + cX_s^{-1}\right) ds,
\end{aligned}
$$

$\forall\, t \geq 0$. The above inequality holds because $(X_{s^-}-x)^\alpha \leq (X_{s^-})^\alpha$, $\forall X_{s^-} \geq M$.

Let $u^* = \max(M, 2c/\sigma^2(1-\rho-\alpha))$. We consider the process X_t on $[u^*, n)$, where n is an integer ($> u^*$), and let

$$\tau_n = \inf\{t > 0 :\ X_t \notin [u^*, n)\}$$

be the first exit time from the interval $[u^*, n)$. Then

$$F(X_{\tau_n}) \leq F(X_0) + \text{ mart.} \tag{2.17}$$

Taking expectation on both sides of the above inequality, and by the Optional Stopping Theorem, we have

$$E[X_{\tau_n}^\alpha] \leq u^\alpha.$$

Since $F(x)$ is increasing, we have

$$E[F(X_{\tau_n})] \geq (u^*-M)^\alpha P(X_{\tau_n} < u^* \,|X_0 = u) + n^\alpha P(X_{\tau_n} = n \,|X_0 = u).$$

Hence

$$(u^*-M)^\alpha P(X_{\tau_n} < u^* \,|X_0 = u) + n^\alpha P(X_{\tau_n} = n \,|X_0 = u) \leq u^\alpha.$$

Therefore

$$P(X_{\tau_n} = n \,|X_0 = u) \leq \left(\frac{u}{n}\right)^\alpha.$$

Let n go to infinity, we have

$$\psi_{u^*}(u) = 1 - \lim_{n\to\infty} P(X_{\tau_n} = n \,|X_0 = u) \geq 1 - \lim_{n\to\infty}\left(\frac{u}{n}\right)^\alpha = 1,\ \ \forall\, u \geq u^*.$$

Lemma 2.3. *Consider the model given by (1.1) and assume that $\sigma > 0$ and $\rho = 1$. Then there exists $u^* > M + 3$, such that*

$$\psi_{u^*}(u) = 1 \ \ \forall\, u \geq u^*.$$

Proof. Let $F(x) = \ln \ln x,\ x > M$. Applying Itô's formula, we have

$$\begin{aligned} F(X_t) - F(X_0) = & \int_0^t (X_s \ln X_s)^{-1}(aX_s + c)\, ds + \int_0^t (X_s \ln X_s)^{-1} \sigma X_s dW_s \\ & + \frac{1}{2} \int_0^t (-\ln X_s - 1)(X_s \ln X_s)^{-2} \sigma^2 {X_s}^2\, ds \\ & + \int_0^{t^+} \int_0^M [\ln \ln (X_{s^-} - x) - \ln \ln X_{s^-}]\, N_p(dsdx). \end{aligned}$$

Hence

$$\begin{aligned} F(X_t) = & F(X_0) + \text{ mart. } + \int_0^t (X_s \ln X_s)^{-1}(aX_s + c)\, ds \\ & + \frac{1}{2} \int_0^t (-\ln X_s - 1)(X_s \ln X_s)^{-2} \sigma^2 {X_s}^2\, ds \\ & + \int_0^{t^+} \int_0^M [\ln \ln (X_{s^-} - x) - \ln \ln X_{s^-}]\, \tilde{N}_p(dsdx) \\ \leq & F(X_0) + \text{ mart. } + \int_0^t \left(cX_s^{-1} - \frac{\sigma^2}{2 \ln X_s} \right) (\ln X_s)^{-1} ds. \end{aligned}$$

The above inequality holds because $\ln \ln (X_{s^-} - x) \leq \ln \ln X_{s^-},\ \forall X_{s^-} \geq M$.

Now let $\tilde{u}$ be the solution of $\sigma^2 x = 2c \ln x$, and $u^* = \max(M + 3, \tilde{u})$. We consider the process X_t on $[u^*, n)$, where n is an integer ($> u^*$), and let

$$\tau_n = \inf\{t > 0 :\ X_t \notin [u^*, n)\}$$

be the first exit time from the interval $[u^*, n)$. Then we have

$$F(X_{\tau_n}) \leq F(X_0) + \text{ mart.} \tag{2.18}$$

Taking expectation on both sides of the above inequality, and by the Optional Stopping Theorem, we have

$$E[\ln \ln X_{\tau_n}] \leq \ln \ln u.$$

Since $F(x)$ is increasing, we have

$$E[\ln\ln X_{\tau_n}] \geq \ln\ln(u^* - M)P(X_{\tau_n} < u^* - M \,|X_0 = u) \\ + \ln\ln n P(X_{\tau_n} = n \,|X_0 = u).$$

Hence

$$\ln\ln(u^* - M)P(X_{\tau_n} < u^* - M \,|X_0 = u) + \ln\ln n P(X_{\tau_n} = n \,|X_0 = u) \leq \ln\ln u.$$

Therefore

$$P(X_{\tau_n} = n \,|X_0 = u) \leq \frac{\ln\ln u}{\ln\ln n}\,.$$

Let n go to infinity, we have

$$\psi_{u^*}(u) = 1 - \lim_{n\to\infty} P(X_{\tau_n} = n \,|X_0 = u) \geq 1 - \lim_{n\to\infty} \frac{\ln\ln u}{\ln\ln n} = 1, \;\; \forall\, u \geq u^*.$$

3 Constantinescu and Thommann's Conjecture

In this section, we will prove that the Constantinescu and Thommann's Conjecture is true.

Lemma 3.1. *Let $u^* > 0$ be any positive real number. Let $M < \infty$ be an essential range for ξ_1. Suppose $\psi_{u^*}(u) = 1$, for all $u \geq u^*$. Then*

$$\psi_K(u) = 1, \;\; \forall\, u \geq K = \max(u^* - \frac{M}{2}, 0).$$

Remark 3.1. *$u^* > 0$ in the above Lemma is any positive real number, it needs not be the one defined in Lemma 2.2 or Lemma 2.3.*

Proof. Our first step is to show that for any $0 < C_1 < 1$, there exists a $\beta_0 = \beta_0(M, C_1)$ such that $P\left(X_t \leq u^* + \frac{M}{8},\, \forall\, 0 \leq t \leq \beta_0 \mid X_0 = u\right) \geq C_1 > 0$, for all $u^* \geq u \geq K$.

Let Y_t, V_t be the same as in lemma 2.1, and $X_t = Y_t^{-1} V_t^u$ the solution of (1.2). Define $Z_t^{u^*} = Y_t^{-1}\left(u^* + c\int_0^t Y_s\, ds\right)$. Since $dZ_t^{u^*} = (aX_t + c)dt + \sigma X_t dW_t$, $Z_t^{u^*}$ is a diffusion process. By the continuity of $Z_t^{u^*}$, $\forall\, \epsilon > 0$, we have

$$P\left(\lim_{\beta\to 0} \sup_{0\leq s\leq\beta} |Z_s^{u^*} - u^*| < \epsilon\right) = 1.$$

Hence for the same $\epsilon > 0$ and $\forall\, 0 < C_1 < 1$. $\exists\, \beta_0 = \beta_0(\epsilon, C_1) > 0,\ s.t.$

$$P\left(\sup_{0 \le s \le \beta_0} |Z_s{}^{u^*} - u^*| < \epsilon \right) \ge C_1 > 0,$$

In particular, choose $\epsilon = \frac{M}{8}$, $\exists\, \beta_0 = \beta_0(M, C_1) > 0, s.t.$

$$P\left(Z_t{}^{u^*} \le u^* + \frac{M}{8},\ \forall\, 0 \le t \le \beta_0 \right) \ge C_1 > 0.$$

Define $Z_t{}^u = Y_t^{-1}\left(u + c\int_0^t Y_s\, ds\right)$, then $Z_t{}^{u^*} \ge Z_t{}^u \ge X_t,\ \forall\, t \ge 0$, and

$$\begin{aligned} P\left(X_t \le u^* + \frac{M}{8},\ \forall\, 0 \le t \le \beta_0 \mid X_0 = u \right) &\ge P\left(Z_t{}^u \le u^* + \frac{M}{8},\ \forall\, 0 \le t \le \beta_0 \right) \\ &\ge P\left(Z_t{}^{u^*} \le u^* + \frac{M}{8},\ \forall\, 0 \le t \le \beta_0 \right) \\ &\ge C_1 > 0, \end{aligned}$$

$\forall\, K \le u \le u^*$.

Let δ be the time that the first jump occurs. Our next step is to show that there exists $C_2 = C_2(C_1, M) > 0$ such that

$$P\left(X_\delta < K \mid X_0 = u \right) \ge C_2 > 0,\ \forall\, K \le u \le u^*.$$

Notes that $\forall\, K \le u \le u^*$,

$$\begin{aligned} &P\left(X_t \le u^* + \frac{M}{8},\ \forall\, 0 \le t \le \beta_0,\ \delta < \beta_0,\ \xi_1 > \frac{3M}{4} \mid X_0 = u \right) \\ &= P\left(X_t \le u^* + \frac{M}{8},\ \forall\, 0 \le t \le \beta_0 \mid X_0 = u \right) P\left(\delta < \beta_0\right) P\left(\xi_1 > \frac{3M}{4} \right) \\ &\ge C_1 P\left(\delta < \beta_0\right) P\left(\xi_1 > \frac{3M}{4} \right) = C_2 > 0, \end{aligned}$$

since M is an essential range of ξ_1 and therefore $P(\xi_1 > \frac{3M}{4}) > 0$. On the other hand,

$$\begin{aligned} &P\left(X_t \le u^* + \frac{M}{8},\ \forall\, 0 \le t \le \beta_0,\ \delta < \beta_0,\ \xi_1 > \frac{3M}{4} \mid X_0 = u \right) \\ &\le P\left(X_t \le u^* + \frac{M}{8},\ \forall\, 0 \le t < \delta,\ \delta < \beta_0,\ \xi_1 > \frac{3M}{4} \mid X_0 = u \right) \\ &\le P\left(X_\delta \le u^* + \frac{M}{8} - \frac{3M}{4} = u^* - \frac{5M}{8} < u^* - \frac{M}{2} \le K \mid X_0 = u \right). \end{aligned}$$

Hence

$$P\left(X_\delta < K \mid X_0 = u\right) \geq C_2 > 0, \ \forall\ K \leq u \leq u^*.$$

Our final step is to show that

$$\psi_K(u) = 1, \ \ \forall\ u \geq K = \max(u^* - \frac{M}{2}, 0).$$

Define

$$T_1 = \begin{cases} \inf\{t > \delta, \ X_t \leq u^*\}, & if\ X_\delta \geq K \\ \\ \infty, & if\ X_\delta < K. \end{cases}$$

Note that the infimum of an empty set is ∞. But by the assumption $\psi_{u^*}(u) = 1$, for all $u \geq u^*$, we have $T_1 = \infty$ if and only if $X_\delta < K$. Let $E = \{X_t \geq K, \ \forall\ 0 \leq t < \infty\}$ and θ_s be the shift operator, then

$$\begin{aligned} P(E|\ X_0 = u^*) &= E[1_E 1_{T_1<\infty} \mid X_0 = u^*] + E[1_E 1_{T_1=\infty} \mid X_0 = u^*] \\ &= E[1_E 1_{T_1<\infty} \mid X_0 = u^*] \\ &= E[1_{T_1<\infty} \theta_{T_1}[1_E] \mid X_0 = u^*]. \end{aligned}$$

In what follows, we denote $E_x[1_E] = E[1_E|\ X_0 = x]$. By the strong Markov property of X_t, we have

$$\begin{aligned} E[1_{T_1<\infty} \theta_{T_1}[1_E] \mid X_0 = u^*] &= E[1_{T_1<\infty} E_{X_{T_1}}[1_E] \mid X_0 = u^*] \\ &\leq E\left[1_{T_1<\infty} E_{u^*}[1_E] \mid X_0 = u^*\right] \\ &= E[1_{T_1<\infty} \mid X_0 = u^*] E_{u^*}[1_E] \\ &\leq (1 - C_2) E[1_E \mid X_0 = u^*] \\ &= P(E|\ X_0 = u^*)(1 - C_2). \end{aligned}$$

The first inequality holds since $K \leq X_{T_1} \leq u^*$ on $\{T_1 < \infty\}$. Hence we have

$$P(E|\ X_0 = u^*) = P(E|\ X_0 = u^*)(1 - C_2).$$

Therefore $P(E|\ X_0 = u^*) = 0$, i.e. $\psi_K(u^*) = 1$. Since $u \leq u^*$, by Lemma 2.1,

$$\psi_K(u) \geq \psi_K(u^*) = 1.$$

The proof is completed.

Theorem 3.1. *Consider the model given by (1.1) and assume that $\sigma > 0$ and $\rho \leq 1$. Suppose also the jump distribution is bounded by an essential range $M > 0$. Then*

$$\psi(u) = 1, \ \ \forall\, u \geq 0.$$

Proof. By Lemma 2.2, Lemma 2.3 and the strong Markov property of X_t, it is sufficient to show that

$$\psi(u) = 1, \ \ \forall\, 0 \leq u \leq u^*.$$

By Lemma 3.1, $\psi_{K_1}(u) = 1, \ \ \forall\, u \geq K_1 = \max(u^* - \frac{M}{2}, 0)$. Applying Lemma 3.1 again, with u^* replaced by K_1, we have

$$\psi_{K_2}(u) = 1, \ \ \forall\, u \geq K_2 = \max(K_1 - M, 0) = \max(u^* - 2\frac{M}{2}, 0).$$

Repeating this argument $N = \lceil \frac{2u^*}{M} \rceil$ times, we have

$$\psi_{K_N}(u) = 1, \ \ \forall\, u \geq K_N = \max(u^* - N\frac{M}{2}, 0) = 0,$$

i.e.,

$$\psi(u) = 1, \ \ \forall\, u \geq 0.$$

Next, we will prove the conjecture true without assuming a cap on the claim size.

Theorem 3.2. *Consider the model given by (1.1) and assume that $\sigma > 0$ and $\rho \leq 1$. Then*

$$\psi(u) = 1, \ \ \forall\, u \geq 0.$$

Proof. Let $M > 0$ be a large constant, define

$$\hat{\xi}_i = \begin{cases} \xi_i, & if\ \xi_i \leq M \\ M, & if\ \xi_i > M, \end{cases}$$

and $\hat{P}_t = \sum_{j=1}^{N(t)} \hat{\xi}_j$. Let Y_t, V_t be the same as in Lemma 2.1, and $X_t = Y_t^{-1} V_t{}^u$ be the solution of (1.2). Define

$$Z_t = Y_t^{-1} \left(u + c \int_0^t Y_s \, ds - \int_0^t Y_s \, d\hat{P}_s \right),$$

then $Z_t \geq X_t, \ \forall\, t \geq 0$. Hence

$$\begin{aligned} \psi(u) &= P(X_t < 0, \text{ for some } \ 0 < t < \infty \mid X_0 = u) && (3.19) \\ &\geq P(Z_t < 0, \text{ for some } \ 0 < t < \infty \mid Z_0 = u). && (3.20) \end{aligned}$$

On the other hand, since $dZ_t = (aX_t + c)dt + \sigma X_t dW_t - d\hat{P}_t$, Z_t satisfies (1.1) with bounded claim size distribution. Hence, by Theorem 3.1,

$$P(Z_t < 0, \text{ for some } 0 < t < \infty \mid Z_0 = u) = 1, \; \forall\, u \geq 0.$$

Therefore

$$\psi(u) = 1, \;\; \forall\, u \geq 0.$$

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