Self-organized model of cascade spreading

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We study simultaneous price drops of real stocks and show that for high drop thresholds they follow a power-law distribution. To reproduce these collective downturns, we propose a self-organized model of cascade spreading based on a probabilistic response of the system's elements to stress conditions. This model is solvable using the theory of branching processes and the mean-field approximation and displays a power-law cascade-size distribution—similar to the empirically observed one—over a wide range of parameters.

I. INTRODUCTION

Cascades spreading is an important emergent property of various complex systems. Real life examples of cascades are numerous and range from infrastructure failures and epidemics to traffic jams and cultural fads [1, 2]. Theoretical models of cascades usually assume that agents can be in one of two states (healthy or failed) and an agent's failure puts some stress on its neighbors which may consequently fail too. See [3] for a recent survey of this field offering a novel unifying view.

In this paper we focus on cascades in economic systems where we can identify them with stock prices suddenly dropping in a major market crash [4] or with companies going bankrupt simultaneously and leading to global recession [5]. Corresponding theoretical models are based on shortage and bankruptcies propagation in production networks [6], default propagation in credit networks [7, 8], interaction of firms through one monopolistic bank [9] or in a complex credit network economy [10], and herding behavior of traders [11, 12]. While these models help us to understand cascade processes in economic systems, they are mostly too involved to allow for analytical solutions—their study relies on numeric simulations and agent-based modeling [13].

A simpler point of view on cascade phenomena is offered by the concept of self-organized criticality (SOC) which has had a deep impact on the science of complexity. First introduced more than twenty years ago to explain the ubiquitous $1/f$ noise [14], it caused a blossoming of toy models, computer simulations, and real life experiments [15]. Used analytical techniques involve scaling arguments [17], mean-field theories [18], renormalization methods [19, 20], and rigorous algebraical techniques [21].

SOC is a mechanism for the emergence of complex behavior in many diverse real world systems [16, 22]. The generic behavior of SOC models is that: (a) they evolve so that they always stay close to the critical point, (b) long periods of robustness and moderate activity are interrupted by sudden breakdowns. This indeed qualitatively resembles "stock markets which expand and grow on relatively long time scales but contract in stock-market crashes on relatively short time scales" [15] and "stock crashes caused by the slow buildup of long-range correlation leading to a global cooperative behavior of the market eventually ending into a collapse in a short time interval" [4]. This similarity provides motivation for the present study.

We begin our work with an empirical investigation of simultaneous price drops of real stocks and show that the size distribution of observed events is broad (for high drop thresholds it follows a power-law distribution). This observation confirms that simultaneous stock downturns are a collective phenomenon. We propose an SOC model of cascades which, unlike most SOC models, assumes a probabilistic response mechanism where a node has only a certain probability of reacting to the current stress conditions. The key premise is that while failed nodes become significantly more resistant in the next time step, healthy nodes become slightly less resistant—this is a close parallel with the slow growth/fast decay picture described above. The model has the advantage of being very simple, analytically solvable in some cases, and easily generalizable to more complicated settings. We analyze it using the formalism of branching processes, the mean-field approximation and, for complex topologies of nodes' interactions, using numerical simulations. Obtained cascade-size distributions exhibit a close similarity to our empirical observations. We conclude our study with a discussion of model's generalizations and possible areas of application.

II. EMPIRICAL MOTIVATION

Here we investigate co-occurring price movements of real stocks. Adapting the vocabulary of cascade models, we say that a stock fails when the relative loss of its price over a given time interval Δt exceeds a certain threshold H. Denoting the price of stock i at time t as $p_i(t)$, its failure occurs when $[p_i(t) - p_i(t + \Delta t)]/p_i(t) > H$. The number of stocks failing at time t, $n_F(t)$, is a direct analog of the cascade size in a model of cascade spreading.

We use daily prices (hence $\Delta t = 1\,\text{day}$) of 500 stocks from the standard U.S. index S&P 500 (this data is freely available at, for example, `finance.yahoo.com`). To achieve a fixed system size, we consider only those 336 companies which are in the stock market since 1991 and use their prices during the 18-years long period 1992-2009 for our analysis. The resulting size distribution is shown

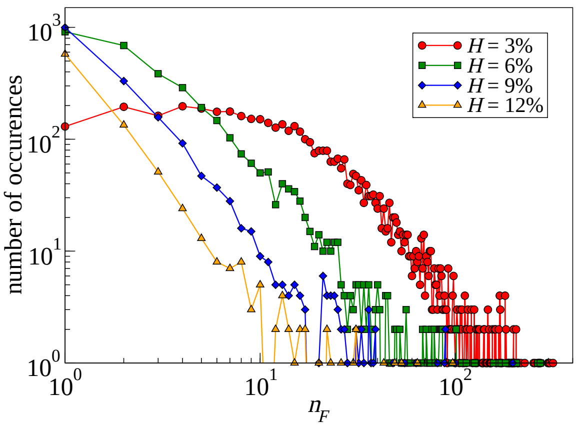

FIG. 1. The empirical size distribution observed with real stock prices for various thresholds H.

in Fig. 1 for different threshold values H. We see that when the threshold value is large enough (which is in line with the notion of stock *failures*), the observed size distribution shows a power-law shape. Using the methodology described in [23], we obtained the power-law exponents 2.23 ± 0.05 ($H = 12\%$, lower bound for the power-law behavior $n_{\min} = 1$) and 2.24 ± 0.06 ($H = 9\%$, $n_{\min} = 3$). The corresponding p-values are 0.83 and 0.34, respectively which confirms that both distributions are consistent with the hypothesis of a power-law distribution (to obtain the p-values, we used the standard Kolmogorov-Smirnov statistic). When $H = 6\%$ and $H = 3\%$, the resulting size distributions are broad but probably not power-law. Finally, when $H = 0$ (*i.e.*, any price drop is interpreted as a failure), the size distribution is roughly symmetric around the value corresponding to half of the system size (not shown in Fig. 1) and it is thus impossible to observe fat tails of the size distribution which is naturally limited by the system size.

III. MODEL AND ITS MEAN-FIELD SOLUTION

Consider a system of N nodes where node i ($i = 1, \dots, N$) has two possible states: failed ($i \in \mathcal{F}$) and healthy ($i \notin \mathcal{F}$). With each node i we further associate fragility $f_i \in [0, 1]$ which measures how this node reacts to failures of its neighbors (the higher the fragility, the more likely is the node to follow a neighbor's failure). The dynamics of the model is governed by the following simple rules. (i) In each time step, the first failed node ("trigger") is chosen at random and may induce failures of other nodes. (ii) If a neighbor of node i fails, node i follows it with the probability f_i. The cascade of failures propagates until all remaining nodes resist the damage. (iii) Fragilities of all nodes are updated according to

$$f_i(t+1) = \begin{cases} \lambda f_i(t) & i \in \mathcal{F} \\ (1+\beta) f_i(t) & i \notin \mathcal{F} \end{cases} \tag{1}$$

where $\beta > 0$ and $\lambda \in (0, 1)$ are parameters of the model. All values $f_i(t+1) > 1$ are truncated to 1 (this may occur when the parameter β is large). After this update is finished, all nodes are again marked as healthy, the current time step ends and the new one begins with point (i). Note that unlike some other models of cascade spreading, failed nodes are not removed from the system in our case. If a long enough equilibration period is applied before measuring the system's behavior, the initial fragility values $f_i(0)$ are of little importance for the system's behavior (see Sec. III E for a detailed discussion); we set them randomly in the range $(0; 1)$ in all our simulations (unless stated otherwise).

According to the rules above, when n neighbors of node i fail, node i resists with the probability $(1-f_i)^n$ and fails with the complementary probability

$$P_F(f_i, n) = 1 - (1 - f_i)^n. \tag{2}$$

This response to failures is "path-independent" in some sense: the probability that a node resists failure of n of its neighbors, $(1 - f_i)^n$, is the same as the probability of resisting two consequent waves of failures of x and $n - x$ neighboring nodes, $(1 - f_i)^x (1 - f_i)^{n-x}$.

We simplify the system by assuming that interactions of all nodes are equally strong (the general case will be studied in Sec. III D). This renders the notion of "node's neighbors" superfluous because each node's failure affects all remaining healthy nodes in the system. Now assume that after the initial failed node is chosen, n_1 nodes respond to this failure and fail too. Each of the remaining $N - n_0 - n_1$ nodes (here $n_0 = 1$ is the initial number of failed nodes) then has some n_1-dependent failure probability which results in n_2 new failures, and so on, until in iteration m, $n_m = 0$ is achieved. The cascade size is then defined as the total number of failures, $S = n_0 + \cdots + n_m$, and node fragilities are consequently updated according to Eq. (1). Since cascade sizes are limited by the system size, $S \le N$.

The dynamics of the system, based on a failure propagation process and fragility updating, is fully contained in the three described simple rules. In the following paragraphs we shall study how these rules drive the system to the critical point and how this self-organized criticality manifests itself in the distribution of cascade sizes $P(S)$.

A. Failure probability

Let P_F be the average failure probability of a given node in one time step (or, equivalently, the average fraction of failed nodes in one time step). Assuming that t_{eq} is some sufficiently long time after which the system is equilibrated (we use $t_{\text{eq}} = 10^4$ for all our simulations), the average fragility values $\langle f_i \rangle$ does not change in time any more. All nodes interact equally strongly, hence $\langle f_i \rangle$ is independent of i and it can be replaced with $\langle f \rangle$. Since in a large number of time steps T each node undergoes

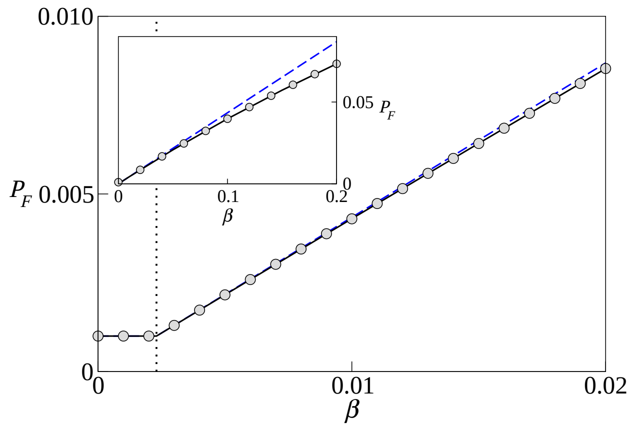

FIG. 2. Average failure probability P_F: analytical (solid black line), approximate analytical (dashed blue line) and numerical (symbols) results for $N = 10^3$, $\lambda = 0.1$, averaged over 10^6 time steps. The vertical dotted line indicates β_0 obtained with Eq. (5).

$P_F T$ failures and $(1 - P_F)T$ non-failures, Eq. (1) implies

$$\langle f(t_{\rm eq} + T)\rangle = \langle f(t_{\rm eq})\rangle \lambda^{P_F T} (1+\beta)^{(1-P_F)T}. \qquad (3)$$

Using the equilibrium condition $\langle f(t_{\rm eq} + T)\rangle = \langle f(t_{\rm eq})\rangle$, we can solve this equation with respect to P_F to get

$$P_F(\beta, \lambda) = -\frac{\ln(1+\beta)}{\ln \frac{\lambda}{1+\beta}} \approx -\frac{\beta}{\ln \lambda} \qquad (4)$$

where the approximate result is valid for $\beta \ll 1$ (see Fig. 2 for a comparison of these results with numerical simulations).

A node may fail because it is selected as the first failed node (which has the probability $1/N$) or due to failure propagation (the probability of which we label as P_P). P_F thus can be written as $P_F = 1/N + P_P$. Since the value of P_F depends solely on β and λ, $P_P = P_F - 1/N$ may be negative for a small system which is, of course, impossible in practice. This situation occurs when for given λ, N, the value of β is smaller than a certain threshold β_0 and hence it does not suffice to compensate for the fragility decay due to λ. Eq. (3) then has only the trivial solution $\langle f\rangle = 0$ and hence $P_F(\beta, \lambda) = 1/N$ (failures do not spread). When β is small, the approximate form of P_F can be used to solve this equation with respect to β and we get

$$\beta_0 \approx \frac{1}{N} \ln \frac{1}{\lambda} \qquad (5)$$

which agrees with numerical simulations (see Fig. 2).

When model parameters are set to extreme values (for example, $N = 10^3$, $\beta = 10^3$, $\lambda = 10^{-3}$), the system exhibits unusual modes of behavior where active turns (with nearly all nodes failed) alternate with calm turns (with nearly all nodes healthy). While Eq. (4) holds also in such conditions, our further analysis focuses on $\beta \ll 1$ which renders more realistic behavior.

B. Average fragility

When $nf_i \ll 1$, Eq. (2) can be approximated with $P_F(f_i, n) = nf_i$ which can be interpreted as independence of stress inflicted by n individual failed nodes. This further means that each failed node has its failing descendants independently of other failed nodes and hence one can use the theory of branching processes [24] to describe the cascade spreading in the limit of small cascade sizes. An elementary result of this theory is that when a branching process is initiated by one failed node and the average number of direct descendants of a failed node is $\langle d\rangle$, the average total progeny of the process is

$$Y = \frac{1}{1 - \langle d\rangle}. \qquad (6)$$

In the limit $\langle d\rangle \to 1$ we recover the so-called *critical branching process* with diverging average progeny. Note that by the use of the theory of branching processes we implicitly assume that the system size is infinite. For a discussion of the finite-size effects on the size of an epidemic outbreak see [25].

As already mentioned, when interactions of all nodes are equal, $\langle f_i\rangle$ is independent of i. If we further neglect fluctuations of f_i, then all nodes have identical fragility $\langle f\rangle$. This is a mean-field-like approximation which replaces the exact cascade spreading with cascade spreading in a homogeneous averaged medium. The number of direct descendants of a failed node, d, now follows the simple binomial distribution

$$P(d\,|\langle f\rangle) = \binom{N}{d} \langle f\rangle^d \big(1 - \langle f\rangle\big)^{N-d} \qquad (7)$$

and its average is $\langle d\rangle = N\langle f\rangle$. By substituting this in Eq. (6), we find the average cascade size in the form $\langle S\rangle = 1/(1 - \langle f\rangle)$. Further, using $\langle S\rangle = NP_F(\beta, \lambda)$ we get the average fragility

$$\langle f\rangle = \frac{1}{N}\left(1 - \frac{\ln\big[(1+\beta)/\lambda\big]}{N\ln(1+\beta)}\right). \qquad (8)$$

Since $\beta > 0$ and $\lambda < 1$, $\langle f\rangle$ is always less than $1/N$. Comparison with numerical simulations (not shown here) confirms that Eq. (8) is valid only for $\beta \ll 1$.

C. Cascade size distribution

The size distribution for a critical branching process is known to have the power-law form $P(S) \sim S^{-3/2}$ (see, for example, [26]). To find the distribution $P(S)$ for general β and λ, we use a theorem from [27]. According to this theorem, if the generating function for the number of direct descendants d is $\pi(x)$, the total progeny of the resulting branching process Y has the distribution

$$P(Y|n_0) = \frac{n_0}{Y}\, p_{Y-n_0}^{(Y)} \qquad (9)$$

where $p_a^{(b)}$ is defined using

$$[\pi(x)]^b = p_0^{(b)} + p_1^{(b)} x + \dots \tag{10}$$

and n_0 is the number of ancestors (in our case, the number of initial failed nodes). Since d obeys the binomial distribution given by Eq. (7), its generating function is simply $\pi(x) = (1 - \langle f \rangle + \langle f \rangle x)^N$ and we get

$$P(S|\beta,\lambda) = \frac{1}{S}\binom{NS}{S-1}\langle f\rangle^{S-1}\big(1-\langle f\rangle\big)^{NS-S+1} \tag{11}$$

where we used $n_0 = 1$ and $\langle f \rangle$ is given by Eq. (8). Note that the resulting $P(S)$ is positive for $S > N$ which contradicts the model assumptions (each node fails at most once in a given turn). This contradiction is a direct consequence of using the theory of branching processes which assumes that the system size is infinite. This problem is of little importance for small values of β when the obtained values of $P(S)$ are negligible for $S > N$.

When $1 \ll S \ll N$, Eq. (11) can be approximated with

$$P(S|\beta,\lambda) = \frac{(N\langle f\rangle)^{S-1} \mathrm{e}^{S(1-N\langle f\rangle)}}{\sqrt{2\pi} S^{3/2}}. \tag{12}$$

According to Eq. (8), $\lim_{N\to\infty} N\langle f \rangle = 1$ for any given β, λ and hence in the limit of large system size is $P(S|\beta,\lambda) \sim S^{-3/2}$ which corresponds to the above-mentioned critical branching process. For any finite-size system, the smaller the value of β, the larger the difference between $N\langle f \rangle$ and 1. Hence with lowering β, we can expect a gradual transition from the power-law distribution to a faster decaying exponential one. From Eq. (12) follows that the power-law scaling appears for $S(1 - N\langle f \rangle) \ll 1$ which implies that $S \ll \beta N$. This condition agrees with Fig. 3 where for $\beta = 10^{-3}$, a deviation from the power-law behavior appears at $S \approx 10$.

A comparison of the obtained analytical results with numerical simulations is shown in Fig. 3. The agreement is good for small values of β ($\beta \lesssim 0.01$) and the initial slope of the distributions (before the finite size effects become apparent) is close to $-3/2$. Results obtained with $\beta = 0.001$ confirm that when β is small enough, $P(S)$ decays faster than as a power law. When β is large, true $P(S)$ deviates from the analytical prediction and exhibits a secondary maximum at a large size value—this effect is well visible in Fig. 3 for $\beta = 0.1$. As β increases, one can expect a gradual transition from a power-law distribution to a bimodal one (when β is large, time steps with a few failing nodes alternate with time steps with almost all nodes failing). The unexpected maximum of a power-law size distribution observed for large values of β resembles so-called meaningful outliers discussed in [28]. Alternatively, one can say that this corresponds to a regime where "bubbles" often form and are followed by large corrections (crashes).

How to estimate the value of β at which the secondary maximum appears and Eq. (11) ceases to hold? The average number of failures can be computed both from

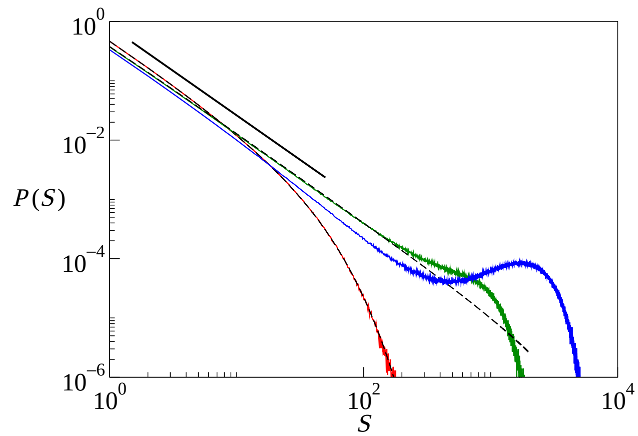

FIG. 3. Cascade size distribution: numerical results (color lines), analytical results according to Eq. (11) (dashed lines) and the power-law decay with exponent $-3/2$ (thick solid line) for $N = 10^4$, $\lambda = 0.1$, 10^7 time steps, and $\beta = 0.001$ (red line, fastest decay), $\beta = 0.01$ (green line, medium decay), $\beta = 0.1$ (blue line, slowest decay). The analytical solution is not plotted for $\beta = 0.1$ because it is very similar to that for $\beta = 0.01$.

Eq. (11) and from Eq. (4). By comparing the two results we obtain

$$NP_F(\beta,\lambda) = \sum_{S=1}^{N} S P(S|\beta,\lambda). \tag{13}$$

When β is small, both sides of this equation depend on β and the equality can hold. However, Eq. (12) shows that when β is sufficiently large, the size distribution is approximately power-law and it does not change with β anymore. As we increase β further, the power-law distribution does not suffice to provide enough failures and for Eq. (13) to hold, an additional contribution must appear on the rights side—this contribution is visible as a secondary maximum (a "hump") in Fig. 3 for $\beta = 0.1$. The value of β when this happens, β^*, can be found by substituting the power-law distribution for $P(S)$ in Eq. (13), leading to

$$NP_F(\beta^*,\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{S=1}^{N} S^{-1/2} \tag{14}$$

where the factor $1/\sqrt{2\pi}$ comes directly from Eq. (12). When N is large, one can replace summation with integration to find that the right side of this equation is approximately $\sqrt{2N/\pi}$. Using the approximate form of Eq. (4) it follows that

$$\beta^* \approx \sqrt{\frac{2}{\pi N}} \ln\frac{1}{\lambda} \tag{15}$$

which complements the previously found threshold β_0 given by Eq. (5). For $N = 10^4$ and $\lambda = 0.1$, we obtain

$\beta^* \approx 0.02$ which agrees with our empirical observation ($\beta \lesssim 0.01$ for Eq. (11) to hold) above.

Finally, by comparing the empirical observations presented in Fig. 1 with the obtained analytical results, we can conclude that the presented model exhibits qualitative agreement with the studied real system.

D. Generalizations

To test how robust are the obtained results, we consider simple generalizations of the proposed model. First of all, when the multiplicative fragility update rule Eq. (1) is replaced by an additive one, the behavior of the system does not change considerably. The second generalization relates to the assumed even influence of a node's failure on all the remaining nodes. Denoting the strength of failure propagation from node i to node j as $C_{i,j}$, the probability that node j fails as a result of i's failure can be generalized to $C_{i,j} f_j$. The probability that node j fails as a result of a group $\mathcal{F}$ of failed nodes (given by Eq. (2) before) generalizes to the form

$$P_F(f_j, \mathcal{F}) = 1 - \prod_{i\in\mathcal{F}} (1 - C_{i,j} f_j). \tag{16}$$

Matrix C encodes the structure of the network of node interactions.

When the elements $C_{i,j}$ are drawn independently from a given distribution and the system size is large, the mean-field approximation is again appropriate to describe the system's behavior and the power-law size distribution with exponent 3/2 results. Similarly when C contains a block structure with inter-block elements drawn from a different distribution than intra-block elements (this mimics the well-known sector structure of the stock correlation matrix), the original power-law size distribution remains largely unchanged (unless either the block division of C or one of the two probabilistic distributions are such that they do not allow to use the mean-field approximation). Analogous behavior results from the "random neighbor approximation" in which node's neighbors are chosen anew repeatedly (see [29] for this kind of analysis of a different model).

When all elements $C_{i,j}$ are either zero or one, matrix C can be represented by a network and a complex topology of node interactions can be introduced by using some of the standard network models [30]. We studied two different types of networks: the Erdös-Rényi network where $C_{i,j} = 1$ with probability p and $C_{i,j} = 0$ otherwise and the growing Barabási-Albert network where each new node is attached to I old nodes. (These two kinds of networks are structurally very distinct as the former consists of nodes of approximately identical degree and the latter exhibits a power-law degree distribution.) Numerical results for both cases are shown in Fig. 4. As expected, for the Erdös-Rényi network with $p > 1/N$, the size distribution exponent remains unchanged. When $p < 1/N$, the network consists of small unconnected components and hence big avalanches cannot occur. The irregular size distribution $P(S)$ observed for $\beta = 5\cdot 10^{-5}$ is due to topological properties of the particular network realization where the model was simulated. These results agree with the previous studies of the sandpile dynamics [31] (see [32] for an extensive recent review of critical phenomena in complex networks). By contrast, Barabási-Albert networks yield avalanche size distributions with significantly higher exponents (approximately 1.65) which is probably due to the strong inhomogeneity of the network. When $I = 1$, the size distribution deviates from the power-law distribution, probably as a consequence of the scale-free network topology.

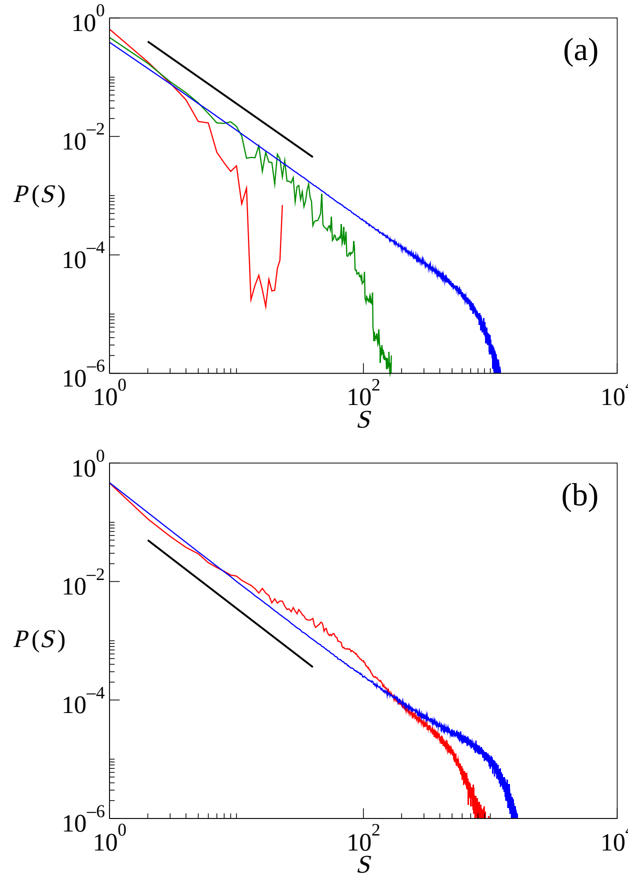

FIG. 4. The cascade size distribution on complex networks: (a) sparse Erdös-Rényi networks with $p = 5\cdot 10^{-5}, 10^{-4}, 10^{-3}$ (red, green, and blue line, respectively; the indicative thick line has slope 1.5), (b) Barabási-Albert networks with $I = 1$ and $I = 10$ (red and blue line, respectively; the indicative thick line has slope 1.65). Parameters of the system: $N = 10^4$, $\beta = 0.005$, $\lambda = 0.1$, 10^7 time steps.

E. Role of the initial fragility values

While it sounds plausible that due to model's stochasticity, the initial fragility values have no influence on the equilibrium fragility distribution, the situation is in fact

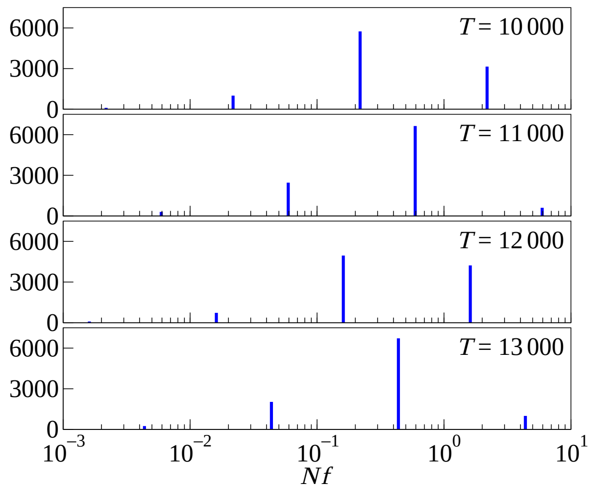

FIG. 5. Fragility distributions at different time steps (the initial fragility values are set to $1/N$, $N = 10^4$, $\beta = 0.001$, $\lambda = 0.1$).

more complicated. For example, a simple numerical simulation with $f_i(0) = 1/N$ for all i shows a case where: (i) no stationary fragility distribution arises, (ii) at any time step, only a small number of distinct fragility values is observed (see Fig. 5). What causes the discreteness of fragility values? Denoting the number of failing and healthy time steps of node i as F_i and H_i, respectively, it must hold that $F_i + H_i = t$ where t is the current time step. This node's fragility now can be written as

$$f_i(t) = f_i(0)(1+\beta)^t \big[\lambda/(1+\beta)\big]^{F_i}. \qquad (17)$$

When all $f_i(0)$ are identical, the possible values of $f_i(t)$ are discrete at any time step t and the ratio of neighboring possible values is $(1+\beta)/\lambda$. If λ is small (as it is in our simulations), this ratio is large and hence the number of actually observed fragility values is small (because values much smaller or greater than the average fragility are very unlikely). Eq. (17) implies that possible fragility values depend on t and hence there can be no stationary fragility distribution—this is confirmed by Fig. 5 where fragility peaks constantly shift to higher values and change their relative heights. Interestingly, even this peculiar setting of $f_i(0)$ does not alter the long-term model's behavior substantially and the aggregate quantities (such as the average failure probability or the cascade size distribution) are similar to those found for randomized initial fragility values before.

Differences between neighboring peaks are $\lambda/(1+\beta)$, hence the time after which the fragility distribution pattern repeats can be estimated as $\ln\big[\lambda/(1+\beta)\big]/\ln(1+\beta)$. Since this is a typical time of fragility evolution, one can use it also as an estimate of the initial equilibration time T_{eq}. For the smallest value of β in our simulations ($\beta = 0.005$) we obtain $T_{\text{eq}} \approx 4\,600$ which ex post confirms our setting of the equilibration time to 10^4. Finally, note that while the random setting of $f_i(0)$ prevents discrete fragility values from appearing, some remnants of the initial fragility values can be preserved by Eq. (17). To obtain a fragility distribution truly independent of the initial values, one has to assume annealed dynamics, *i.e.* fragility updating by randomized values of β and λ.

IV. DISCUSSION

We have studied the proposed model of cascade spreading by means of the mean-field approximation and the branching process theory. Despite the noticeable success of this approach, some problems remain open. Firstly, since the cascade sizes corresponding to the secondary maximum in Fig. 3 are comparable with the system size, this mode of behavior obviously cannot be described using the formalism of branching processes (where an infinite system size is assumed). While we found an approximate condition for the appearance of the secondary maximum, how to proceed further to the description of the resulting size distribution is an open question. Secondly, the size distribution exponent is different for scale-free networks and it would be interesting to find its analytical expression. Thirdly, it would be interesting to know whether the model can be modified to produce power-law size distributions with exponents considerably different from 1.5–1.65 observed here. One opportunity for such a generalization is to assume that the network structure is dynamic and its evolution depends on the failures of its nodes as it is done for different models in [33, 34]. It is also possible to generalize the model by introducing memory or delayed stress propagation as in, for example, [35].

We stress that the spreading mechanism proposed here is a general one and in its use it is not limited to market crashes or firm bankruptcies. For example, when nodes are aligned in a two- or three-dimensional lattice, one could use a similar mechanism to model earthquakes where the failure at one place of the Earth's crust exerts some stress on the neighboring places. (The number of failed nodes is then equivalent to the size of the earthquake.) We are looking forward to future studies and applications of this model.

ACKNOWLEDGMENTS

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