The Underlying Dynamics of Credit Correlations

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Abstract

We propose a hybrid model of portfolio credit risk where the dynamics of the underlying latent variables is governed by a one factor GARCH process. The distinctive feature of such processes is that the long-term aggregate return distributions can substantially deviate from the asymptotic Gaussian limit for very long horizons. We introduce the notion of correlation surface as a convenient tool for comparing portfolio credit loss generating models and pricing synthetic CDO tranches. Analyzing alternative specifications of the underlying dynamics, we conclude that the asymmetric models with TARCH volatility specification are the preferred choice for generating significant and persistent credit correlation skews. The characteristic dependence of the correlation skew on term to maturity and portfolio hazard rate in these models has a significant impact on both relative value analysis and risk management of CDO tranches.

1 Introduction

The latest advances in credit correlation modeling were in part motivated by the growth and sophistication of the so called correlation trading strategies, in particular those involving the standard tranches referencing the Dow Jones CDX (US) and iTraxx (Europe) broad market CDS indexes. The synthetic CDO market allows investors to take views on the shape of the credit loss distribution of the underlying collateral portfolio. The market implied portfolio loss distribution is now well exposed through the pricing of liquid standard tranches, which in turn are expressed through their implied correlations.

The pricing of credit derivatives has been based on either structural Merton style models or reduced form credit migration models. See Lando [21] and Schonbucher [27] for a survey of these techniques. In both cases, rather ad hoc models of dependence are needed to explain why correlations vary over time and across tranches. Frequently, specific copulas are postulated to model prices at

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a point in time. However, this approach does not easily generalize to dynamic situations where new information is continually being revealed and prices of tranches of different maturities evolve consistently with each other and with the underlying reference portfolio.

This paper brings the standard time series methodology into the portfolio credit risk setting. Specifically, we consider a structural credit model where the latent asset variables evolve according to a one factor multi-variate asymmetric GARCH model. This model is formulated and estimated on high frequency data and the implications for long horizon loss distributions are derived by time aggregation. We demonstrate that the asymmetric GARCH specification can generate multivariate return distributions with both significant lower tail dependence and asymmetry that persist for very long time horizons. We then show that a multiname credit portfolio loss distribution derived from our model can produce implied correlation skews similar to the ones observed in the synthetic CDO market.

The article is organized as follows. In section 2 we give a brief overview of the portfolio credit modeling in a general copula framework, and introduce the notion of correlation surface as a generalization of base correlation to a dynamic setting. We derive analytical formulas expressing the portfolio loss distribution directly in terms of the shape of the correlation surface. In section 3 we apply time series models to the portfolio credit risk problem. We develop new implications of the well known asymmetric threshold GARCH (TARCH) models and show their power in explaining persistent non-Gaussian features of long horizon market returns. We then demonstrate the implications of using such latent variable specification for default correlation estimates. In section 4 we use the correlation surface estimates to assess the ability of various static and dynamic credit loss-generating models to produce realistic pricing of CDO tranches. In section 5 we summarize the results and outline possible applications and extensions of our approach.

2 Defining Portfolio Credit Risk Models

In this section we will introduce the notion of correlation surface as a general tool for comparing portfolio credit risk models. We will first give a brief overview of the general copula framework, focusing particularly on the symmetric one factor latent variable assumption and the large homogeneous portfolio approximation which will be used throughout this paper. We will then formalize the definition of the correlation surface and show that, under the above approximation, it contains sufficient information to recover the full portfolio loss distribution and to price CDO tranches.

2.1 General Copula Framework

Consider a portfolio of M credit-risky obligors in a static setup with a fixed time horizon $[0, T]$. To simplify notations we will skip the time subscript for time dependent variables. At time $t = 0$ all M obligors are assumed to be in non-default state and at time T firm i is in default with probability p_i. We assume we know the individual default probabilities $\mathbf{p} = [p_1, ..., p_M]'$ (either risk-neutral, e.g. inferred from default swap quotes, or actual, e.g. estimated by rating agencies). Let $\tau_i \geq 0$ be the random default time of obligor i and $Y_i = 1_{\{\tau_i \leq T\}}$ the default dummy variable which is equal to 1 if default happened before T and 0 otherwise.

The loss generated by obligor i conditional on its default is denoted as $l_i > 0$. The loss l_i is a product of the total exposure n_i and percentage loss given default $1 - \bar{R}_i$ where $\bar{R}_i \in [0, 1]$ is the recovery rate. We assume that all l_i are constant (see [1] for discussion on stochastic recoveries). Portfolio loss L_M at time T is the sum of the individual losses for the defaulted obligors

$$L_M = \sum_{i=1}^{M} l_i 1_{\{\tau_i \leq T\}} = \sum_{i=1}^{M} l_i Y_i \tag{1}$$

The mean loss of the portfolio can be easily calculated in terms of individual default probabilities:

$$E(L_M) = \sum_{i=1}^{M} l_i E(Y_i) = \sum_{i=1}^{M} l_i p_i \tag{2}$$

Risk management and pricing of derivatives contingent on the loss of the credit portfolio, such as CDO tranches, require knowing not only the mean but the whole distribution of losses with cdf $F_L(x) = P(L_M \leq x)$. Portfolio loss distribution depends on the joint distribution of default indicators $\mathbf{Y} = [Y_1, ..., Y_M]'$ and in a static setup can be conveniently modeled using the latent variables approach [14]. Particularly, to impose structure on the joint distribution of default indicators we assume that there exists a vector of M real-valued random variables $\mathbf{R} = [R_1, ..., R_M]'$ and M dimensional vector of non-random default thresholds $\mathbf{d} = [d_1, ..., d_M]'$ such that

$$Y_i = 1 \Longleftrightarrow R_i \leq d_i \text{ for } i = 1, ..., M \tag{3}$$

Denote $F : \mathbb{R}^M \to [0, 1]$ as a cdf of $\mathbf{R}$ and assume that it is a continuous function with marginal cdf $\{F_i\}_{i=1}^M$. For each obligor i the default threshold d_i is calibrated to match the obligor's default probability p_i by inverting the cdf of its aggregate returns $R_i : d_i = F_i^{-1}(p_i)$. According to Sklar's theorem [32], under the continuity assumption F can be uniquely decomposed into marginal cdfs $\{F_i\}_{i=1}^M$ and the M-dimensional copula $C : [0, 1]^M \Rightarrow [0, 1]$

$$F(\mathbf{d}) = C(F_1(p_i), .., F_M(p_M)) \tag{4}$$

The most popular copula choices are the Gaussian copula model [22], Student-t [23], and double-t [20]. The choice of copula C defines the joint distribution of default indicators from which the portfolio loss distribution can be calculated.

The number of names in the portfolio can be large and therefore the calibration of the copula parameters can be problematic. To reduce the number of parameters some form of symmetry is usually imposed on the distribution of default indicators. Gordy [18] and Frey and McNeil [14] discuss the mathematics behind the modeling of credit risk in homogeneous groups of obligors and the equivalence of the homogeneity assumption to the factor structure of default generating variables.

Assumption 1(Symmetric One Factor Model)*: Assume that loss given default* $l_i = (1-\bar{R}_i) \cdot n_i$ *and individual default probabilities* p_i *are the same for all* M *names in the portfolio and that the latent variables admit symmetric linear one factor representation:*

$$n_i = n \tag{5a}$$

$$\bar{R}_i = \bar{R} \tag{5b}$$

$$p_i = p \tag{5c}$$

$$R_i = bR_m + \sqrt{1-b^2}E_i \text{ with } 0 \le b \le 1 \tag{5d}$$

where R_m *and* E_i *are independent zero mean, unit variance random variables.* E_i'*-s are identically distributed with cdf* $G(\bullet)$*.*

Within this framework, denote:

- $F(d_i) \equiv P(R_i \le d_i)$ cdf of aggregate total returns R_i
- $G(d_i) \equiv P(E_i \le d_i)$ cdf of aggregate idiosyncratic returns E_i
- $F(\mathbf{d}) \equiv P(\mathbf{R} \le \mathbf{d})$ joint cdf of $\mathbf{R}$
- $C(\mathbf{u}) \equiv F\left(F^{-1}(u_1), ..., F^{-1}(u_M)\right)$ copula of $\mathbf{R}$

Note than the assumption of one factor structure implies that equity returns $\mathbf{R}$ are independent conditional on the market return R_m and therefore $F(\mathbf{d})$ can be computed as expectation of the product of conditional cdfs:

$$F(\mathbf{d}) = E\left(\prod_{i=1}^{M} P(R_i \le d_i | R_m)\right) = E\left(\prod_{i=1}^{M} G(d_i - b_i R_m)\right) \tag{6}$$

Parameter b defines the pairwise correlation of latent variables with the market factor. The correlation of latent variables, ρ, which is often referred to as "asset correlation" (this naming reflects the interpretation of latent variables as asset returns in Merton-style structural default models), is constant across all pairs of assets in a symmetric single factor model:

$$\rho_{ij} = \rho = b^2 \tag{7}$$

In addition to the asset correlation, which reflects the co-movement of returns on small scale, multivariate distributions can be also characterized by measures that reflect joint extreme movements for a pair of assets – the tail dependence coefficient λ_{ij}^d and the pairwise default correlation coefficient $\rho_{ij}^d(p)$. Suppose R_i and R_j are the stock returns for companies i and j over the $[0,T]$ time horizon. The coefficient of lower tail dependence and the default correlation coefficient for two random variables with the same continuous marginal cdfs, $F(R)$, and the same default probabilities, p,are defined as:

$$\lambda_{ij}^d = \lim_{p\to+0} P\left(R_i \le d_p | R_j \le d_p\right) = \lim_{p\to+0} \frac{C(p,p)}{p} \tag{8}$$

$$\rho_{ij}^d(p) = corr(1_{\{R_i\le d_p\}}, 1_{\{R_j\le d_p\}}) = \frac{C(p,p)-p^2}{(1-p)p} \tag{9}$$

where p is the probability of crossing the threshold (also interpreted as the default probability), and is related to the latter via the relationship $d_p = F^{-1}(p)$. Both these measures depend only on the bivariate copula of the two random variables and are asymptotically equal: $\lim_{p\to+0}\rho_{ij}^d(p) = \lambda_{ij}^d$. Generally speaking, the measures of small-scale and extreme co-movement of assets are independent of each other. For example, it is quite possible to have $\rho_{ij} = 0$ and $\lambda_{ij}^d \ne 0$ and vice versa for a non-Gaussian multi-variate distribution. Embrecht *et al* [11] provide very detailed introduction to the properties of those dependence measures.

To simplify the calculations even more, the large homogenous portfolio (LHP) approximation is often used. Suppose that we increase the number of names in the portfolio while keeping the total exposure size of the portfolio constant so that $n_i = N/M$. Conditional on R_m the loss of the portfolio contains the mean of independent identically distributed random variables, $L_M = \left(1-\bar{R}\right) N \frac{1}{M}\sum_{i=1}^{M} 1_{\{R_i\le d\}}$, which a.s. converges to its conditional expectation as M increases to infinity. We use L without subscript to denote the portfolio loss under LHP assumption.

Proposition 1 ***(LHP Loss)*** *Under Assumption 1*

$$\begin{aligned} L &\equiv \lim_{M\to\infty}\left[\left(1-\bar{R}\right) N \frac{1}{M}\sum_{i=1}^{M} 1_{\{R_i\le d\}}\right] = \left(1-\bar{R}\right) NP\left(R_i \le d | R_m\right) \qquad (10)\\ &= \left(1-\bar{R}\right) NG\left(\frac{d-bR_m}{\sqrt{1-b^2}}\right) \text{ a.s. for any } R_m \in supp\,(G) \end{aligned}$$

Proof. see proposition 4.5 in [14] ■

Based on (10) cdf of L can be expressed in terms of the cdf of R_m

$$P(L\le l) = P\left(R_m \ge d_1(l)\right) \tag{11}$$

$$d_1(l) = \frac{d}{b} - \frac{\sqrt{1-b^2}}{b} G^{-1}\left(\frac{l}{\left(1-\bar{R}\right)N}\right) \tag{12}$$

We use the following notation for the Gaussian distribution:

Notation 2 *Let $\Phi(.)$ and $\phi(.)$ with one agrument denote the cdf and pdf, correspondingly, of a standard normal random variable. Let $\Phi(.,.;\rho)$ and $\phi(.,.;\rho)$ with three arguments denote cdf and pdf of two standard normal random variables with linear correlation ρ, and $\Phi_i(.,.;\rho)$ denote the partial derivative of $\Phi(.,.;\rho)$ with respect to the i'th agrument, e.g. $\Phi_3(.,.;\rho) \equiv \frac{\partial}{\partial\rho}\Phi(.,.;\rho)$.*

For the Gaussian copula we have the familiar formula for LHP loss first derived by Vasicek [33], where we have substituted the asset correlation parameter ρ in place of the factor loading b using the relation (7):

$$L^G = \left(1-\bar{R}\right) N \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho} R_m}{\sqrt{1-\rho}}\right) \tag{13}$$

$$P(L \leq l) = 1 - \Phi\left(d_1^G(l)\right) \tag{14}$$

$$d_1^G(l) = \frac{\Phi^{-1}(p)}{\sqrt{\rho}} - \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \Phi^{-1}\left(\frac{l}{\left(1-\bar{R}\right)N}\right) \tag{15}$$

Vasicek [33] and Schonbucher and Shubert [28] show that LHP approximation is quite accurate for upper tail of the loss distribution even for mid-sized portfolios of about 100 names. We will use a symmetric one factor LHP approximation in this paper for analytical tractability.

2.2 From Loss Distribution to Correlation Surface

It is intuitively clear that the choice of the dependence structure affects the degree of uncertainty about the portfolio loss. Indeed, if all issuers in the portfolio are completely independent of each other and any common driving factor, then the law of large numbers assures that the portfolio loss under the LHP approximation is a well determined number with little uncertainty about it. On the other extreme, if the issuers are highly dependent such that they all default or survive at the same time, then the portfolio loss has a binary outcome – it is either zero or equal to the maximum loss.

It is easy to quantify this intuitive result. While the mean of the loss distribution is not affected by the choice of copula, one can show that the second and higher moments of the loss distribution depend on the copula characteristics. In particular, the variance of the loss can be expressed in terms of bivariate default correlation coefficient $\rho^d(p)$ defined in section 2.1. Under assumption of equal default probabilities for all obligors, it is given by:

$$Var(L) = (1-\bar{R})^2 N^2 p(1-p)\rho^d(p) \tag{16}$$

Thus, in line with the intuition, the uncertainty of the default loss distribution is directly proportional to the default correlation coefficient.

To characterize the shape of the entire distribution of L, we must look at the particular slices of portfolio loss. Let $(K_d, K_u]$ denote a tranche with attachment point K_d and detachment point K_u expressed as fractions of the reference portfolio notional so that $0 \leq K_d < K_u \leq 1$. The notional of the tranche is defined as $N_{(K_d,K_u]} = N(K_u - K_d)$ where N is the notional of the portfolio. The loss $L_{(K_d,K_u]}$ of the tranche is the fraction of L that falls between K_d and K_u. For simplicity, assume that total notional N is normalized to 1.

$$L_{(K_d,K_u]} = f_{(K_d,K_u]}(L) \tag{17}$$

$$f_{(K_d,K_u]}(x) \equiv (x - K_d)^+ - (x - K_u)^+ \tag{18}$$

Tranches with zero attachment point, $(0, K_u]$, and unit detachment point, $(K_d, 1]$, are called equity and senior tranches, respectively. Loss of any tranche can be decomposed into losses of either two equity or two senior tranches tranches $L_{(K_d,K_u]} = L_{(0,K_u]} - L_{(0,K_d]} = L_{(K_d,1]} - L_{(K_u,1]}$ This is similar to representing a spread option as a long/short position using either calls or puts.

The expected loss of the equity tranche $L_{(0,K]}$ depends on the portfolio loss distribution and under the LHP approximation can be computed using only the distribution of the market factor:

$$\begin{aligned} EL_{(0,K]} &= Ef_{(0,K]}(L) \\ &= \left(1 - \bar{R}\right) E\left[G\left(\frac{d - bR_m}{\sqrt{1-b^2}}\right) 1_{\{R_m \geq d_1(K)\}}\right] + KP\left(R_m < d_1(K)\right) \end{aligned} \tag{19}$$

The expectation in (19) can be computed by Monte Carlo simulation or numerical integration if we know the cdf of residuals G and the distribution of R_m (see appendix C). For the Gaussian copula, the integral can be calculated in a closed form:

$$E^G L_{(0,K]} = \left(1 - \bar{R}\right) \Phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) + K\Phi(d_1) \tag{20}$$

$$d_1 = \frac{1}{\sqrt{\rho}}\Phi^{-1}(p) - \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\Phi^{-1}\left(\frac{K}{1-\bar{R}}\right) \tag{21}$$

Because of its analytical tractability, it is convenient to use the Gaussian copula as a benchmark model when comparing different choices of dependence structure. By finding the asset correlation level ρ that replicates the results of more complex portfolio loss generating models in the context of a Gaussian copula framework, we can translate the salient features of such models into mutually comparable units. More specifically, we define the correlation surface as follows:

Definition 3 *Suppose the loss distribution of a large homogeneous portfolio is generated by a model* $\left\{C, p, \bar{R}\right\}$ *with copula* C*, identical individual default probabilities* p *and recovery rates* $\bar{R}$*. Let* $L_{(0,K]} \in \left[pf_{(0,K]}\left(1-\bar{R}\right),\ f_{(0,K]}\left(\left(1-\bar{R}\right)p\right)\right]$

be the expected loss of the equity tranche $(0,K]$ *. We define the* ***correlation surface*** $\rho(K,p,\bar{R})$ *of the model* $\left\{C,p,\bar{R}\right\}$ *as the correlation parameter of the Gaussian copula that produces the same expected loss* $EL_{(0,K]}$ *for the tranche* $(0,K]$ *for the given horizon* T *and given single-issuer cumulative default probability* p*:*

$$\begin{gathered}\rho(K,p,\bar{R}) \text{ solves } E^G L_{(0,K]}(\rho) = EL_{(0,K]} \text{ for all } K \in [0,1] \\ \text{where } EL_{(0,K]} \text{ is expected loss of the tranche} \\ (0,K] \text{ generated by model } \left\{C,p,\bar{R}\right\}\end{gathered} \tag{22}$$

where $E^G L_{(0,K]}$ *is defined in (20).*

The correlation surface as defined above is closely related but not identical to the notion of the base correlation used by many practitioners [25]. The difference is that the base correlation is defined using the prices of the equity tranches, which in turn depend on interest rates, term structure of losses, etc. By contrast, the correlation surface is defined without a reference to any market price. It characterizes the portfolio loss generating model, rather than the supply/demand forces in the market.

One could, of course, take another logical step, and instead of deriving the correlation surface from the parameters of the dynamic loss generating model, go in the opposite direction – find such parameters of the loss generating model that result in the closest match to the market prices of CDO tranches. It would be natural to call this solution "implied parameters", and the corresponding function $\rho(K,p,\bar{R})$ "implied correlation surface". The latter would, in fact, coincide with the conventionally defined base correlation.

To ensure that the correlation surface is well defined we need to prove that (22) has a unique solution. Let us first prove the following:

Proposition 4 *For the Gaussian copula, the expected loss of an equity tranche,* $E^G L_{(0,K]}$*, is a monotonically decreasing function of* ρ *and attains its maximum when* ρ *is equal to 0 and minimum when* ρ *is 1.*
Proof. *Using Notation 2, Eq. (20) and the properties of Gaussian distribution*[1]

[1] The following properties of two dimensional Gaussian cdf are used in the calculation

$$\Phi_2(x,y;\rho) \equiv \frac{\partial}{\partial y}\Phi(x,y;\rho) = \phi(y)\,\Phi\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)$$
$$\Phi_3(x,y;\rho) \equiv \frac{\partial}{\partial \rho}\Phi(x,y;\rho) = \phi(x,y;\rho)$$

First formula is derived by taking the derivative and re-arranging the terms. The proof of the second can be found in Vasicek([34])

we derive

$$
\begin{aligned}
E_\rho^G L_{(0,K]} &\equiv \frac{\partial}{\partial \rho} E^G L_{(0,K]} &(23)\\
&= -\left(1-\bar{R}\right)\left[\frac{1}{2\sqrt{\rho}}\Phi_3\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) + \Phi_2\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right)\frac{\partial}{\partial b}d_1\right]\\
&+ K\phi\left(d_1\right)\frac{\partial}{\partial b}d_1\\
&= -\frac{1-\bar{R}}{2\sqrt{\rho}}\phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) < 0 &(24)
\end{aligned}
$$

for any $\rho \in (0,1)$. ■

Therefore, there is a one-to-one mapping between loss distribution and correlation surface, and our transformation does not lead to any loss of information. The next proposition shows how to calculate the loss cdf using the correlation surface and its slope along the K-dimention.

Proposition 5 *Suppose* $\rho(K,p,\bar{R})$ *is the correlation surface for model* $\left\{C,p,\bar{R}\right\}$ *and the probability distribution function of the portfolio loss is a continuous function then the loss cdf can be computed from the correlation surface:*

$$
P\left(L \le K\right) = P^G\left(L \le K\right) + \rho_K E_\rho^G L_{(0,K]} \tag{25}
$$

where $\rho \equiv \rho(K,p,\bar{R})$ *is the level of the correlation surface,* $\rho_K \equiv \frac{\partial}{\partial K}\rho(K,p,\bar{R})$ *is the correlation surface slope and*

$$
P^G\left(L \le K\right) = 1 - \Phi\left(d_1\right) \tag{26}
$$

$$
E_\rho^G L_{(0,K]} = \left(1-\bar{R}\right)\frac{1}{2\sqrt{\rho}}\phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) \tag{27}
$$

$$
d_1 = \frac{1}{\sqrt{\rho}}\Phi^{-1}(p) - \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\Phi^{-1}\left(\frac{K}{1-\bar{R}}\right) \tag{28}
$$

Proof. *first note that the derivative with respect to* K *of the expected tranche's loss under true copula* C *is related to the cdf of the loss*

$$
\begin{aligned}
\frac{d}{dK}EL_{(0,K]} &= \frac{d}{dK}E\left(L - \left(L-K\right)_+\right) &(29)\\
&= -\frac{d}{dK}E\left(L-K\right)_+ = E1_{\{L-K\ge 0\}} = 1 - P\left(L \le K\right)
\end{aligned}
$$

therefore

$$\begin{aligned}
P(L \le K) &= 1 - \frac{d}{dK} EL_{(0,K]} && (30)\\
&= 1 - E_K^G L_{(0,K]} - \rho_K E_\rho^G L_{(0,K]}\\
&= 1 - \Phi(d_1) + \frac{1-\bar{R}}{2\sqrt{\rho}} \phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) \rho_K\\
&= P^G(L \le K) + \left(1-\bar{R}\right) \frac{1}{2\sqrt{\rho}} \phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) \rho_K
\end{aligned}$$

where partial derivative with respect to K is computed as

$$\begin{aligned}
E_K^G L_{(0,K]} &\equiv \frac{\partial}{\partial K} E^G L_{(0,K]} && (31)\\
&= -\left(1-\bar{R}\right) \Phi_2\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) \frac{\partial}{\partial K} d_1 + K\phi(d_1) \frac{\partial}{\partial K} d_1 + \Phi(d_1)\\
&= \Phi(d_1)
\end{aligned}$$

■

Another important point is that the correlation surface depends implicitly on the term to maturity via the cumulative default probability p. However, this is not the only dependence – potentially, the dependence structure characterized by the copula C also exhibits some time dependence when viewed within the context of the Gaussian copula. This statement needs a clarification – the copula C itself is defined in a manner that encompasses all time horizons and therefore cannot depend on any particular horizon. However, when we translate the tranche loss generated with this dependence structure into the simpler Gaussian model the transformation that is required may depend on the horizon T. In other words, the shape of the correlation surface $\rho(K, p, \bar{R})$ may depend on the horizon. We study this dependence in detail in section 4.

These results allow us to further develop the often mentioned analogy between the role that default correlation plays in tranche pricing on one hand and the role that implied volatility plays in equity derivatives pricing, on the other. The starting point of this analogy is the result (16), which shows that default correlation is directly related to the uncertainty of portfolio loss. If we recall that the CDO equity tranche is essentially a call option on portfolio survival, it becomes clear that the price of the equity tranche should be positively related to the default (asset) correlation, just as the price of any equity option is positively related to the implied volatility of its underlying stock.

The correlation surface takes this analogy one step further. Just as the implied volatility surface is sufficient to derive the risk-neutral distribution of the underlying stock (see [8], [9], [30], [31]) and price any European option, we have shown that the correlation surface is sufficient to derive the full portfolio loss distribution and price any CDO tranche.

3 Time Series Approach to Tail Risk

3.1 Motivation for the time series approach

The analogy between the correlation surface and implied volatility surface introduced in the previous section leads to further insight about the origins of the large credit portfolio loss risks. Both equity and credit derivatives pricing exhibits substantial deviations from the simplest Gaussian models of the underlying assets. In particular, the equity index options implied volatility exhibits a steep downward skew of implied volatility, and the CDO tranches exhibit a steep upward skew of implied (base) correlations. To further underline the similarities between these skews, note that a senior CDO tranche with an attachment point that is higher than the expected loss on the underlying portfolio can be thought as an out-of-money put option on portfolio losses. When looked from this angle, the equity implied volatility skew and credit correlation skew are tilted the same way – towards the farther out-of-money options.

Recall that the empirical distribution of returns does indeed exhibit significant downside tails, and that a large part of the implied volatility skew can be explained by the properties of the empirical distribution [7]. Given the above mentioned analogies between the synthetic CDO tranches and equity index options, it is quite natural to look for a similar explanation of the implied correlation skew.

The standard Gaussian copula framework implicitly relies on the Merton-style structural model for definition of default correlations. Therefore, if we are to give an empirical explanation to the observed base correlation skew we must start by giving an empirical meaning to the variables in this model. Our working hypothesis in this paper will be that the meaning of the "market factor" in the factor copula framework is the same as the market factor used in the equity return modeling. As such, it is often possible to use an observable broad market index such as S&P 500 as a proxy for the economic market factor, with an added convenience that there exists a long historical dataset for its returns and a rich set of equity options data from which one can glean independent information about their implied return distribution.

This hypothesis is not uncommon in portfolio credit risk modeling – for example, the authors of [23] emphasized the importance of using a fat-tailed distribution of asset returns in the copula framework in part by citing the empirical evidence from equity markets. However, most researchers have focused on the single-period return distribution characteristics.

In contrast, we focus in this paper on the long-run cumulative returns, and prove that their distribution is quite distinct from that of the short-term (single-period) returns. As we will show in the rest of this paper, it is the time aggregation properties and the compounding of the asymmetric volatility responses that make it possible to explain the credit correlation skew for 5- or even 10-year horizons. Moreover, this dynamic explanation of the skew allows one to make rather specific predictions for the dependence of this skew on both the term to maturity and on the hazard rates and other model parameters.

We first specify time series properties of stock returns for high frequency time intervals (daily or weekly) and then derive the distribution of stock prices over longer horizons measured in months or even years. Assuming geometric Brownian motion of stock prices on short intervals leads to the same log-normal shape for the distribution of stock prices for all future horizons. Models with more realistic dynamics can lead to richer distributions of the time aggregated returns even if the high frequency shocks are Gaussian.

In the GARCH family of models [12], [3], [6], there have been many investigations of the difference between the conditional and the unconditional distributions. These models reveal an important explanation for the excess kurtosis in financial returns but generally show no reason to expect skewness in returns. Diebold [5] investigates the implications of time aggregation of GARCH models concluding that eventually they lose their excess kurtosis as the central limit theorem leaves the average distribution Gaussian.

Asymmetric volatility models were introduced by Nelson [26] and further investigated by Glosten, Jaganathan and Runkle [15] and by Zakoian [35]. The model to be discussed in this paper is essentially the GJR model and will be called the Threshold-ARCH or TARCH. These models all show that negative returns forecast higher volatility than positive returns of the same magnitude. This observation is very widespread and is sometimes called the leverage effect following Black [2]. However, it is most likely due to risk aversion as in Campbell and Hentschell [4] and in this paper we will refer to it simply as asymmetric volatility.

The first indication that skewness could arise from time aggregation was presented in Engle [13]. He showed that with asymmetric volatility, the skewness of time aggregated returns could be more negative than the skewness of the individual innovations. In the next section we show analytically how this negative skewness depends upon time aggregation. We then examine the multivariate distribution when the single common factor has TARCH dynamics.

3.2 Univariate model: TARCH(1,1)

Let r_t be the log-return of a particular stock or an index such as SP500 from time $t-1$ to time t . $\digamma_t$ denotes the information set containing realized values of all the relevant variables up to time t. We will use the expectation sign with subscript t to denote the expectation conditional on time t information set: $E_t(.) = E(.|\digamma_t)$. The time step that we use in the empirical part is 1 day or 1 week. Predictability of stock returns is negligible over such time horizons and therefore we assume that the conditional mean is constant and equal to zero:

$$m_t \equiv E_{t-1}(r_t) = 0 \tag{32}$$

The conditional volatility $\sigma_t^2 \equiv E_{t-1}(r_t^2)$ of r_t in TARCH(1,1) has the autoregressive functional form similar to the standard GARCH(1,1) but with an additional asymmetric term [15], [35]:

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \omega + \alpha r_{t-1}^2 + \alpha_d r_{t-1}^2 1_{\{r_{t-1} \le 0\}} + \beta \sigma_{t-1}^2 \end{aligned} \tag{33}$$

where $\{\varepsilon_t\}$ are iid , have zero mean, variance normalized to 1, finite skewness s_ε and finite kurtosis k_ε. We also assume that $\omega > 0$ and α, α_d, β are non-negative so that the conditional variance σ_t^2 is guaranteed to be positive.

The persistence of volatility in the model is governed by the parameter ζ :

$$\zeta \equiv E\left(\beta + \alpha \varepsilon_t^2 + \alpha_d \varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right) = \beta + \alpha + \alpha_d v_\varepsilon^d \tag{34}$$

where $v_\varepsilon^d \equiv E\left(\varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right)$ is the "right truncated variance" of ε_t . If $\zeta \in [0,1)$ then conditional variance mean-reverts to its unconditional level $\sigma^2 = E\left(\sigma_t^2\right) = \frac{\omega}{1-\zeta}$. The following parameter ξ will also be useful in describing the higher moments of TARCH(1,1) returns and volatilities:

$$\xi \equiv E\left(\beta + \alpha \varepsilon_t^2 + \alpha_d \varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right)^2 = \beta^2 + \alpha^2 k_\varepsilon + \alpha_d^2 k_\varepsilon^d + 2\alpha\beta + 2\alpha_d \beta v_\varepsilon^d + 2\alpha\alpha_d k_\varepsilon^d \tag{35}$$

where $k_\varepsilon^d \equiv E\left(\varepsilon_t^4 1_{\{\varepsilon_t \le 0\}}\right)$ is the "right truncated kurtosis" of ε_t. We can rewrite (33) in terms of the increments of the conditional volatility $\Delta\sigma_{t+1}^2 \equiv \sigma_{t+1}^2 - \sigma_t^2$ and the volatility shocks η_t

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \Delta\sigma_{t+1}^2 &= (1-\zeta)\left(\sigma^2 - \sigma_t^2\right) + \sigma_t^2 \eta_t \\ \eta_t &\equiv \alpha\left(\varepsilon_t^2 - 1\right) + \alpha_d\left(\varepsilon_t^2 1_{\{\varepsilon_t \le 0\}} - v_\varepsilon^d\right) \end{aligned} \tag{36}$$

The speed of mean reversion in volatility is $1-\zeta$ and is small when ζ is close to one which is usually true for daily and weekly equity returns – hence the persistence of the volatility. The TARCH(1,1) volatility shocks η_t are iid, with zero mean and constant variance $var(\eta_t) = var(\alpha\varepsilon_t^2 + \alpha_d \varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}) = \xi - \zeta^2$. The correlation of conditional volatility with the return in the previous period depends on the covariance of return and volatility innovations:

$$corr_{t-1}\left(r_t, \sigma_{t+1}^2\right) = corr_{t-1}\left(\varepsilon_t, \eta_t\right) = \frac{\alpha s_\varepsilon + \alpha_d s_\varepsilon^d}{\sqrt{\xi - \zeta^2}} \tag{37}$$

where $s_\varepsilon^d = E\left(\varepsilon_t^3 1_{\{\varepsilon_t \le 0\}}\right) < 0$ is the "right truncated" skewness of ε_t.The negative correlation of return and volatility shocks, often cited as the "leverage effect"[2], is the main source of the asymmetry in the return distribution. We can see from formula (37) that negative return-volatility correlation can be

[2] Though we note here that the magnitudfe of this "leverage effect" in return time series for stocks of most investment grade issuers far exceeds the amount that would be reasonable based purely on their capital structure leverage.

achieved either through negative skewness of return innovations $s_\varepsilon < 0$, through asymmetry in volatility process $\alpha_d > 0$ or combination of the two. We call these static and dynamic asymmetry, respectively.

In this paper we are interested in the effects of the volatility dynamics on the distribution of long horizon returns which in the log representation is a sum of short term log returns $R_{t,t+T} \equiv \ln S_{t+T} - \ln S_t = \sum_{u=t+1}^{t+T} r_u$. While a closed form solution for the probability density function of TARCH(1,1) aggregated returns is not available, we can still derive some analytical results for its conditional and unconditional moments: volatility, skewness and kurtosis. Since TARCH is linear autoregressive volatility process, the conditional variance $V_{t,t+T}$ of the log return $R_{t,t+T}$ is linear in σ_{t+1}^2 :

$$V_{t,t+T} = E_t R_{t,t+T}^2 = E_t \left(\sum_{t+1 \le u \le t+T} \sigma_u^2 \right) = T \left(\sigma^2 + \left(\sigma_{t+1}^2 - \sigma^2 \right) \frac{1}{T} \frac{1-\zeta^T}{1-\zeta} \right) \tag{38}$$

The new result of this paper for the TARCH(1,1) model is the representation of the skewness term structure derived in the following proposition.

Proposition 6 *Suppose $0 \le \zeta < 1$ and the return innovations have finite skewness, s_ε, and finite "truncated" third moment, s_ε^d. Then the conditional third moment of $R_{t,t+T}$ has the following representation for TARCH(1,1)*

$$E_t R_{t,t+T}^3 = s_\varepsilon \sum_{u=1}^{T} E_t \left(\sigma_{t+u}^3 \right) + 3 \left(\alpha s_\varepsilon + \alpha_d s_\varepsilon^d \right) \sum_{u=1}^{T} \frac{1-\zeta^{T-u}}{1-\zeta} E_t \left(\sigma_{t+u}^3 \right) \tag{39}$$

In addition, if $E\sigma_t^3$ is finite, then unconditional skewness of $R_{t,t+T}$ is given by

$$S_T \equiv \frac{E R_{t,t+T}^3}{E(R_{t,t+T}^2)^{3/2}} = \left[\frac{1}{T^{1/2}} s_\varepsilon + 3 \frac{1}{T^{3/2}} \left(\alpha s_\varepsilon + \alpha_d s_\varepsilon^d \right) \frac{T(1-\zeta) - 1 + \zeta^T}{(1-\zeta)^2} \right] E \left(\frac{\sigma_t}{\sigma} \right)^3 \tag{40}$$

Proof. *See appendix A for the details.* ■

The conditional third moment is a function of the conditional term structure of σ_t^3, term horizon T and volatility parameters. The conditional skewness can be computed using second and third conditional moments derived above. The asymmetry in the return distribution arises from two sources - skewness of return innovations and asymmetry of the volatility process. Note that the second term in the formulas for conditional and unconditional skewness is directly related to the correlation of return and volatility innovations. If return-volatility correlation is zero $\left(\alpha s_\varepsilon + \alpha_d s_\varepsilon^d = 0 \right)$ then $S_T = \frac{1}{T^{1/2}} s_\varepsilon E \left(\frac{\sigma_t}{\sigma} \right)^3$. If return innovations are symmetric then asymmetric volatility drives the asymmetry in the return distribution. In figure 1 we show conditional and unconditional skewness term structures. For realistic parameters corresponding approximately to

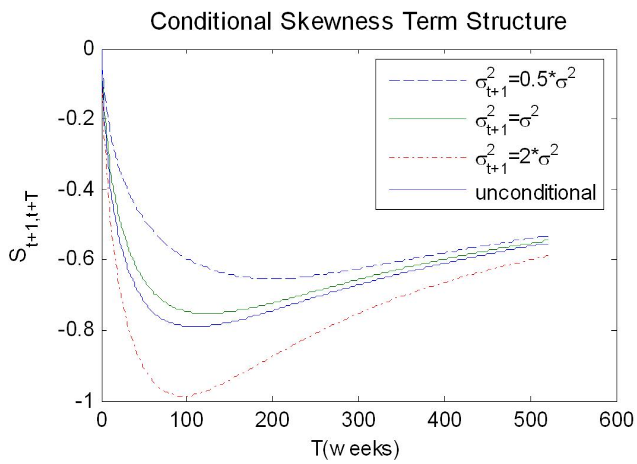

Figure 1: Term structure of conditional skewness of time aggregated return $R_{t+1,t+T}$.TARCH(1,1) has persistence coefficient $\zeta = 0.98$ and the following parametrization: $\sigma^2 = 1$, $\alpha = 0.01$, $\alpha_d = 0.10$, $\beta = 0.92$, $\varepsilon_t \sim N(0,1)$. We plotted unconditional skewness term structure and conditional for three different initial volatilities: $\sigma^2/2$, $\sigma^2$and $2\sigma^2$. The term structure of $E_t\sigma^3_{t+u}$was computed from 10,000 independent simulations.

parameters of the TARCH(1,1) estimated for weekly SP500 log returns, both conditional and unconditional skewness is negative. It decreases in the medium term, attains the minimum at approximately the 2 year point and then decays to zero as T increases. The skewness term structure conditional on the high/low current volatility is above/below the unconditional skewness.

To provide some empirical context to the theoretical discussion above, let us consider the time series of SP500 returns. The results of the estimation of various TARCH(1,1) specification are shown in the appendix B. Figure 2 shows the estimate of skewness for overlapping returns of different aggregation horizons measured in days. The full sample shows high negative skewness for one day return because of the 1987 crash. On the post 1990 sample negative skewness rises with aggregation horizon up to 1 year and then slowly decays toward zero. Both samples show significant skewness for horizons of several years. We should note that confidence bounds around skewness curves are quite wide due to the persistence and high volatility of the squared returns and serial correlation of the overlapping observations. Nevertheless, both the shape and the level of the empirical skewness in figure 2 and the theoretical estimate shown in figure 1 are quite similar.

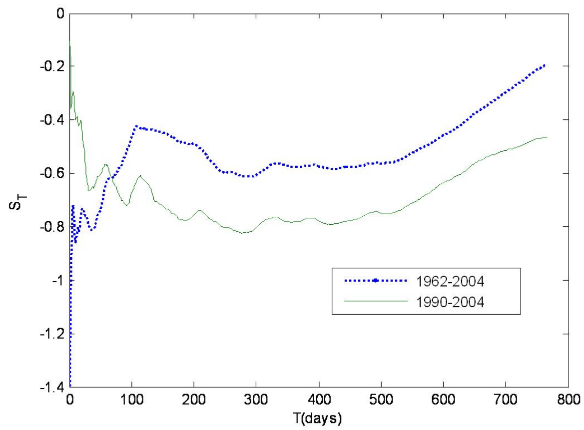

Figure 2: Term structure of skewness for SP500 time aggregated log returns estimated with overlapping samples moments for full and post-1990 data.

3.3 Multivariate model with TARCH(1,1) factor volatility dynamics

Let us now turn to a multi-variate model of equity returns for M companies, with a simple dynamic factor structure decomposing the returns into a common (market) and idiosyncratic components. To concentrate on the time dimension of the model we assume a homogeneity of cross-sectional return properties, namely that factor loadings and volatilities of idiosyncratic terms are constant and identical for all stocks. Thus, our homogeneous one factor ARCH model has the following form.

$$r_{i,t} = br_{m,t} + \sigma\varepsilon_{i,t}$$

where

- $b \geq 0$ is the constant market factor loading and it is the same for all stocks
- $r_{m,t}$ is the market factor return with zero conditional mean $E_{t-1}(r_{m,t}) = 0$, conditional volatility $\sigma^2_{m,t} \equiv E_{t-1}(r^2_{m,t})$ that has TARCH(1,1) parametrization (33)
- $\sigma\varepsilon_{i,t}$ are the idiosyncratic return components with constant volatilities σ^2 and zero conditional means $E_{t-1}(\sigma\varepsilon_{i,t}) = 0$
- $\{\varepsilon_{i,t}, \varepsilon_{m,t}\}$ are unit variance iid shocks for each t and all i

Because of the simple linear factor structure and constant market loadings time aggregated equity returns $R_{i,T} = \sum_{u=1}^{T} r_{i,u}$ also have a one factor representation[3]

$$R_{i,T} = bR_{m,T} + E_{i,T} \tag{41}$$

where $R_{m,T} = \sum_{u=1}^{T} r_{m,u}$ and $E_{i,T} = \sigma \sum_{u=1}^{T} \varepsilon_{i,u}$ are independent conditional on $\mathcal{F}_0$.

3.4 Tail risk and default correlation estimates

Assuming that the latent variables in the copula framework follow the one factor TARCH(1,1) dynamics, we can calculate the coefficient of lower tail dependence $\lambda^d_{i,j}$ and the default correlation coefficient $\rho^d_{i,j}(p)$, which were defined in equations (8) and (9), respectively. Default correlation coefficient for Gaussian and Student-t can be computed in closed form, while the factor GARCH/TARCH models require Monte Carlo simulation, which is desribed in appendix C. Figure 3 shows the numerical estimates of the default correlation $\rho^d_{1,2}$ as a function of p for 4 different models - TARCH, GARCH and Student-t and Gaussian copulae.

The linear correlation of latent returns is set to 0.3 for all 3 models. TARCH and GARCH are calibrated to have volatility dynamics parameters corresponding approximately to the weekly SP500 returns and the time aggregation horizon is set to 5 years. The degrees of freedom parameter for the Student-t idiosyncracies and T-Copula is set to be equal to 12.

The GARCH distribution is symmetric and has smaller tail dependence for both upper and lower tails. We can see on the graph that it also has lowest default correlation for all default probabilities in the range of [0.01,0.2]. The Student-t copula is also symmetric but has fatter joint tails compared to the GARCH. Its default correlation is above GARCH for all p and converges to a positive number (the tail dependence coefficient) as p decreases to zero.

We can see that TARCH has higher default correlation than other 2 models and is upward sloping for very low quantiles. The upturn for the extreme tails is a consequence of the left tail shape of the common factor. The default correlation for very low default probabilities should be close to 1 since the left tail of the factor is fatter than the left tail of the idiosyncratic shocks. As we showed in the previous sections, both kurtosis and skewness of the market factor declines faster for GARCH than TARCH given the same level of volatility persistence.

We also show the 95% confidence bounds for default correlation, which are calculated using 1000 independent repetitions of 10,000 Monte Carlo simulations of the common factor. TARCH, T-Copula and GARCH models have virtually non-overlapping confidence bounds for $\rho^d_{i,j}(p)$ for sufficiently low values of the default probability [0.01,0.1]. Since the correlation skew for sufficiently large values of the detachment point K is also related to the limit of low default probabilities, we can therefore rely on this result in the next section to assert that the differences between the correlation surfaces generated by these three models are statistically significant.

[3] To simplify the notations we assume that the initial time $t = 0$ and use only subscipt for the time aggregation horizon T.

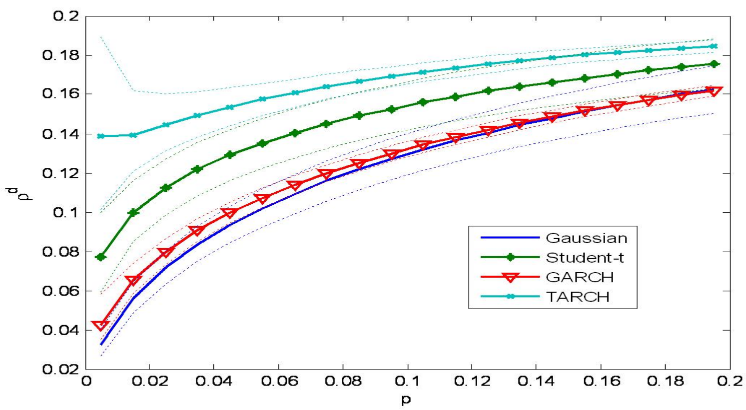

Figure 3: Default correlation as a function of p for Gaussian copula, T-Copula, GARCH and TARCH models. The linear correlation parameter is 0.3 for all four models. TARCH(GARCH) parameters are $\alpha = 0.01(0.06)$, $\alpha_d = 0.1(0)$, $\beta = 0.92(0.92)$. The degrees of freedom parameter for T-Copula and TARCH Student-t idiosyncracies is 12.

4 Comparing Credit Portfolio Loss Generating Models

Our goal in this paper is to provide a general framework for judging the versatility of various portfolio credit risk models. All such models, whether defined via dynamic multivariate returns model like in this paper or in various versions of the static copula framework ([1], [14], [16], [22], [23] and [28]), can be characterized by the full term structure of loss distributions. Thus, without loss of generality, we can refer to all models of credit risk as loss generating models, with an implicit assumption that any two models that produce identical loss distributions for all terms to maturity are considered to be equivalent.

The correlation surface, introduced in section 2.2, conveniently transforms specific choice of a loss generating model into a two dimensional surface $\rho(K, T)$ of the Gaussian copula correlation parameter, with the main dimensions being the loss threshold (detachment level) K and the term to maturity T. All other inputs such as the recovery rate R, the term structure of (static) hazard rates h, the level of linear asset correlation ζ, the Student-t degrees of freedom ν, various GARCH model coefficients, etc. – are considered as model parameters upon which the two-dimensional correlation surface itself depends.

Note that in the previous sections we have expressed the correlation surface as a function of detachment level and the underlying portfolio's cumulative expected default probability p rather than the term to maturity T. Given our assumption of the static term structure of the hazard rates h these two formu-

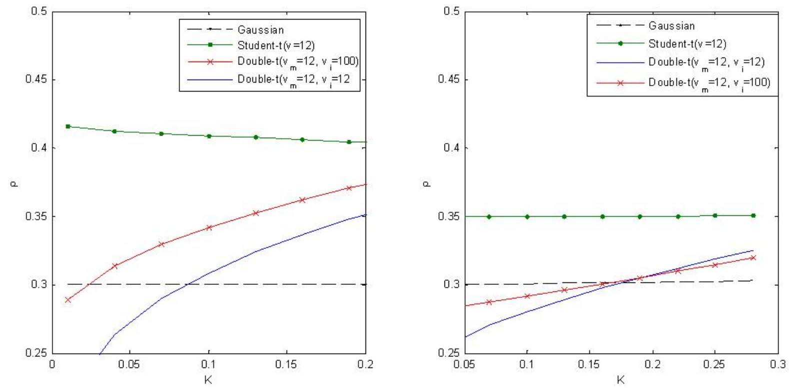

Figure 4: Correlation surface slices corresponding to 1-year (left chart) and 5-year (right chart) horizons with default probabilities 0.02 and $1-(1-0.02)^5 = 0.0961$ for Gaussian, Student-t Copula with 12 degrees of freedom, and Double-t Copulas with (v_m =12, v_i =100) and (v_m =12, v_i =12) degrees of freedom. Linear correlation is 0.3 for all copulas.

lations are equivalent. In this section we prefer to emphasize the dependence on maturity horizon in order to facilitate the comparison with base correlation models and also to analyze the dependence on the level of hazard rates separately from the term to maturity dimension.

Of course, the correlation surface of a static Gaussian copula model [22] is a flat surface with constant correlation across both detachment level K and term to maturity T. Any deviation from a flat surface is therefore an indication of a non-trivial loss generating model, and we can judge which features of the model are the important ones by examining how strong a deviation from flatness they lead to.

4.1 Models with static dependence structure

Let us begin with the analysis of one of the popular static loss generation models. On figure 4 we show the correlation surface computed for the Student-t copula with linear correlation $\rho = 0.3$ and $\nu = 12$ degrees of freedom. Student-t copula is in the same elliptic family as the Gaussian copula but has non-zero tail dependence governed by the degrees of freedom parameter. As a model of single-period asset returns the Student-t distribution has been shown to provide a significantly better fit to observations than the standard normal [23].

However, from the figure 4 we can see that the static Student-t copula does not generate a notable skew in the direction of detachment level K, and in fact generates a mild downward sloping skew for very short terms, which is contrary to what is observed in the market. The main reason for this is the rigid structure of this model, with the tails of the idiosyncratic returns tied closely to the tails of the market factor. This follows from the representation of the Student-t copula as a mixture model [29]. Instead of producing a varying degree of correlation depending on the default threshold, the Student-t copula model simply produces a higher overall level of correlation.

On the other hand, the more flexible double-t copula model [20] produces a steep upward sloping skew, as can be seen from figure 4. The main feature of the double-t copula that is responsible for the skew is the cleaner separation between the common factor and idiosyncratic returns – there is no longer a single mixing variable which ties the two sources of risk together. As a result, the idiosyncratic returns get efficiently diversified in the LHP framework and their contribution becomes progressively smaller for farther downside returns. Since the higher value of the detachment level K corresponds to farther downside tails, the greater dominance of the market factor translates into higher effective correlation for higher K, i.e. upward sloping correlation skew.

Furthermore, by making the fully independent idiosyncratic returns more fat tailed one achieves a steeper skew – compare the two examples of the double-t copula, with the degrees of freedom of the idiosyncratic returns set to 100 (i.e. nearly Gaussian case) and to 12 (i.e. strongly fat-tailed case), respectively. Indeed, for the same detachment level K the idiosyncratic returns with lower degrees of freedom (stronger fat tails) are less dominated by the market factor, resulting in relatively lower effective correlation. Since the difference between two cases diminishes as K grows, this translates into steeper correlation skew for fatter-tailed idiosyncratic returns.

Finally, we observe that the slope of the correlation skew gets flatter as the time horizon grows. Within the context of double-t copula this is simply because the same detachment level K corresponds to less extreme tails when the term to maturity is greater. Following the same logic as above, this means less steep correlation skew.

All of these features will have their close counterparts in the dynamic models which we will consider next.

4.2 Multi-period (dynamic) loss generating models

Let us now turn to loss generating models based on latent variables with multi-period dynamics. We have concluded in the previous section that a clean separation of the market factor and the idiosyncratic returns appears to be a prerequisite for producing an upward sloping correlation skew. Fortunately, the dynamic multi-variate models which we considered in section 3 all have this property, both for single-period and for aggregated returns.

On figure 5 we show the correlation surface computed for a loss generating model based on GARCH dynamics with Gaussian residuals, with a linear correlation set to $\rho = 0.3$, and GARCH model parameters taken from the weekly SP500 estimates in appendix B. The loss distributions and the correlation surfaces are calculated using a Monte Carlo simulation with 100,000 trials.

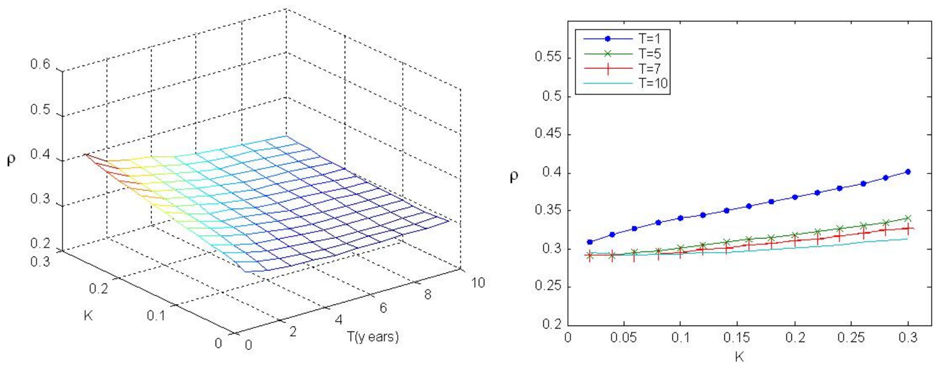

Figure 5: Correlation surface for GARCH model (α=0.045, β=0.948) with Gaussian shocks and the slices of the surface for 1, 3, 5 and 7 year maturities.

As we can see, this model does exhibit a visible deviation from the flat correlation surface for short maturities. However, as we already noted in section 3, the distribution of aggregate returns for the symmetric GARCH model quickly converges to normal. Therefore, it is not surprising to see that the correlation surface also flattens out fairly quickly and becomes virtually indistinguishable from a Gaussian copula for maturities beyond 5 years. Thus, we conclude that the symmetric GARCH model with Gaussian residuals is inadequate for description of liquid tranche markets where one routinely observes steep correlation skews at maturities as long as 7 and 10 years.

Based on the empirical results of Section 3.3 we know that a GARCH model with Student-t residuals provides a better fit to historical time series of equity returns. A natural question is whether allowing for such volatility dynamics can lead to a persistent correlation skew commensurate with the levels observed in synthetic CDO markets.

The results of section 3.3 suggest that the additional kurtosis of the single-period returns represented by the Student-t residuals does not matter very much for aggregate return distributions at sufficiently long time horizons. Indeed, figure 6 shows that the GARCH model with Student-t residuals exhibits a correlation skew that is quite a bit steeper at the short maturities, yet is almost as flat and featureless at the long maturities as its non-fat-tailed counterpart – there is a small amount of skew at 10 years, but it is too small compared to the steepness observed in the liquid tranche markets. Thus, we conclude that one has to focus on the dynamic features of the market factor process in order to achieve the desired correlation skew effect.

Our next candidates are the TARCH models with either Gaussian or Student-t return innovations. We have seen in section 3 that the asymmetric volatility dynamics of these models leads to a much more persistent skewness and kurtosis of aggregated equity returns that actually grow rather than decay at very

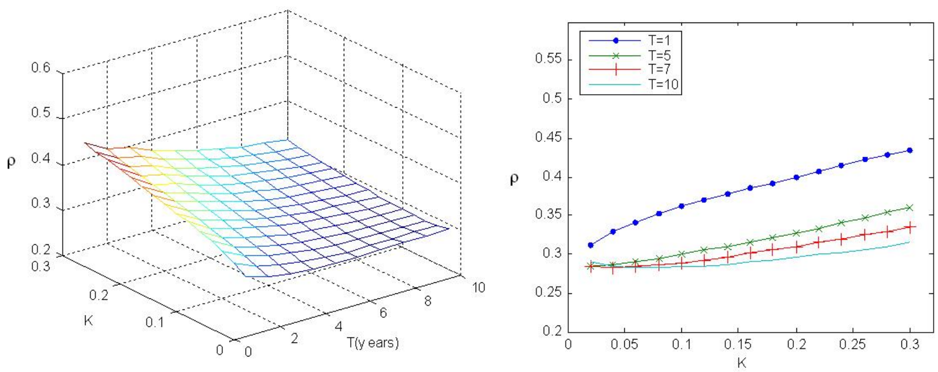

Figure 6: Correlation surface for GARCH model (α=0.045, β=0.948) with Student-t shocks (v=8.3) and the slices of the surface for 1, 3, 5 and 7 year maturities.

short horizons, and survive for as long as 10 years for the range of parameters corresponding to the post-1990 sample of SP500 weekly log-returns. Hence, our hypothesis is that a latent variable model with TARCH market dynamics might be capable of producing a non-trivial credit correlation skew for maturities of up to 10 years.

Figures 7 and 8 show the correlation surfaces for the TARCH-based loss generating models. The most immediate observation is that both versions of the model produce a rather persistent correlation skew. Although the correlation surface flattens out with growing term to maturity, the steepness of the skew is still quite significant even at 10 years. Just as in the case of the symmetric GARCH model, the fat-tailed residuals lead only to marginal steepening of the correlation surface compared to the case with Gaussian residuals.

Contrast these properties of the dynamic GARCH-based models with the features of the static double-t copula. Upon a closer inspection of figures 4 and 7 we can see that the TARCH model with Gaussian shocks and Gaussian idiosyncrasies produces a slightly steeper 5-year correlation skew than the double-t copula, even when the latter is taken with fat-tailed idiosyncrasies. When we turn on the Student-t return residuals for the market factor dynamics (see figure 8) the differences in the 5-year skew become quite significant.

The explanation of the correlation skew in the dynamic TARCH-based models is similar to the static double-t copula when one considers a particular time horizon. The separation of aggregate returns for the common market factor and idiosyncratic factors remains valid for all time horizons. The diminishing importance of the idiosyncratic returns compared to the market factor for the greater values of detachment level K explains most of the steepness of the correlation skew. Practitioners using the static models often have to assume heuristic term structure dependence for base correlations, typically without fundamental reasons why one choice or another is preferred, and consequently leading to

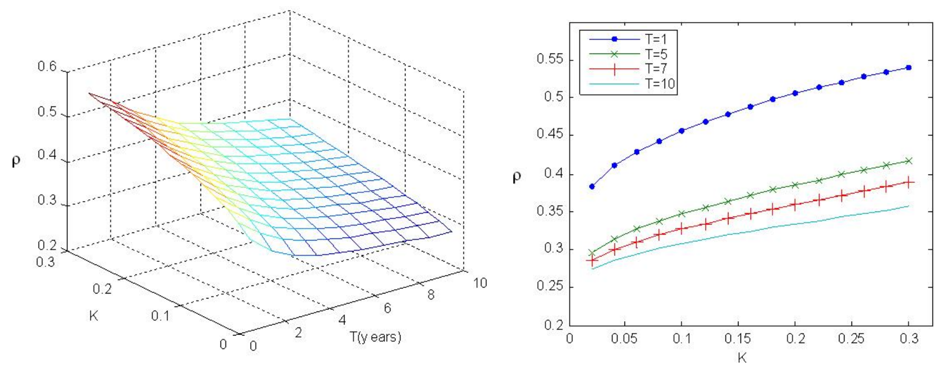

Figure 7: Correlation surface for TARCH model (α=0.004, α_d=0.094, β=0.927) with Gaussian shocks and the slices of the surface for 1, 3, 5 and 7 year maturities.

biased relative value assessments between tranches of different maturities. Our TARCH-based model, in contrast, produces a characteristic pattern of correlation skew dependence on T which is driven by the speed of convergence of the idiosyncratic factors to normal and can serve as a starting point for such relative value assessments. We confine our discussion of this subject to highlighting of this possibility, since its detailed analysis in the context of actual market prices is well beyond the scope of our paper.

4.3 Sensitivity to model parameters and hedging applications

The predictable shape of the correlation surface, demonstrated above, is complemented by an equally important feature of the dynamic loss generating models – they lead to a well-defined dependence of the correlation skew on model parameters. The ability to calculate the sensitivity of expected losses to underlying parameters is crucial in risk management applications. In particular, sensitivity of the correlation surface with respect to the underlying portfolio hazard rate leads to a non-trivial adjustment of the CDO tranche deltas.

First, let us demonstrate this sensitivity for TARCH model with Gaussian idiosyncrasies. Figure 9 shows the correlation skew $\rho\left(K, p_t\left(h\right), \bar{R}\right)$ of the 5-year tranches with various detachment levels K as a function of varying portfolio hazard rate h. The range of variation is chosen from 100bp to 500bp, which corresponds roughly to portfolio spreads ranging from 40bp to 200bp, with $\bar{R} = 0.4$. From the visual comparison of figures 9 and 7 it appears that the dependence of the correlation skew for a fixed term to maturity but varying level of hazard rates is very similar to the dependence of the correlation surface on the term to maturity. This similarity is natural, since the first order effect is the dependence on the level of the cumulative default probability $p_t\left(h\right) = 1 - e^{-ht}$

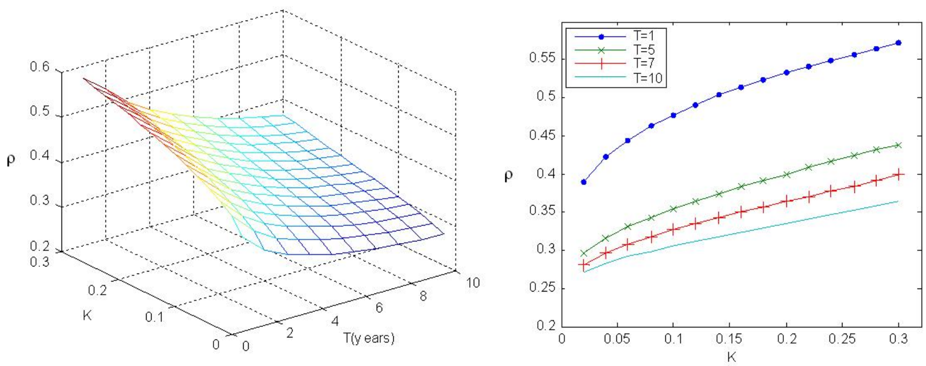

Figure 8: Correlation surface for TARCH model (α=0.004, α_d=0.094, β=0.927) with Student-t shocks(v=8.3) and the slices of the surface for 1, 3, 5 and 7 year maturities.

which depends on the product of $h \cdot t$ rather than on the hazard rate or the term to maturity separately. For each level of this product, we get a specific level of the default threshold in the structural credit risk model. The higher this threshold, the closer is the sampled region to the center of the latent variables distribution and the less it is affected by the tail risk – thus leading to a lower level and flatter skew of the credit correlation.

However, there is a second order effect which makes these two dependencies somewhat different. Let us recall first that idiosyncrasies with less fat tails (higher degrees of freedom) correspond to flatter correlation skew, as we argued in the previous section. Since the idiosyncrasies converge to normal distribution faster than the market factor as the return aggregation horizon grows, we can deduce that the dependence on the term to maturity with fixed hazard rate should exhibit a faster flattening of the correlation surface than the dependence on the hazard rate with fixed term to maturity.

The right hand side figure in 9 shows a comparison of the change in correlation when going from 5-year horizon to 10-year horizon with constant hazard rate set at 100bp, against the change in correlation of fixed 5-year slice when hazard rate goes from 100bp to 200bp. One can observe that the term-to-maturity extension indeed causes a greater degree of flattening than the hazard rate increase.

The dependence of the correlation surface on hazard rate has a strong effect on CDO tranche deltas. The precise calculation of tranche deltas in our framework requires one to derive the tranche loss probabilities from the shape of the correlation surface, as outlined in section 2.2, and then use the standard pricing techniques described in Schonbucher [27]. However, the magnitude of the adjustment can be estimated more easily by neglecting the interest rates and looking only on the protection leg of the given equity tranche. The tranche delta is dominated by the hazard rate sensitivity of the present value of its protection

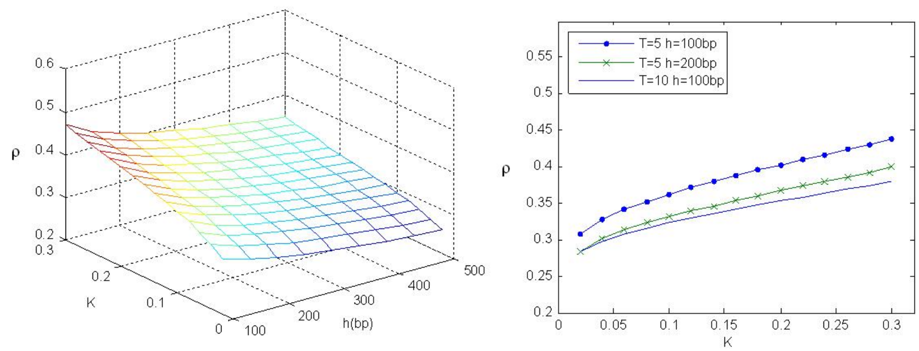

Figure 9: Left figure: the dependence of the 5-year correlation skew on the level of the hazard rates. Right figure shows 3 correlation surface slices, contrasting the flattening of the skew with growing hazard rates and term to maturity.

leg, which in turn is proportional to the expected tranche loss:

$$\Delta_{(0,K]}\left(p_t\right) \propto \frac{dEL_{(0,K]}(t)}{dh} = E_h^G L_{(0,K]}(t) + \rho_h\left(K, p_t, \bar{R}\right) E_\rho^G L_{(0,K]}(t) \qquad (42)$$

where, in accordance with the definition of the correlation surface, we have:

$$EL_{(0,K]}\left(t\right) = E^G L_{(0,K]}\left(\rho\left(K, p_t\left(h\right), \bar{R}\right), p_t\left(h\right)\right) \qquad (43)$$

Let us define the delta adjustment factor as the percentage adjustment to the tranche delta compared to fixed-correlation Gaussian delta of the tranche:

$$\Delta_{(0,K]}\left(p_t\right) = \Delta_{(0,K]}^G\left(p_t\right)\left(1 + \delta_{adj}\left(K, p_t\right)\right) \qquad (44)$$

Since the first term in eq. (42) corresponds to the Gaussian delta of the tranche, we can see that the delta adjustment factor is equal to the correlation surface sensitivity times the tranche loss sensitivity ratio:

$$\delta_{adj}\left(K, p_t\right) = \rho_h\left(K, p_t, \bar{R}\right) \frac{E_\rho^G L_{(0,K]}(t)}{E_h^G L_{(0,K]}(t)} = \rho_h\left(K, p_t, \bar{R}\right) \Psi\left(K, p_t\right) \qquad (45)$$

where

$$E_\rho^G L_{(0,K]}(t) = -\frac{1-\bar{R}}{2\sqrt{\rho}} \phi\left(\Phi^{-1}(p_t), -d_1; -\sqrt{\rho}\right) \tag{46}$$

$$E_h^G L_{(0,K]}(t) \equiv \frac{dp_t}{dh} E_p^G L_{(0,K]}(t) = (1-p_t)\, t \left(1-\bar{R}\right) \Phi\left(\frac{-d_1 + \sqrt{\rho}\Phi^{-1}(p_t)}{\sqrt{1-\rho}}\right) \tag{47}$$

$$d_1 = \frac{1}{\sqrt{\rho}} \Phi^{-1}(p_t) - \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \Phi^{-1}\left(\frac{K}{1-\bar{R}}\right) \tag{48}$$

and the functions are evaluated at $\rho = \rho\left(K, p_t(h), \bar{R}\right)$. Thus, the tranche loss sensitivity ratio is given by:

$$\Psi(K, p_t) = \frac{E_\rho^G L_{(0,K]}(t)}{E_h^G L_{(0,K]}(t)} = -\frac{1}{2(1-p_t)\, t\sqrt{\rho}} \frac{\phi\left(\Phi^{-1}(p_t), -d_1; -\sqrt{\rho}\right)}{\Phi\left(\frac{-d_1+\sqrt{\rho}\Phi^{-1}(p_t)}{\sqrt{1-\rho}}\right)} \tag{49}$$

Figure 10 illustrates these calculations for the case of TARCH-based dynamic loss generating model. We can see that the systematic delta estimates in our model are anywhere from 40% to 100% greater than the conventional fixed-correlation delta estimates stemming from the Gaussian copula model. The adjustment is greatest for the lowest values of K, and drops quickly as the detachment level grows. While comparison with the actual tranche market price sensitivity is beyond the scope of our paper, we would like to note that both the the significant under-estimation of the equity tranche deltas and the relatively smaller amount of the delta error for more senior tranches are in line with the market experience during the correlation dislocation in May-September of 2005.

5 Summary and Conclusions

In this paper we have introduced and studied a new class of credit correlation models defined as an extension of the structural credit model where the latent variables follow a factor-ARCH process with asymmetric volatility dynamics. To build the foundation for our model, we have studied the time aggregation properties of the multivariate dynamic models of equity returns. We showed that the dynamics of equity return volatilities and correlations leads to significant departures from the Gaussian distribution even for horizons measured in several years. The asymmetry appears to "survive aggregation" longer than fat tails based on the parameters estimated from the real data. The main source of skewness and kurtosis of the return distribution for long horizons is the dynamic asymmetry of volatility response to return shocks.

We introduced the notion of the correlation surface as a tool for comparing loss generating models, whether defined via a single-period (static) copula, or via multi-period (dynamic) latent-variable framework, and for simple and consistent approach to non-parametric pricing of CDO tranches. We showed that

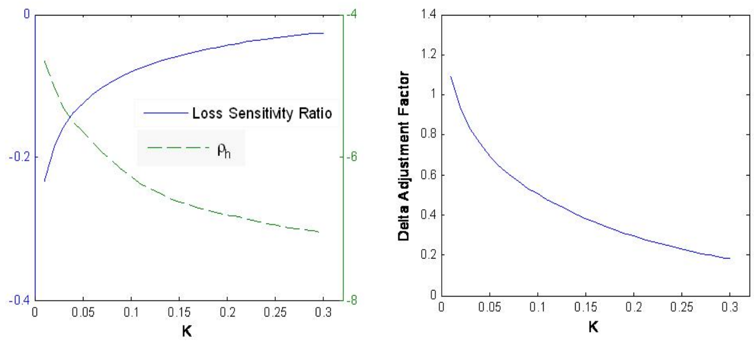

Figure 10: Left figure: the dependence of the tranche loss sensitivity ratio and ρ_h on K. Right figure shows the product of the delta adjustment factor as a function of K. The base case corresponds to TARCH model (α=0.004, α_d=0.094, β=0.927) with Gaussian idiosyncracies, aggregated over 5 year horizon. Portfolio hazard rate is equal to 100bps.

portfolio loss distributions with smooth pdf can be easily reconstructed from the correlation surface using its level and slope along the K-dimention.

We considered the differences in the correlation surfaces generated by static models, including Gaussian, Student-t and Double-t copula, and dynamic models including GARCH(1,1) with Gaussian and Student-t shocks, and TARCH(1,1) with Gaussian and Student-t shocks. From this comparison we can conclude that the most relevant stylized facts for explanation of the market observable correlation skew are, in order of their importance:

- the independence of the market factor and idiosyncratic returns (no common mixing variables);
- the persistent asymmetry of aggregate return distribution of the market factor, which in case of TARCH models occurs as a consequence of the asymmetric volatility dynamics;
- the fat tails in the market factor returns;
- the fat tails in the idiosyncratic returns;
- the slower convergence to Normal distribution of the market factor compared to idiosyncratic returns;

Importantly, in our dynamic framework, the correlation surface is not only explained, but predicted – based on empirical parameters of the TARCH process and the parameters describing the reference credit portfolio. The model also predicts a specific sensitivity of the correlation surface to changes in various parameters, including the average hazard rate of the underlying portfolio.

The inability of static models to incorporate changing base correlations are at the heart of the difficulties faced by these models during the credit market dislocations. In particular, our model reveals that the systematic deltas of equity tranches are understated by the industry standard static copula models, since the growing portfolio spread (hazard rate) should lead to an additional drop in equity tranche prices due to decreasing implied correlation level. Similarly, the static models require making additional assumptions about the correlation skew at different maturities. The early market convention of keeping this skew constant which some practitioners still adhere to is very far from realistic as can be seen from the results of section 4.2. The more reasonable assumption is that the correlation level decreases and the skew flattens for longer horizons.

We should note however, that these conclusions are based on an implicit assumption that the model includes a single common return factor, and that the parameters of the dynamic model are constant over time. While this is a weaker assumption that an outright imposition of the constant correlation skew, it may still be too strict in some circumstances. A possible direction for generalization of our model is to move from a single market factor to a multi-factor framework which can make the model much more flexible in terms of both the detachment level K and term structure T dependence of the correlation surface $\rho = \rho(K, T)$. The well-documented importance of both macro and industry factors for explanation of equity returns suggests that such a generalization is not only desirable from calibration point of view but also warranted empirically. While the analytical tractability of the model will suffer, the numerical accuracy will likely remain intact when using Monte Carlo simulations.

Whether in a single factor or a multi-factor setting, many of our conclusions reflect the limitations of the large homogeneous portfolio approximation which we have adopted in this paper. In particular, it is clear that even deterministic but heterogenous idiosyncrasies, market factor loadings and hazard rates could lead to significant changes in portfolio loss distribution and consequently to the correlation surface of the model. An extension of our model to such heterogenous case is possible, although the computational efforts will increase very significantly.

In conclusion we note that the ARCH family of time series models [12] had proven quite successful in explaining the behavior of implied volatility smile and skew and stock index option pricing [13]. Given the similar empirical motivation of our model, and multi-faceted analogies with equity derivatives pricing outlined throughout the paper, we believe that our approach can lead to similar advances in the portfolio credit risk modeling, and shed new light on pricing of CDO tranches.

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A Kurtosis and Skewness of Aggregated TARCH Returns

In this notes we analyze kurtosis and skewness of aggregated returns $R_T = \sum_{t=1}^{T} r_t$ when r_t is assumed to follow TARCH(1,1) process

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= (1-\zeta)\,\sigma^2 + \alpha r_{t-1}^2 + \alpha_d r_{t-1}^2 1_{\{r_{t-1} \leq 0\}} + \beta \sigma_{t-1}^2 \end{aligned}$$

where returns innovations ε_t are assumed to be iid, have zero mean and unit variance. We are interested in variance, skewness and kurtosis of time aggregated returns. To make sure that those moments are finite we need corresponding moments of the return innovations to be finite. Particularly, we assume that ε_t has finite kurtosis. Let us introduce the following notations for the central and truncated moments of ε_t

$$\begin{aligned} m_\varepsilon &\equiv E(\varepsilon_t) = 0 \\ v_\varepsilon &\equiv E\left(\varepsilon_t^2\right) = 1 \\ v_\varepsilon^d &\equiv E\left(\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}\right) \\ s_\varepsilon &\equiv E\left(\varepsilon_t^3\right) \\ s_\varepsilon^d &\equiv E\left(\varepsilon_t^3 1_{\{\varepsilon_t \leq 0\}}\right) \\ k_\varepsilon &\equiv E\left(\varepsilon_t^4\right) \\ k_\varepsilon^d &\equiv E\left(\varepsilon_t^4 1_{\{\varepsilon_t \leq 0\}}\right) \end{aligned}$$

Lemma 7 *The following recursions hold for TARCH(1,1) model*

$$\begin{aligned} cov_{t-1}\left(r_t^k, r_{t+u}^2\right) &= \rho cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right) \text{ for } u>1 \\ cov_{t-1}\left(r_t^k r_{t+1}^2\right) &= \alpha var_{t-1}\left(r_t^{k+2}\right) + \alpha_d var_{t-1}\left(r_t^{k+2} 1_{\{r_t \leq 0\}}\right) \end{aligned}$$

Proof.

$$\begin{aligned} &cov_{t-1}\left(r_t^k, r_{t+u}^2\right) = cov_{t-1}\left(r_t^k\left[(1-\zeta)\,\sigma^2 + \alpha r_{t+u-1}^2 + \alpha_d r_{t+u-1}^2 1_{\{r_{t+u-1} \leq 0\}} + \beta \sigma_{t+u-1}^2\right]\right) \\ &= 0 + \alpha cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right) + \alpha_d cov_{t-1}\left(r_t^k, r_{t+u-1}^2 1_{\{r_{t+u-1} \leq 0\}}\right) + \beta cov_{t-1}\left(r_t^k, \sigma_{t+u-1}^2\right) \end{aligned}$$

if u>1 then

$$cov_{t-1}\left(r_t^k, r_{t+u-1}^2 1_{\{r_{t+u-1}\le 0\}}\right) = v_\varepsilon^d cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right)$$
$$cov_{t-1}\left(r_t^k, \sigma_{t+u-1}^2\right) = cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right)$$

If u=1 then

$$cov_{t-1}\left(r_t^k, \sigma_{t+u-1}^2\right) = 0$$

■

Proposition 8 *Suppose* $0 \le \zeta < 1$ *and the return innovations have finite skewness,* s_ε*, and finite "truncated" third moment,* s_ε^d*, then conditional third moment of T-period aggregate return* $R_{t,t+T}$ *has the following representation for TARCH(1,1)*

$$E_t R_{t,t+T}^3 = s_\varepsilon \sum_{u=1}^{T} E_t\left(\sigma_{t+u}^3\right) + 3\left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right) \sum_{u=1}^{T} \frac{1-\zeta^{T-u}}{1-\zeta} E_t\left(\sigma_{t+u}^3\right)$$

In addition if $E\sigma_t^3$ *is finite then unconditional skewness of* $R_{t,t+T}$ *is given by*

$$S_T \equiv \frac{ER_{t,t+T}^3}{E(R_{t,t+T}^2)^{3/2}} = \left[\frac{1}{T^{1/2}} s_\varepsilon + 3\frac{1}{T^{3/2}}\left(\alpha s_\varepsilon + \alpha_d s_\varepsilon^d\right)\frac{T(1-\zeta) - 1 + \rho^T}{(1-\zeta)^2}\right] E\left(\frac{\sigma_t}{\sigma}\right)^3$$

Proof. *Using Lemma 7 we have*

$$\begin{aligned}
E_t\left(\sum_{u=t+1}^{t+T} r_u\right)^3 &= E_t\left(\sum_{t+1\le t_1\le t_2\le t_3\le t+T} r_{t_1} r_{t_2} r_{t_3}\right) \\
&= \sum_{u=1}^{T} E_t r_{t+u}^3 + \sum_{t+1\le t_1<t_2\le t+T} 3E_t\left(r_{t_1} r_{t_2}^2\right) \\
&= \sum_{u=1}^{T} E_t\left(r_{t+u}^3\right) + 3\sum_{t+1\le t_1<t_2\le t+T} \zeta^{t_2-t_1-1}\left(\alpha E_t\left(r_{t_1}^3\right) + \alpha_d E_t\left(r_{t_1}^3 1_{\{r_{t_1}\le 0\}}\right)\right) \\
&= \sum_{u=1}^{T} E_t\left(r_{t+u}^3\right) + 3\sum_{u=1}^{T}\frac{1-\zeta^{T-u}}{1-\zeta}\left(\alpha E_t\left(r_{t+u}^3\right) + \alpha_d E_t\left(r_{t+u}^3 1_{\{r_{t+u}\le 0\}}\right)\right) \\
&= s_\varepsilon \sum_{u=1}^{T} E_t\left(\sigma_{t+u}^3\right) + \left(\alpha s_\varepsilon + \alpha_d s_\varepsilon^d\right)\sum_{u=1}^{T}\frac{1-\zeta^{T-u}}{1-\zeta} E_t\left(\sigma_{t+u}^3\right)
\end{aligned}$$

Using the law of iterated expectations

$$E\left(\sum_{u=t+1}^{t+T} r_u\right)^3 = E\left(E_t\left(\sum_{u=t+1}^{t+T} r_u\right)^3\right) = \left[T s_\varepsilon + 3\left(\alpha s_\varepsilon + \alpha_d s_\varepsilon^d\right)\frac{T(1-\zeta) - 1 + \zeta^T}{(1-\zeta)^2}\right] E\left(\sigma_t\right)^3$$

S_T *is then computed using the simple formula for the unconditional variance* $E(R_{t,t+T}^2) = \sigma^2$. ■

To derive unconditional kurtosis we define the following unconditional autocorrelations

$$\gamma_n = \gamma_{-n} = corr(r_{t-n}^2, r_t^2)$$
$$\varphi_n = corr(r_{t-n}, r_t^2) \text{ for } n \geq 1$$
$$\psi_{i,j} \equiv E\left(r_{t-i}r_{t-j}r_t^2\right) \text{ for } 1 \leq j < i$$

Lemma 9 *γ_n, φ_n and $\psi_{i,j}$ decay exponentially as n and $i-j$ increase*

$$\gamma_n = \zeta\gamma_{n-1} = \zeta^{n-1}\gamma_1 \text{ for } n \geq 1$$
$$\varphi_n = \zeta\varphi_{n-1} = \zeta^{n-1}\varphi_1 \text{ for } n \geq 1$$
$$\psi_{i,j} = \zeta\psi_{i-1,j-1} = \zeta^{j-1}\psi_{i-j+1,1} \text{ for } 1 \leq j < i$$

where γ_1, φ_1 and $\psi_{k,1}$ are given by

$$\gamma_1 = \alpha\left(k_r - 1\right) + \alpha_d\left(k_r^d - v_r^d\right) + \beta k_r/k_\varepsilon$$
$$\varphi_1 = \alpha s_r + \alpha_d s_r^d$$
$$\psi_{k,1} = \alpha E\left(r_{t-k+1}r_t^3\right) + \alpha_d E\left(r_{t-k+1}r_t^3 1_{\{r_t \leq 0\}}\right)$$

with $v_\varepsilon^d = \frac{E\left(r_t^2 1_{\{r_t \leq 0\}}\right)}{Er_t^2}$, $s_r = \frac{E\left(r_t^3\right)}{\left(Er_t^2\right)^{3/2}}$, $s_r^d = \frac{E\left(r_t^3 1_{\{r_T < 0\}}\right)}{\left(Er_t^2\right)^{3/2}}$, $k_r = \frac{E\left(r_t^4\right)}{\left(Er_t^2\right)^2}$ *and* $k_r^d = \frac{E\left(r_t^4 1_{\{r_t \leq 0\}}\right)}{\left(Er_t^2\right)^2}$.

Proposition 10 *If*

$$\zeta \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}\right) = \beta + \alpha + \alpha_D v_\varepsilon^d < 1$$
$$\xi \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}\right)^2 = \beta^2 + \alpha^2 k_\varepsilon + \alpha_D^2 k_\varepsilon^d + 2\alpha\beta + 2\alpha_D\beta v_\varepsilon^d + 2\alpha\alpha_D k_\varepsilon^d < 1$$

then unconditional kurtosis of r_t, K_1, is finite and

$$K_1 \equiv \frac{Er_t^4}{\left(Er_t^2\right)^2} = k_\varepsilon \frac{1-\zeta^2}{1-\xi}$$

Proof. *If the 4th moment of r_t exists then the following equation must hold*

$$\begin{aligned}
Er_t^4 &= E\left(\varepsilon_t^4\right)E\left(\sigma_t^4\right) \\
&= k_\varepsilon E\left(\left(1-\zeta\right)\sigma^2 + \alpha r_{t-1}^2 + \alpha_d r_{t-1}^2 1_{\{r_{t-1} \leq 0\}} + \beta\sigma_{t-1}^2\right)^2 \\
&= k_\varepsilon(\left(1-\zeta\right)^2\sigma^4 + 2\left(1-\zeta\right)\sigma^2 E\left(\alpha r_{t-1}^2 + \alpha_d r_{t-1}^2 1_{\{r_{t-1} \leq 0\}} + \beta\sigma_{t-1}^2\right) \\
&+ \left(\alpha r_{t-1}^2 + \alpha_d r_{t-1}^2 1_{\{r_{t-1} \leq 0\}} + \beta\sigma_{t-1}^2\right)^2) \\
&= k_\varepsilon\left(\left(1-\zeta\right)^2\sigma^4 + 2\left(1-\zeta\right)\rho\sigma^4 + \xi E\sigma_{t-1}^4\right)
\end{aligned}$$

Therefore Er_t^4 necessarily solves

$$Er_t^4 = k_\varepsilon\left(1-\zeta^2\right)\sigma^4 + \xi Er_t^4$$

■

Proposition 11 *If the distribution of ε_t is symmetric and $\alpha_d = 0$ then unconditional kurtosis of R_T, if exists, is given by the following formula:*

$$K_T = 3 + \frac{1}{T}(K_1 - 3) + 6\frac{\gamma_1}{T^2}\frac{T(1-\zeta) - 1 + \zeta^T}{(1-\zeta)^2} \text{for } T > 1 \tag{50}$$

$$K_1 = k_\varepsilon \frac{1-\zeta^2}{1-\xi} \tag{51}$$

where k_ε is unconditional kurtosis of ε_t and

$$\xi \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_d \varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right)^2 = \beta^2 + \alpha^2 k_\varepsilon + \alpha_d^2 k_\varepsilon^d + 2\alpha\beta + 2\alpha_d\beta v_\varepsilon^d + 2\alpha\alpha_d k_\varepsilon^d.$$

$$\gamma_1 \equiv corr\left(r_{t-1}^2, r_t^2\right) = \alpha\left(k_r - 1\right) + \alpha_d\left(k_r^d - v_r^d\right) + \beta k_r / k_\varepsilon$$

Proof.

$$\begin{aligned}
E\left(\sum_{u=t+1}^{t+T} r_u\right)^4 &= \sum_{u=1}^{T} E\left(r_{t+u}^4\right) + 6\sum_{t+1\le t_1 < t_2 \le t+T} E\left(r_{t_1}^2 r_{t_2}^2\right) \\
&= \sum_{u=1}^{T} E\left(r_{t+u}^4\right) + 6\sum_{t+1\le t_1 < t_2 \le t+T} \left[cov\left(r_{t_1}^2, r_{t_2}^2\right) + E\left(r_{t_1}^2\right)E\left(r_{t_2}^2\right)\right] \\
&= TE\left(r_t^4\right) + 6\frac{T(T-1)}{2}E\left(r_t^2\right)^2 + 6cov\left(r_{t-1}^2, r_t^2\right)\sum_{t+1\le t_1 < t_2 \le t+T} \zeta^{t_2 - t_1 - 1} \\
&= TE\left(r_t^4\right) + 6\frac{T(T-1)}{2}E\left(r_t^2\right)^2 + 6cov\left(r_{t-1}^2, r_t^2\right)\frac{T(1-\zeta) - 1 + \zeta^T}{(1-\zeta)^2}
\end{aligned}$$

substituting the derived 4th moment into the definition of the kurtosis $K_T = E\left(\sum_{u=t+1}^{t+T} r_u\right)^4 / E\left(r_t^2\right)^2$ *completes the proof.* ■

B Estimation Results for SP500

In this appendix consider the estimation results of several TARCH(1,1) specifications for SP500 weekly returns, to provide empirical context for the rest of the paper. We obtained the daily levels of SP500 from CRSP database. The total number of observations is 10,699 and covers the period from 07/02/1962 till 12/31/2004. We constructed weekly log returns and estimated the parameters of TARCH and GARCH models with Gaussian and Student-t shocks for 2 samples - full and post-1990.

Tables 1 and 2 shows estimated parameters and various data statistics. Note that the Student-t distribution has an additional parameter, degrees of freedom ν, that adjusts the tails of the error distribution. Since the Gaussian distribution is nested within the Student-t as a limit of large degrees of freedom, and since the estimates of the full unconstrained model result in a relatively small and statistically significant value of the degrees of freedom, we conclude that the data points toward the fat-tailed return shock distribution.

On the other hand, the asymmetric TARCH model is nested within the symmetric GARCH in the limiting case $\alpha_d = 0$. The estimated asymmetric coefficient α_d in the TARCH model is not only non-zero, but significantly higher than the symmetric coefficient α for both complete and post 1990 samples, both daily and weekly frequencies and Gaussian and Student-t shock distributions. Thus, we conclude that the asymmetric volatility is prominently present in the data. The best fit model among those considered is the TARCH(1,1) with Student-t distribution of return innovations. The additional parameters of this model are statistically significant.

To make sure that asymmetry in volatility is not a result of several extreme negative returns like 1987 crash we provide data statistics and re-estimated parameters of TARCH models for trimmed full and post 1990 samples. The trimming is done by cutting excess volatility in the most extreme 0.1% observations of both positive and negative return.

Table 1: SP500 moments.

Sample period	Daily				Weekly			
	s_r	s_r^d	k_r	v_r^d	s_r	s_r^d	k_r	v_r^d
1962-2004	-1.40	-2.43	39.83	0.53	-0.55	-1.35	7.01	0.55
1990-2004	-0.11	-1.14	6.67	0.51	-0.64	-1.36	6.10	0.56
SP500 moments(After trimming 0.1% of extreme positive and negative returns)								
1962-2004	0.05	-1.03	5.95	0.50	-0.39	-1.18	5.26	0.54
1990-2004	0.04	-1.01	5.56	0.50	-0.50	-1.24	5.22	0.55

Table 2.

Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/Student-t shocks on weekly SP500 returns. The total number of return observations is 756 for post-1990 sample and 2,139 for the full sample starting in 1962.. Number below each parameter estimate in parenthesis is an asymptotic standard deviation, LogL is corresponding loglikelihood value. GARCH parameters correspond to the volatility specification: $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \alpha_d r_{t-1}^2 1_{\{r_{t-1} \leq 0\}} + \beta \sigma_{t-1}^2$. $\nu$is degrees of freedom of Student-t distributed return innovation ε_t. The constant term of the volatility process is not shown since it is not used in the simulations.

Sample Dates	01/01/1990-12/31/2004			01/01/1962-12/31/2004		
Model	GARCH	TARCH	TARCH + t	GARCH	TARCH	TARCH + t
α	0.044	0.007	0.004	0.107	0.037	0.032
	(0.0073)	(0.024)	(0.019)	(0.013)	(0.016)	(0.0125)
α_d	-	0.112	0.094	-	0.136	0.106
		(0.046)	(0.033)		(0.033)	(0.0223)
β	0.953	0.918	0.927	0.886	0.877	0.894
	(0.0003)	(0.0022)	(0.0013)	(0.0022)	(0.0031)	(0.0015)
ν	-	-	8.31	-	-	10.19
			(2.75)			(2.42)
LogL	1855.6	1861.7	1877.4	5347.4	5368.6	5397.7

C Monte Carlo Simulations

Most of the numerical estimates for credit risk in this paper are obtained by Monte Carlo simulation. Here we outline the simulation procedure for two such calculations, the estimation of the pairwise default correlation coefficient, and the estimation of the tranche losses under the LHP assumption.

The default correlation coefficient, $\rho^d(p)$ for the factor GARCH and TARCH models is calculated based on the simulated factor time series and closed form formulas of conditional default probabilities:

- simulate the common factor, $R_{m,T}$, $I = 10,000$ times and normalize it to have variance 1
- for each p find $d_T(p)$ that solves $\frac{1}{I}\sum_{i=1}^{I}\Phi\left(\frac{d_T - bR^{(i)}_{m,T}}{\sqrt{1-b^2}}\right) = p$
- calculate $\rho^d(p) = \frac{p_{12}-p^2}{p(1-p)}$ where $p_{12} = \frac{1}{I}\sum_{i=1}^{I}\Phi\left(\frac{d_T(p) - bR^{(i)}_{m,T}}{\sqrt{1-b^2}}\right)^2$

The likelihood bounds in Figure 3 were obtained by repeating this procedure 1000 times.

When simulating portfolio loss distribution under the large homogeneous portfolio (LHP) assumption we again begin by simulating the aggregated market factor return. The latent variables are assumed to have symmetric one factor structure with the factor following TARCH(1,1) model. For each realization of the market factor the portfolio loss is given by the LHP formula (10)

$$L_T = \left(1 - \bar{R}\right)\Phi\left(\frac{d_T - bR_{m,T}}{\sqrt{1-b^2}}\right)$$

where

- $R_{m,T} = \sum_{u=1}^{T} r_{m,u}/std\left(\sum_{s=1}^{T} r_{m,s}\right)$ is a normalised return over horizon T generated using time aggregation of simulated TARCH(1,1) returns with unconditional volatility equal to 1

- d_T is calibrated so that the probability of $R_{i,T} = bR_{m,T} + \sqrt{1-b^2}E_T$ hitting d_T is equal to single name default probability p_T

$$P\left(bR_{m,T} + \sqrt{1-b^2}E_T \leq d_T\right) = p_T$$

- b is the factor loading that is chosen to match a given unconditional linear correlation $\rho = b^2$

To calculate the expected tranche losses generated by the model and to calibrate d_T we use $I = 100,000$ independent Monte Carlo simulations of the factor and then use corresponding sample moments:

$$d_T \text{ solves } \frac{1}{I}\sum_{i=1}^{I}\Phi\left(\frac{d_T - bR_{m,T}^{(i)}}{\sqrt{1-b^2}}\right) = p_T$$

$$EL_{(0,K]} = \frac{1}{I}\sum_{i=1}^{I} f_{(0,K]}\left(\left(1-\bar{R}\right)\Phi\left(\frac{d_T - bR_{m,T}^{(i)}}{\sqrt{1-b^2}}\right)\right)$$