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BLOCKED REGULAR FRACTIONAL FACTORIAL DESIGNS WITH MINIMUM ABERRATION[1]

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This paper considers the construction of minimum aberration (MA) blocked factorial designs. Based on coding theory, the concept of minimum moment aberration due to Xu [*Statist. Sinica* **13** (2003) 691–708] for unblocked designs is extended to blocked designs. The coding theory approach studies designs in a row-wise fashion and therefore links blocked designs with nonregular and supersaturated designs. A lower bound on blocked wordlength pattern is established. It is shown that a blocked design has MA if it originates from an unblocked MA design and achieves the lower bound. It is also shown that a regular design can be partitioned into maximal blocks if and only if it contains a row without zeros. Sufficient conditions are given for constructing MA blocked designs from unblocked MA designs. The theory is then applied to construct MA blocked designs for all 32 runs, 64 runs up to 32 factors, and all 81 runs with respect to four combined wordlength patterns.

1. Introduction. Fractional factorial designs are widely used in scientific and industrial experiments. Blocking is an effective method for reducing systematic variations and therefore increasing precision of effect estimation. Experimenters often face the practical problem of choosing good fractional factorial designs and blocking schemes.

Fractional factorial designs are typically chosen according to the *minimum aberration* (MA) criterion [12], which includes the *maximum resolution* criterion [1] as a special case. The study of blocking in fractional factorial designs is complicated by the presence of two defining contrast subgroups, one for defining the fraction and another for defining the blocking scheme, therefore, resulting in two types of wordlength patterns, one for treatment

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and another for block. The MA criterion can be applied to the treatment and block wordlength patterns separately. However, MA designs with respect to one wordlength pattern may not have MA with respect to the other wordlength pattern. One approach, that taken by Sun, Wu and Chen [19] and Mukerjee and Wu [15], is to consider the concept of admissible blocking schemes, but it often leads to too many admissible designs. Another approach is to combine the treatment and block wordlength patterns into one single wordlength pattern so that the criterion of MA can be applied to it in the usual way. Sitter, Chen and Feder [17], Chen and Cheng [5] and Cheng and Wu [10] have proposed four combined sequences, resulting in four MA criteria (to be defined later). See [9] for a related approach.

A practical and important issue is how to construct MA blocked designs with respect to one or more criteria. This question is not adequately addressed in the literature. Most of the existing MA blocked designs rely on the work of Sun, Wu and Chen [19], who obtained the complete catalog of blocked designs with 8, 16, 32, 64 and 128 runs for up to nine factors. MA criteria rank blocked designs according to the treatment and block wordlength patterns, which are often obtained by counting words in the treatment defining contrast subgroups and alias sets. When the number of factors is large, there are a huge number of words to be counted, causing considerable difficulties in computation. For example, when a design with 64 runs and 25 factors is arranged in 8 blocks, there are $2^{22} - 1 = 4{,}194{,}303$ words to be counted. It is cumbersome and sometimes even impossible to do so for thousands or millions of different designs. This calls for alternative computational methods.

To avoid the aforementioned computational difficulties, we take a coding theory approach and propose new methods to compare and rank blocked designs without using defining contrast subgroups and alias sets. The idea is originally due to Xu [22], who proposed the concept of *minimum moment aberration* and established its equivalence to MA for unblocked designs. We extend the concept of minimum moment aberration to blocked designs for three of the four MA criteria in Section 2.

To further ease the computation burden, we study relationships among MA blocked designs under different criteria and develop a general theory on MA blocked designs. The coding theory approach studies designs in a row-wise fashion and therefore links blocked designs with nonregular and supersaturated designs. Results on nonregular and supersaturated designs are used to establish an important lower bound on blocked wordlength pattern. It is shown that a blocked design has MA with respect to all four criteria if it originates from an unblocked MA design and achieves the lower bound. It is also shown that a regular design can be partitioned into maximal blocks if and only if it contains a row (i.e., treatment combination) without zeros. Sufficient conditions are given for constructing MA blocked designs from unblocked MA designs. Some technical lemmas are presented in Section 3 and the main results are given in Section 4. We shall point out that, for simplicity, we focus entirely on regular designs, even though most of the results can be easily extended to nonregular designs.

With the concept of minimum moment aberration and developed theory, we present methods to construct MA blocked designs in Section 5. We obtain MA blocked designs for all 8, 16, 27 and 32 runs, 64 runs up to 32 factors, and all 81 runs with respect to four combined wordlength patterns. The difference among MA blocked designs under different criteria is summarized.

The rest of this section introduces some background. A *regular* s^{n-k} design is defined by k treatment defining words, which form the *treatment defining contrast subgroup*. The *resolution* [1] is the length of the shortest word in the treatment defining contrast subgroup. For $i = 1, \ldots, n$, let $A_{i,0}$ denote the number of words of length i in its treatment defining contrast subgroup. For two unblocked regular s^{n-k} designs D_1 and D_2, let r be the smallest integer such that $A_{r,0}(D_1) \neq A_{r,0}(D_2)$. Then D_1 is said to have less aberration than D_2 if $A_{r,0}(D_1) < A_{r,0}(D_2)$. If there is no design with less aberration than D_1, then D_1 has MA. In short, the MA criterion sequentially minimizes $A_{1,0}, A_{2,0}, \ldots, A_{n,0}$.

To arrange a regular s^{n-k} design in s^p blocks of size s^{n-k-p}, one can choose p independent block defining words, which form the *block defining contrast subgroup*. There are $(s^p - 1)/(s - 1)$ block effects, each confounded with s^k treatment effects. For $i = 1, \ldots, n$, let $A_{i,1}$ denote the number of treatment words of length i that are confounded with some block effects.

As done in the literature, we shall only consider regular main effect (RME) designs where none of the main effects is aliased with another main effect or confounded with a block effect. It is evident that, for RME designs, $A_{1,0} = A_{2,0} = A_{1,1} = 0$. The vectors $W_t = (A_{3,0}, \ldots, A_{n,0})$ and $W_b = (A_{2,1}, \ldots, A_{n,1})$ are called the treatment and block wordlength pattern, respectively. Let $A_{0,1} = 0$ for convenience.

MA criteria for blocked designs differ in how the treatment and block wordlength patterns are combined. Sitter, Chen and Feder [17] first proposed the combined wordlength pattern

$$W_{\mathrm{scf}} = (A_{3,0}, A_{2,1}, A_{4,0}, A_{3,1}, A_{5,0}, A_{4,1}, \ldots), \tag{1}$$

where $A_{i,1}$ is ranked after $A_{i+1,0}$ for $i = 2, 3, \ldots.$ Chen and Cheng [5] pointed out that the ordering of wordlength patterns in (1) violates the hierarchical assumption, and proposed the sequence

$$W_{\mathrm{cc}} = (3A_{3,0} + A_{2,1}, A_{4,0}, 10A_{5,0} + A_{3,1}, A_{6,0}, \ldots), \tag{2}$$

where the sum of $\binom{2i-1}{i} A_{2i-1,0}$ and $A_{i,1}$ is ranked before $A_{2i,0}$ for $i = 2, 3, \ldots.$ Cheng and Wu [10] proposed the two combined wordlength patterns

$$W_1 = (A_{3,0}, A_{4,0}, A_{2,1}, A_{5,0}, A_{6,0}, A_{3,1}, \ldots), \tag{3}$$

$$(4) \qquad W_2 = (A_{3,0}, A_{2,1}, A_{4,0}, A_{5,0}, A_{3,1}, A_{6,0}, \dots),$$

where $A_{i,1}$ is ranked after $A_{2i,0}$ in W_1 and after $A_{2i-1,0}$ in W_2 for $i = 2, 3, \dots$. We shall mention that sequence (4) was first proposed by Chen and Cheng [5] and later independently by Zhang and Park [25] and Cheng and Wu [10].

Four MA criteria result from sequentially minimizing the corresponding combined wordlength patterns. MA blocked designs under the W sequence are called MA W designs.

An *orthogonal array* (OA) of N runs, n columns, s levels and strength t, denoted by $OA(N, n, s, t)$, is an $N \times n$ matrix in which all possible s^t level combinations appear equally often as rows for any set of t columns.

2. A coding theory approach: minimum moment aberration. For a prime power s, let $GF(s)$ be the finite field of s elements. Let V_n be the n-dimensional row vector space over $GF(s)$, that is, $V_n = \{(v_1, \dots, v_n) : v_i \in GF(s) \text{ for } i = 1, \dots, n\}$.

An $[n, m]$ *linear code* over $GF(s)$ is a vector subspace of V_n with dimension m so that it has s^m distinct vectors. An $[n, m]$ linear code D can be specified by an $m \times n$ *generator matrix* G whose rows form a basis for the code. Then $D = \{u \in V_n : u = vG, v \in V_m\}$. A regular s^{n-k} design is an $[n, n-k]$ linear code over $GF(s)$. For an introduction to coding theory, see [13], Chapter 4, and [20].

Consider arranging a regular s^{n-k} design in s^p equal-sized blocks. A design of this kind is called a regular $(s^{n-k} : s^p)$ design. Such a design is specified by a pair of matrices T and B, defined over $GF(s)$ and of orders $(n-k) \times n$ and $(n-k) \times p$, respectively, such that T has full row rank and B has full column rank. Then a typical block of the design consists of all level combinations of the form uT, with $u \in V_{n-k}$ and $uB = v$, where v is any fixed vector in V_p. Different blocks correspond to different choices of v. Since B has full column rank p, there are s^p choices of v, leading to a division of the s^{n-k} level combinations into s^p blocks. See [9] and [15].

Let $L_p = (s^p - 1)/(s - 1)$ throughout this paper. Suppose that the columns of B are $b_1, \dots, b_p$. Let F be the $(n-k) \times L_p$ matrix whose columns are $\lambda_1 b_1 + \dots + \lambda_p b_p$, where $\lambda_i \in GF(s)$, at least one $\lambda_i \neq 0$ and the first nonzero λ_i is 1.

The columns of T and F can be viewed as points of $PG(n-k-1, s)$, the projective geometry of dimension $n-k-1$ over $GF(s)$. In the terminology of projective geometry, F is a $(p-1)$-flat in $PG(n-k-1, s)$. Then a regular $(s^{n-k} : s^p)$ design is an RME design if and only if T and F are disjoint; see [5] and [15].

Let $G = (T, F)$ be the $(n-k) \times (n + L_p)$ matrix and D be the linear code generated by G. For convenience, write $D = (D_T, D_F)$, where D_T is the $N \times n$ treatment matrix and D_F is the $N \times L_p$ block matrix, with $N = s^{n-k}$. For integers $t \geq 0$, define moments

$$K_{t,0}(D) = N^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} [\delta_{ij}(D_T)]^t, \tag{5}$$

$$K_{t,1}(D) = N^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} [\delta_{ij}(D_T)]^t \delta_{ij}(D_F), \tag{6}$$

where $\delta_{ij}(D_T)$ and $\delta_{ij}(D_F)$ are the number of coincidences between the ith and jth rows of D_T and D_F, respectively. For two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, the number of coincidences is the number of i's such that $u_i = v_i$. We take $0^0 = 1$ throughout the paper.

REMARK 1. The definitions of $K_{t,0}(D)$ and $K_{t,1}(D)$ given in (5) and (6) work for both regular and nonregular designs. For regular designs, the double summation can be replaced with a single summation; for example, (6) can be simplified to

$$K_{t,1}(D) = N^{-1} \sum_{i=1}^{N} [\delta_{ij}(D_T)]^t \delta_{ij}(D_F), \tag{7}$$

where j can be any row number.

REMARK 2. Note that D_F is a replicated $OA(s^p, L_p, s, 2)$. It follows from Lemma 1 of [14] that $\delta_{ij}(D_F)$ takes on only two different values. Specially, let $y_1, \ldots, y_N$ be the rows of D_F. Then

$$\delta_{ij}(D_F) = \begin{cases} L_p = (s^p - 1)/(s-1), & \text{if } y_i = y_j, \\ L_{p-1} = L_p - s^{p-1}, & \text{otherwise.} \end{cases} \tag{8}$$

For an integer k, let $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$ if $k > 0$, $\binom{x}{0} = 1$ and $\binom{x}{k} = 0$ if $k < 0$. For integers $k, j \geq 0$, let $S(k,j)$ be a Stirling number of the second kind, that is, the number of ways of partitioning a set of k elements into j nonempty sets. It is well known that $S(k,j) = (1/j!) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^k$ for $k \geq j \geq 0$. For integers $k, i \geq 0$, define

$$Q_k(i; n, s) = (-1)^i \sum_{j=0}^{k} j! S(k,j) s^{-j} (s-1)^{j-i} \binom{n-i}{j-i}. \tag{9}$$

For integers $t, i \geq 0$, define

$$c_t(i; n, s) = (s-1) \sum_{k=0}^{t} (-1)^k \binom{t}{k} n^{t-k} Q_k(i; n, s). \tag{10}$$

It is easy to show that $S(k,k)=1$, $Q_k(k;n,s)=(-1)^k s^{-k}k!$ and $Q_k(i;n,s)=0$ when $i>k$. Therefore, $c_t(t;n,s)=s^{-t}(s-1)t!$ and $c_t(i;n,s)=0$ when $i>t$.

The following two lemmas regarding unblocked designs are from Xu [22, 23].

LEMMA 1. *For a regular s^{n-k} design D and integers $t\geq 0$,*

$$K_{t,0}(D)=\sum_{i=0}^{\min(t,n)} c_t(i;n,s)A_{i,0}(D), \tag{11}$$

where $c_t(i;n,s)$ are constants defined in (10) and $A_{0,0}(D)=1/(s-1)$.

LEMMA 2. *Sequentially minimizing $K_{1,0},K_{2,0},\dots,K_{n,0}$ is equivalent to sequentially minimizing $A_{1,0},A_{2,0},\dots,A_{n,0}$.*

The *minimum moment aberration* criterion [22] sequentially minimizes $K_{1,0},K_{2,0},\dots,K_{n,0}$. Lemma 2 implies that the minimum moment aberration criterion is equivalent to the MA criterion for unblocked designs.

Extending Lemma 1 to blocked designs, we have the following result.

THEOREM 1. *For a regular $(s^{n-k}:s^p)$ design D and integers $t\geq 0$,*

$$K_{t,1}(D)=s^{-1}\sum_{i=0}^{\min(t,n)} c_t(i;n,s)[A_{i,1}(D)+L_pA_{i,0}(D)], \tag{12}$$

where $L_p=(s^p-1)/(s-1)$, $c_t(i;n,s)$ are constants defined in (10) and $A_{0,0}(D)=1/(s-1)$.

The proof of Theorem 1 requires the generalized *Pless power moment identities*, a fundamental result in coding theory. For clarity, all proofs are given in the Appendix.

For an RME design D, $A_{1,0}(D)=A_{2,0}(D)=A_{0,1}(D)=A_{1,1}(D)=0$. From (11) and (12), we obtain $K_{1,0}(D)=s^{-1}n$, $K_{2,0}(D)=s^{-2}n(n+s-1)$, $K_{0,1}(D)=s^{-1}L_p$ and $K_{1,1}(D)=s^{-2}nL_p$. Furthermore,

$$K_{3,0}(D)=6s^{-3}(s-1)A_{3,0}(D)+s^{-3}n(n^2+3ns+s^2-3n-3s+2), \tag{13}$$

$$K_{2,1}(D)=2s^{-3}(s-1)A_{2,1}(D)+s^{-3}n(n+s-1)L_p. \tag{14}$$

We can define three minimum moment aberration criteria for blocked designs by replacing $A_{i,0}$ and $A_{i,1}$ with $K_{i,0}$ and $K_{i,1}$ in (1), (3) and (4). Because $c_t(t;n,s)$ is a positive constant, it follows from (11) and (12) that the minimum moment aberration criterion with respect to W_{scf}, W_1 or W_2 is equivalent to its corresponding MA criterion.

The MA W_{cc} criterion defined in (2) is more complicated than the other three criteria. Nevertheless, from (13) and (14), we obtain

$$K_{3,0}(D)+K_{2,1}(D)=2s^{-3}(s-1)[3A_{3,0}(D)+A_{2,1}(D)] \\ +s^{-3}n[(n^2+3ns+s^2-3n-3s+2)+(n+s-1)L_p].$$

Therefore, minimizing $K_{3,0}+K_{2,1}$ is equivalent to minimizing $3A_{3,0}+A_{2,1}$.

3. Some lemmas. Suppose that $D=(D_T,D_F)$ is a regular $(s^{n-k}:s^p)$ design. Let $x_1,\dots,x_N$ be the rows of D_T and $y_1,\dots,y_N$ be the rows of D_F, where $N=s^{n-k}$. For $m=1,\dots,s^p$, let D_m be the $s^{n-k-p}\times n$ treatment matrix corresponding to the mth block. For integers $t\ge 0$, define moments

$$K_t(D_m)=s^{-2(n-k-p)}\sum_{i=1}^{s^{n-k-p}}\sum_{j=1}^{s^{n-k-p}}[\delta_{ij}(D_m)]^t,$$

where $\delta_{ij}(D_m)$ is the number of coincidences between the ith and jth rows of D_m. Let $B_m=\{i:x_i \text{ is a row of } D_m, 1\le i\le N\}$. It is evident that $i\in B_m$ and $j\in B_m$ for some m if and only if $y_i=y_j$. It is useful to express $K_t(D_m)$ in terms of the original design D_T as

$$K_t(D_m)=s^{-2(n-k-p)}\sum_{i\in B_m}\sum_{j\in B_m}[\delta_{ij}(D_T)]^t. \tag{15}$$

Without loss of generality, assume that D_1 contains the null treatment (i.e., a row of zeros) and call D_1 the *principal block*. Then D_1 is an $[n,n-k-p]$ linear code over $GF(s)$ and other blocks D_m, $2\le m\le s^p$, are cosets of D_1; therefore,

$$K_t(D_m)=K_t(D_1)\qquad \text{for } m=2,\dots,s^p. \tag{16}$$

Note that D_1 is possibly a supersaturated design in which the number of columns is larger than the number of rows.

The next result shows that $K_{t,1}(D)$ is determined by $K_{t,0}(D)$ and $K_t(D_1)$.

LEMMA 3. *Suppose that D is a regular $(s^{n-k}:s^p)$ design and D_1 is its principal block. For integers $t\ge 0$, $K_{t,1}(D)=L_{p-1}K_{t,0}(D)+s^{-1}K_t(D_1)$.*

LEMMA 4. *Suppose that D is an $(s^{n-k}:s^p)$ RME design and D_1 is its principal block. Let $J=n(s^{n-k-p-1}-1)(s^{n-k-p}-1)^{-1}$ and η be the fractional part of J.*

(i) *$K_1(D_1)=s^{-1}n$ and D_1 is an $OA(s^{n-k-p},n,s,1)$.*

(ii) *$K_2(D_1)\ge s^{-(n-k-p)}[n^2+(s^{n-k-p}-1)(J^2+\eta(1-\eta))]$. The equality holds if and only if the difference among all $\delta_{ij}(D_1)$, $i<j$, does not exceed one.*

(iii) $K_{2,1}(D) \geq L_{p-1}s^{-2}n(n+s-1) + s^{-(n-k-p+1)}[n^2 + (s^{n-k-p}-1)(J^2 + \eta(1-\eta))]$.

A regular $(s^{n-k}:s^p)$ design $D = (D_T, D_F)$ can be viewed as an unblocked regular $s^{(n+L_p)-(k+L_p)}$ design. For clarity, denote this unblocked design as D_{un}. For integers $t \geq 0$, define moments $K_t(D_{\mathrm{un}}) = N^{-2}\sum_{i=1}^N \sum_{j=1}^N [\delta_{ij}(D_{\mathrm{un}})]^t$, where $N = s^{n-k}$ and $\delta_{ij}(D_{\mathrm{un}}) = \delta_{ij}(D_T) + \delta_{ij}(D_F)$ is the number of coincidences between the ith and jth rows of D_{un}. The next result shows that $K_t(D_{\mathrm{un}})$ is related to $K_{t,0}(D)$, $K_{t-1,1}(D)$, $K_{t-2,1}(D)$, $K_{t-2,0}(D)$ and so on.

LEMMA 5. *For a regular $(s^{n-k}:s^p)$ design D and integers $t \geq 0$,*

$$\begin{aligned} K_t(D_{\mathrm{un}}) &= K_{t,0}(D) + tK_{t-1,1}(D) \\ &\quad + s^{-p+1}\sum_{r=2}^{t}\binom{t}{r}[(L_p^r - L_{p-1}^r)K_{t-r,1}(D) \\ &\qquad - L_pL_{p-1}(L_p^{r-1} - L_{p-1}^{r-1})K_{t-r,0}(D)]. \end{aligned}$$

The following two lemmas are useful to know when MA blocked designs are the same under different criteria.

LEMMA 6. *If D has MA with respect to both W_{scf} and W_1, then D has MA with respect to W_2.*

LEMMA 7. *Suppose there exists some constant $0 \leq \alpha < 3$ such that $\alpha A_{3,0} + A_{2,1}$ is minimized for D. If D has MA with respect to both W_{scf} and W_2, then D has MA with respect to W_{cc}.*

Lemma 7 is very useful to show the MA W_{cc} optimality. The condition $\alpha < 3$ is necessary; see Section 5 for counterexamples.

4. Main results. Lemma 4(iii) and (14) together yield a lower bound of $A_{2,1}$ as follows.

THEOREM 2. *For an $(s^{n-k}:s^p)$ RME design D,*

$$\begin{aligned} A_{2,1}(D) &\geq [2(s-1)]^{-1} \\ &\quad \times\{-n(n+s-1) + s^{-(n-k-p-2)} \\ &\qquad \times [n^2 + (s^{n-k-p}-1)(J^2 + \eta(1-\eta))]\}, \end{aligned}$$

where $J = n(s^{n-k-p-1}-1)(s^{n-k-p}-1)^{-1}$ and η is the fractional part of J.

Theorem 2 plays an important role in the theoretical development and construction of MA blocked designs. The lower bound is tight for $p = n - k - 1$ and $n - k - 2$. Note that an RME design achieving the lower bound does not always have MA. When $s = 2$ and $p < n - k - 2$, the lower bound can be improved in some cases if the results in [4] and [2] are used. However, the improvement is usually negligible, noting that $A_{2,1}$ must be an integer for RME designs.

COROLLARY 1. *If D has MA with respect to W_{scf} and W_2, and D achieves the lower bound in Theorem 2, then D has MA with respect to W_{cc}.*

The next result provides a sufficient condition when MA blocked designs are the same under four criteria.

THEOREM 3. *If D_T has MA among all regular s^{n-k} designs and D achieves the lower bound in Theorem 2, then D has MA with respect to W_{scf}, W_1, W_2 and W_{cc}.*

When the lower bound in Theorem 2 is achieved, the principal block D_1 has minimum moment aberration among all $s^{n-k-p} \times n$ designs. Theorem 3 can be generalized as follows.

COROLLARY 2. *If D_T has MA among all regular s^{n-k} designs and the principal block D_1 has minimum moment aberration among all $s^{n-k-p} \times n$ designs, then D has MA with respect to W_{scf}, W_1, W_2 and W_{cc}.*

The next result gives a simple necessary and sufficient condition when a regular design can be partitioned into maximal blocks as an RME design.

THEOREM 4. *A regular s^{n-k} design containing the null treatment can be partitioned into maximal s^{n-k-1} blocks as an RME design if and only if it contains a row without zeros.*

Mukerjee and Wu [15] previously studied the maximal blocking problem with a projective geometric approach. They managed to obtain a complete solution for s^{n-1} and s^{n-2} designs. Our approach appears to be more pleasant than theirs. Theorem 4 gives a simple answer to the question.

When $s = 2$, a row without zeros is necessarily a row of all 1's. Then a row and its fold-over forms a block. The unblocked design must be a fold-over design. A regular fold-over design is also called an *even design* [11], because it contains only words of even length. Whether or not a design is an even design can be simply checked by its wordlength pattern. The following corollary is a special case of Theorem 4.

COROLLARY 3. *A regular 2^{n-k} design containing the null treatment can be partitioned into maximal 2^{n-k-1} blocks as an RME design if and only if it is an even design.*

It is of special interest to know when an unblocked MA design can be partitioned into maximal blocks. Unblocked MA 2^{n-k} designs were given by Chen and Wu [8] for $k = 1, 2, 3, 4$ and by Chen [6] for $k = 5$. Combining their results and Corollary 3, we have the following result. An unblocked MA 2^{n-k} design can be partitioned into maximal 2^{n-k-1} blocks as an RME design as follows:

(i) when $k = 1$ and n is even,
(ii) when $k = 2$ and n is a multiple of 3,
(iii) when $k = 3$ and $n = 7t + q$ for integers $t \geq 0$ and $q = 7, 11$,
(iv) when $k = 4$ and $n = 15t + q$ for integers $t \geq 0$ and $q = 8, 12, 15, 20$,
(v) when $k = 5$ and $n = 31t+q$ for integers $t \geq 0$ and $q = 16, 21, 24, 28, 31, 37, 40, 44$.

Furthermore, it is known from coding theory that even designs are the only designs of resolution IV for $5N/16 < n \leq N/2$ with $N = 2^{n-k}$; see [3]. For such n, an unblocked MA 2^{n-k} design can always be partitioned into maximal 2^{n-k-1} blocks as an RME design.

To describe the next result, let $\tilde{F}$ be an $(n - k - 2)$-flat and $\tilde{T}$ be the complement of $\tilde{F}$ in $PG(n-k-1, s)$. Let $\tilde{H}_{n-k}$ be the linear code generated by $\tilde{T}$. Note that $\tilde{H}_{n-k}$ is unique up to isomorphism. It is evident that $\tilde{H}_{n-k}$ and its projection designs (i.e., subsets of columns) can be partitioned into maximal s^{n-k-1} blocks as RME designs. The reverse is also true in the following sense. If an s^{n-k} design can be partitioned into s^{n-k-1} blocks as an RME design, then it is isomorphic to a projection design of $\tilde{H}_{n-k}$. The next result characterizes MA $(s^{n-k} : s^{n-k-1})$ RME designs.

THEOREM 5. *If D_T has MA among all projection designs of $\tilde{H}_{n-k}$, then D_T can be partitioned as an $(s^{n-k} : s^{n-k-1})$ RME design D that has MA with respect to W_{scf}, W_1, W_2 and W_{cc}.*

Theorem 5 shows that MA blocked $(s^{n-k} : s^{n-k-1})$ designs are the same for all four criteria when they exist. As a numeric illustration, consider $s = 3$. For 27 runs, $\tilde{H}_3$ is the unique MA 3^{9-6} design. According to Xu [23], for $4 \leq n < 9$, MA $3^{n-(n-3)}$ designs are projection designs of the MA 3^{9-6} design; therefore, they can be partitioned into maximal 9 blocks and resulting RME designs have MA with respect to all four criteria. For 81 runs, $\tilde{H}_4$ is the unique MA 3^{27-23} design. According to Xu [23], for $5 \leq n \leq 9$ and $12 \leq n < 27$, MA $3^{n-(n-4)}$ designs are projection designs of the MA 3^{27-23} design; therefore, they can be partitioned into maximal 27 blocks and resulting RME designs have MA with respect to all four criteria. For $n = 10, 11$, MA $3^{n-(n-4)}$ designs are not projection designs of the MA 3^{27-23} design; therefore, they cannot be partitioned as RME designs with 27 blocks. The second best designs are projection designs of the MA 3^{27-23} design; therefore, they can be partitioned into maximal 27 blocks and the resulting RME designs have MA with respect to all four criteria.

When $s = 2$, $\tilde{H}_{n-k}$ is an even design with resolution IV. We have the following result.

COROLLARY 4. *If a regular 2^{n-k} design has MA among all even designs, then it can be partitioned as a $(2^{n-k} : 2^{n-k-1})$ RME design that has MA with respect to W_{scf}, W_1, W_2 and W_{cc}.*

Recall that a regular $(s^{n-k} : s^p)$ design D can be viewed as an unblocked regular $s^{(n+L_p)-(k+L_p)}$ design D_{un}. The next result provides a sufficient condition when an MA blocked design originates from an unblocked MA design.

THEOREM 6. *If D_T has MA among all regular s^{n-k} designs and the unblocked design D_{un} has MA among all regular $s^{(n+L_p)-(k+L_p)}$ designs, then the blocked $(s^{n-k} : s^p)$ RME design D has MA with respect to W_{scf}, W_1, W_2 and W_{cc}.*

Theorem 6 is most useful when $p = 1$. It happens frequently that an unblocked MA s^{n-k} design can be extended to an unblocked MA $s^{(n+1)-(k+1)}$ design by adding an extra column. For example, according to Chen, Sun and Wu [7] and Xu [23], for 8 and 27 runs, MA unblocked designs are in sequential order for all n. Whenever this happens, the extra column can be used as the block generator, and the resulting $(s^{n-k} : s^1)$ design has MA with respect to W_{scf}, W_1, W_2 and W_{cc}.

Theorem 6 is less useful when $p > 1$ because D_{un} usually does not have MA. The following result is interesting in this regard.

THEOREM 7. *If D_T has MA among all regular s^{n-k} designs and D has MA with respect to W_{scf}, then D has MA with respect to both W_1 and W_2. If, in addition, $\alpha A_{3,0}(D) + A_{2,1}(D)$ is also minimized for some constant $0 \le \alpha < 3$, then D has MA with respect to W_{cc}.*

Theorem 7 implies that when MA blocked designs are different under W_{scf}, W_1 and W_2, an MA W_{scf} blocked design must not originate from an unblocked MA design.

5. MA blocked designs. MA blocked designs with respect to W_{scf}, W_1 or W_2 can be obtained by computing and comparing moments $K_{t,0}$ and $K_{t,1}$ for all possible blocking schemes. This is a feasible task when the number of blocking schemes is not too large; see [24] for details, where MA blocked designs for all 32 runs, 64 runs up to 32 factors, and all 81 runs with respect to W_{scf}, W_1 and W_2 are given.

However, this method cannot be used to construct MA W_{cc} designs because there is no equivalent minimum moment aberration criterion with respect to (2). Furthermore, an essential difference exists between the MA W_{cc} criterion and the other three criteria. Because the MA W_{scf}, W_1 and W_2 criteria minimize $A_{3,0}$ first, there is no need to search over resolution III designs whenever blocking schemes from resolution IV designs exist. However, the MA W_{cc} criterion minimizes $3A_{3,0} + A_{2,1}$ first. Combining $A_{3,0}$ with $A_{2,1}$ makes it more difficult to construct MA W_{cc} designs than other types of MA designs. To determine the minimum of $3A_{3,0} + A_{2,1}$, a simple strategy is to search over all resolution III designs. This requires a complete catalog of resolution III designs, but such a catalog is not available for 64-run designs.

Combining the developed theory and computer search, we obtain MA W_{cc} designs for all 8, 16, 27 and 32 runs, 64 runs up to 32 factors, and all 81 runs. Previously, Chen and Cheng [5] developed a theory to characterize MA W_{cc} designs in terms of their blocked residual designs and obtained MA W_{cc} designs for all 8 and 16 runs and 32 runs up to 20 factors.

Here we explain how to construct MA W_{cc} designs for 64 runs and $n \le 32$ with the results of Xu and Lau [24]. First, for $p = 5$ and $6 \le n \le 32$, by Theorem 5, MA $(2^{n-(n-6)} : 2^5)$ designs are the same under all four criteria; therefore, MA designs given by Xu and Lau [24] have MA with respect to all four criteria. Indeed, they can be easily constructed by searching over MA projection designs of the unique even 2^{32-26} design. Next, for $p = 1$, because unblocked MA $2^{n-(n-6)}$ designs are in sequential order for $n = 6$–7, 8–12, 14–20 and 21–33, by Theorem 6, we obtain MA $(2^{n-(n-6)} : 2^1)$ designs with respect to all four criteria for all $6 \le n \le 32$ but $n = 7$, 12, 13 and 20. For $n = 7$, 12, 13 or 20, according to [24], MA W_2 and W_{scf} designs coincide and have $A_{2,1} = 0$; therefore, by Lemma 7, MA W_{cc} designs also coincide with MA W_2 and W_{scf} designs.

The situation for $p = 2, 3, 4$ is more complicated than that for $p = 1$ and 5. We first compute the lower bounds of $A_{2,1}$ in Theorem 2, which are given in Table 1. It is evident that a lower bound can be replaced by the smallest nonnegative integer that exceeds it if it is negative or not an integer. According to Xu and Lau [24], MA W_2 designs achieve the modified lower bounds of $A_{2,1}$ except for the following 22 cases: $p = 2$, $n = 19$–26, 31, 32; $p = 3$, $n = 29$–32; and $p = 4$, $n = 25$–32. Furthermore, MA W_{scf} and W_2 designs coincide except for $p = 2$ and $n = 7, 12$. Then, by Corollary 1, MA

TABLE 1
Lower bound of $A_{2,1}$ in Theorem 2 for 64 runs, $6 \le n \le 32$ and $p = 2, 3, 4$

p	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
2	−1.5	−1.5	−1.5	−1.5	−1.2	−1.3	−0.8	−0.7	0	0	1	1.2	2.3	2.7
3	0	0	1	1.5	2.5	3.5	4.5	6	7	9	10.5	12.5	14.5	16.5
4	3	5	7	9	12	15	18	22	26	30	35	40	45	51
p	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	
2	3.8	4.5	5.5	6.5	7.5	8.7	9.8	11.3	12.2	14	15	17	18.2	
3	19	21	24	26.5	29.5	32.5	35.5	39	42	46	49.5	53.5	57.5	
4	57	63	70	77	84	92	100	108	117	126	135	145	155	

W_{scf} and W_2 designs also have MA with respect to W_{cc} except for the 24 special cases, which require additional computer search.

For the 24 special cases, Theorem 2 and Lemma 7 are again used to ease computation. Consider, for example, $p = 4$ and $n = 29$. According to Xu and Lau [24], MA W_{scf}, W_1 and W_2 designs coincide and have $A_{3,0} = 0$ and $A_{2,1} = 196$. The lower bound of $A_{2,1}$ is 126. To determine the minimum of $3A_{3,0} + A_{2,1}$, we only need search over all designs with $A_{3,0} \le (196 - 126)/3 = 23.3$, leading to $A_{3,0} \le 23$. This is a feasible task. A complete enumeration (to be explained later) shows that there are exactly 17 regular 2^{29-23} designs with $A_{3,0} \le 23$, among which one has resolution IV. It is straightforward to verify that $2.9A_{3,0} + A_{2,1}$ has minimum 196 among all 17 2^{29-23} designs with $A_{3,0} \le 23$. Then, by Lemma 7, MA W_{cc} designs coincide with MA W_2 and W_{scf} designs.

When MA W_2 and W_{scf} designs are different or when they do not minimize $\alpha A_{3,0} + A_{2,1}$ for all α with $0 \le \alpha < 3$, Lemma 7 cannot be used; then MA W_{cc} designs are determined by sequentially comparing the complete sequence in (2). Fortunately, this happens only for the following five cases: $(n, p) = (7, 2), (12, 2), (25, 4), (26, 4), (29, 3)$. For the first two cases, MA W_{cc} designs coincide with MA W_2 designs; for the last three cases, MA W_{cc} designs are different from MA W_2 designs, which coincide with MA W_{scf} and W_1 designs.

Now we explain how to enumerate all 2^{29-23} designs with $A_{3,0} \le 23$. Note that a 3-letter word consists of three factors and there are 29 factors in a 2^{29-23} design. Therefore, for any 2^{29-23} design with $A_{3,0} = 23$, there must exist a column appearing in at least $3 \times 23/29 = 2.4$ or 3 words of length 3. Deleting that column yields a 2^{28-22} design with $A_{3,0} \le 20$. Therefore, all 2^{29-23} designs with $A_{3,0} \le 23$ can be enumerated by adding a column to all 2^{28-22} designs with $A_{3,0} \le 20$, which in turn can be enumerated by adding a column to all 2^{27-21} designs with $A_{3,0} \le 17$. This can be done sequentially in the same way as in Chen, Sun and Wu [7] and Xu [23], as long as the number of designs is not too large at each step. We shall point out the importance of the lower bound of $A_{2,1}$ in Theorem 2. Without this bound, one has to search over all 2^{29-23} designs with $A_{3,0} \leq 196/3 = 65.3$. This is not a feasible task because there are more than 100,000 2^{29-23} designs with $A_{3,0} \leq 65$ and it is impossible to enumerate all of them with the current method and computer.

Finally, we summarize the differences of MA blocked designs under different criteria for all 8, 16, 27, 32 runs, 64 runs up to 32 factors, and all 81 runs. We observed that MA blocked designs under all four criteria are the same in most cases. This occurs for all 8 and 27 runs, which can be easily verified with Theorems 5 and 6. When MA blocked designs under four criteria are not all the same, one of the following four situations occurs:

1. MA W_1, W_2 and W_{cc} designs are the same, but they differ from MA W_{scf} designs.
2. MA W_2, W_{scf} and W_{cc} designs are the same, but they differ from MA W_1 designs.
3. MA W_1, W_2 and W_{scf} designs are the same, but they differ from MA W_{cc} designs.
4. MA W_2 and W_{cc} designs are the same, but they differ from MA W_1 or W_{scf} designs.

Situation 1 occurs once for 32 runs with $(n,p) = (6,1)$ and once for 64 runs with $(n,p) = (7,2)$, and does not occur for 16 and 81 runs. Situation 2 occurs twice for 16 runs with $(n,p) = (5,1),(5,2)$, 12 times for 32 runs, 35 times for 64 runs, and twice for 81 runs with $(n,p) = (11,1),(11,2)$. Situation 3 occurs once for 32 runs with $(n,p) = (13,3)$, three times for 64 runs with $(n,p) = (25,4),(26,4),(29,3)$, and three times for 81 runs with $(n,p) = (9,2),(17,2),(21,2)$. Situation 4 occurs only once for 64 runs with $(n,p) = (12,2)$.

Except for situation 3, MA W_{cc} designs coincide with MA W_2 designs, which are given by Xu and Lau [24]. Table 2 gives MA designs for situation 3 with treatment and block columns in the same fashion as Cheng and Wu [10] and Xu and Lau [24]. The designs are labeled as n-$k\cdot i/\text{B}p(W)$, where i denotes the rank of the unblocked s^{n-k} design under the MA criterion, p denotes the number of block variables, and W denotes the MA W-criterion. See Xu and Lau [24] for generator matrices and column labels. To save space, in Table 2 independent columns are omitted in the treatment columns; only generators are given in the block columns, treatment wordlength pattern is truncated as $W_t = (A_{3,0}, A_{4,0}, A_{5,0}, A_{6,0})$ and block wordlength pattern is truncated as $W_b = (A_{2,1}, A_{3,1}, A_{4,1}, A_{5,1})$. The last two columns in Table 2 give the numbers of clear main effects ($C1$) and of clear two-factor interactions ($C2$). A main effect or two-factor interaction is *clear* if it is not aliased with any other main effect or two-factor interaction and is not confounded with any block effect [19].

TABLE 2
MA blocked designs for situation 3

Design	Treatment	W_t	Block	W_b	$C1$	$C2$
			32 runs			
13-8.1/B3(*)	31 7 11 21 25 13 14 19	0 55 0 96	3 5 17	36 0 310 0	13	0
13-8.4/B3(W_{cc})	31 7 11 21 13 14 26 3	4 39 32 48	5 10 19	22 76 124 288	4	0
			64 runs			
25-19.1/B4(*)	31 35 13 52 14 55 37 61 11 19 21 44 7 62 25 49 22 41 38	0 435 0 5440	3 5 9 48	144 0 5923 0	25	0
25-19.17/B4(W_{cc})	31 35 13 52 14 55 21 37 11 19 25 38 7 26 49 22 28 50 9	8 378 336 4032	3 5 17 41	92 568 2688 13104	8	0
26-20.1/B4(*)	31 35 13 52 14 55 37 61 11 19 21 44 7 62 25 49 22 41 38 26	0 515 0 7062	3 5 9 48	156 0 6999 0	26	0
26-20.50/B4(W_{cc})	31 35 13 52 14 55 21 37 11 19 25 38 7 26 49 22 28 50 9 33	16 386 672 4368	3 5 17 41	100 632 3248 15960	0	0
29-23.1/B3(*)	31 35 13 52 14 55 37 61 11 19 21 44 7 62 25 49 22 41 38 26 28 42 47	0 819 0 14560	5 17 33	91 0 5187 0	29	0
29-23.4/B3(W_{cc})	31 35 13 52 14 55 37 61 11 19 21 44 7 62 25 49 22 41 26 28 42 56 3	12 707 640 11536	9 20 38	46 484 2252 14016	4	0
			81 runs			
9-5.1/B2(*)	22 9 24 31 34	0 18 36 12	4 20	9 30 117 162	9	0
9-5.2/B2(W_{cc})	22 9 24 31 3	1 18 27 28	6 18	6 44 90 186	6	5
17-13.1/B2(*)	22 9 24 31 3 25 13 37 6 18 7 35 12	20 336 1014 5072	4 15	40 210 2079 9256	0	0
17-13.2/B2(W_{cc})	22 9 24 31 3 25 13 37 6 18 7 35 16	23 306 1107 4952	12 15	28 303 1782 9814	0	0
21-17.1/B2(*)	22 9 24 31 3 25 13 37 6 18 7 35 12 38 15 16 19	51 729 3717 21819	4 26	48 550 4590 32418	0	0
21-17.2/B2(W_{cc})	22 9 24 31 3 25 13 37 15 23 16 34 6 38 7 18 26	52 720 3735 21876	11 30	45 573 4545 32310	0	0

NOTES. (*) MA W_{scf}, W_1 and W_2 designs.

For all designs given in Table 2, MA W_{cc} designs have a larger $A_{3,0}$ value but a smaller $A_{2,1}$ value than corresponding MA designs under the other three criteria. Indeed, these MA W_{cc} designs achieve the lower bound of $A_{2,1}$ in Theorem 2, whereas MA designs under the other three criteria originate from unblocked MA designs.

Note that for 81 runs, MA W_{scf}, W_1 and W_2 designs 9-5.1/B2 have the same $3A_{3,0} + A_{2,1}$ value as the MA W_{cc} design 9-5.2/B2. This also happens with 21-17.1/B2 and 21-17.2/B2. These examples show that the condition $\alpha < 3$ in Lemma 7 is necessary.

APPENDIX

Further notation and results in coding theory are necessary in order to prove Theorem 1. The *Hamming weight* of a vector $u = (u_1, \ldots, u_n)$, denoted by $wt(u)$, is the number of its nonzero components u_i.

Associated with any linear code D is another linear code, called its *dual* and denoted by $D^\perp$. Suppose D is an $[n, m]$ linear code with generator matrix G over $GF(s)$. The dual $D^\perp$ is the null space of G, that is, $D^\perp = \{u \in V_n : uG' = 0\}$, where G' is the transpose of G. The dual $D^\perp$ is indeed the defining contrast subgroup of D.

Suppose D is an $[n_1 + n_2, m]$ linear code over $GF(s)$ and $D^\perp$ is its dual code. Each vector u in D and $D^\perp$ can be written as $u = (u_1, u_2)$, where $u_1 \in V_{n_1}$ and $u_2 \in V_{n_2}$. Let $B_{i_1,i_2}(D)$ and $B_{i_1,i_2}(D^\perp)$ be the number of vectors in D and respectively in $D^\perp$ with $wt(u_1) = i_1$ and $wt(u_2) = i_2$.

The following result, a special case of Lemma 4.3 of Xu [21], generalizes the Pless power moment identities [16].

LEMMA A.1. *For integers $k_1, k_2 \geq 0$,*

$$s^{-m} \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} i_1^{k_1} i_2^{k_2} B_{i_1,i_2}(D) = \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} B_{j_1,j_2}(D^\perp) Q_{k_1}(j_1; n_1, s) Q_{k_2}(j_2; n_2, s),$$

where $Q_k(j; n, s)$ is defined in (9).

PROOF OF THEOREM 1. Let $N = s^{n-k}$ and $n_2 = L_p$. Then $D = (D_T, D_F)$ is an $[n + n_2, n - k]$ linear code. Let $D^\perp$ be the dual code of D. Each vector u in D and $D^\perp$ can be written as $u = (u_1, u_2)$, where $u_1 \in V_n$ and $u_2 \in V_{n_2}$. It is known that the wordlength patterns are proportional to the split weight distributions of $D^\perp$ as follows: for $i = 0, \ldots, n$,

$$A_{i,0}(D) = B_{i,0}(D^\perp)/(s - 1) \quad \text{and} \quad A_{i,1}(D) = B_{i,1}(D^\perp)/(s - 1); \tag{A.1}$$

see [18] and [5]. By (7),

$$\begin{aligned} K_{t,1}(D) &= N^{-1} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n_2} (n - i_1)^t (n_2 - i_2) B_{i_1,i_2}(D) \\ &= N^{-1} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n_2} \sum_{k=0}^{t} \binom{t}{k} (-1)^k n^{t-k} i_1^k (n_2 - i_2) B_{i_1,i_2}(D) \\ &= N^{-1} \sum_{k=0}^{t} \binom{t}{k} (-1)^k n^{t-k} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n_2} (i_1^k n_2 - i_1^k i_2) B_{i_1,i_2}(D). \end{aligned}$$

By Lemma A.1,

$$K_{t,1}(D) = \sum_{k=0}^{t} \binom{t}{k} (-1)^k n^{t-k} \sum_{j_1=0}^{n} \sum_{j_2=0}^{n_2} B_{j_1,j_2}(D^\perp) Q_k(j_1; n, s) \times [Q_0(j_2; n_2, s) n_2 - Q_1(j_2; n_2, s)].$$

Recall that $Q_k(j; n, s) = 0$ for $j > k$. Then

$$K_{t,1}(D) = \sum_{k=0}^{t} \binom{t}{k} (-1)^k n^{t-k} \sum_{j_1=0}^{n} Q_k(j_1; n, s) \Delta(D^\perp, j_1; n_2, s),$$

where $\Delta(D^\perp, j_1; n_2, s) = B_{j_1,0}(D^\perp) Q_0(0; n_2, s) n_2 - B_{j_1,0}(D^\perp) Q_1(0; n_2, s) - B_{j_1,1}(D^\perp) Q_1(1; n_2, s)$. Note that $Q_0(0; n_2, s) = 1$, $Q_1(0; n_2, s) = n_2(s-1)s^{-1}$ and $Q_1(1; n_2, s) = -s^{-1}$. Then

$$\begin{aligned} K_{t,1}(D) &= \sum_{k=0}^{t} \binom{t}{k} (-1)^k n^{t-k} \sum_{j_1=0}^{n} Q_k(j_1; n, s) \times [B_{j_1,0}(D^\perp) n_2 s^{-1} + B_{j_1,1}(D^\perp) s^{-1}] \\ &= \sum_{j_1=0}^{n} \sum_{k=0}^{t} \binom{t}{k} (-1)^k n^{t-k} Q_k(j_1; n, s) \times [B_{j_1,0}(D^\perp) n_2 s^{-1} + B_{j_1,1}(D^\perp) s^{-1}] \\ &= \sum_{j_1=0}^{n} c_t(j_1; n, s)(s-1)^{-1} [B_{j_1,0}(D^\perp) n_2 + B_{j_1,1}(D^\perp)] s^{-1}. \end{aligned}$$

Then (12) follows from (A.1) and the fact that $c_t(j_1; n, s) = 0$ when $j_1 > t$. □

Proof of Lemma 3. By (8) and (15),

$$\begin{aligned} K_{t,1}(D) &= N^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} [\delta_{ij}(D_T)]^t L_{p-1} \\ &\quad + N^{-2} \sum_{m=1}^{s^p} \sum_{i \in B_m} \sum_{j \in B_m} [\delta_{ij}(D_T)]^t (L_p - L_{p-1}) \\ &= L_{p-1} K_{t,0}(D) + (L_p - L_{p-1}) s^{-2p} \sum_{m=1}^{s^p} K_t(D_m). \end{aligned}$$

Then the result follows from (16). □

Proof of Lemma 4. Let $N_1 = s^{n-k-p}$ and $J_t(D_1) = \sum_{1\le i<j\le N_1}[\delta_{ij}(D_1)]^t/[N_1(N_1-1)/2]$ for $t \ge 0$. It is easy to verify that, for $t \ge 0$,

$$K_t(D_1) = N_1^{-1}[(N_1-1)J_t(D_1) + n^t]. \tag{A.2}$$

(i) Recall that for an $(s^{n-k}: s^p)$ RME design D, $K_{1,0}(D) = s^{-1}n$ and $K_{1,1}(D) = s^{-2}nL_p$. Then, by Lemma 3, $K_1(D_1) = s[K_{1,1}(D) - L_{p-1}K_{1,0}(D)] = s^{-1}n$. By (A.2), $J_1(D_1) = (N_1-1)^{-1}[N_1K_1(D_1) - n] = n(N_1-s)[(N_1-1)s]^{-1} = J$. On the other hand, Xu [22] showed that $J_1(D_1) \ge J$, with equality if and only if D_1 is an $OA(N_1, n, s, 1)$. Therefore, D_1 must be an $OA(N_1, n, s, 1)$.

(ii) Since the number of coincidences, $\delta_{ij}(D_1)$, must be an integer, it is easy to verify that, given $J_1(D_1) = J$, $J_2(D_1)$ achieves the minimum value of $J^2 + \eta(1-\eta)$ when all $\delta_{ij}(D_1)$, $i<j$, take on only one of the two values, $\lfloor J\rfloor$ and $\lfloor J\rfloor + 1$, where $\lfloor x\rfloor$ is the largest integer that does not exceed x. Then the result follows from (A.2).

(iii) By Lemma 3, $K_{2,1}(D) = L_{p-1}K_{2,0}(D) + s^{-1}K_2(D_1)$. The result follows from (ii) and the fact that $K_{2,0}(D) = s^{-2}n(n+s-1)$. □

Proof of Lemma 5. By the binomial theorem,

$$\begin{aligned}
K_t(D_{\mathrm{un}}) &= N^{-2}\sum_{i=1}^{N}\sum_{j=1}^{N}[\delta_{ij}(D_T) + \delta_{ij}(D_F)]^t \\
&= N^{-2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{r=0}^{t}\binom{t}{k}[\delta_{ij}(D_T)]^{t-r}[\delta_{ij}(D_F)]^r.
\end{aligned}$$

By (8), (15) and (16),

$$\begin{aligned}
K_t(D_{\mathrm{un}}) &= N^{-2}\sum_{r=0}^{t}\binom{t}{k}\sum_{i=1}^{N}\sum_{j=1}^{N}[\delta_{ij}(D_T)]^{t-r}L_{p-1}^r \\
&\quad + N^{-2}\sum_{r=0}^{t}\binom{t}{k}\sum_{m=1}^{s^p}\sum_{i\in B_m}\sum_{j\in B_m}[\delta_{ij}(D_T)]^{t-r}(L_p^r - L_{p-1}^r) \\
&= \sum_{r=0}^{t}\binom{t}{k}L_{p-1}^r K_{t-r,0}(D) + \sum_{r=0}^{t}\binom{t}{k}(L_p^r - L_{p-1}^r)s^{-2p}\sum_{m=1}^{s^p}K_{t-r}(D_m) \\
&= K_{t,0}(D) + \sum_{r=1}^{t}\binom{t}{k}L_{p-1}^r K_{t-r,0}(D) \\
&\quad + s^{-p}\sum_{r=1}^{t}\binom{t}{k}(L_p^r - L_{p-1}^r)K_{t-r}(D_1).
\end{aligned}$$

Then the result follows from Lemma 3 with some algebra. □

Proof of Lemma 6. First $A_{3,0}(D)$, $A_{2,1}(D)$ and $A_{4,0}(D)$ are minimized sequentially because D has MA with respect to W_{scf}. Next, among designs with the same values of $A_{3,0}(D)$, $A_{2,1}(D)$ and $A_{4,0}(D)$, $A_{5,0}(D)$ is minimized because D has MA with respect to W_1, and $A_{3,1}(D)$ is minimized because D has MA with respect to W_{scf}. Continuing this type of argument shows that D has MA with respect to W_2. □

Proof of Lemma 7. First $3A_{3,0}(D) + A_{2,1}(D) = (3-\alpha)A_{3,0}(D) + [\alpha A_{3,0}(D) + A_{2,1}(D)]$ is minimized because both $A_{3,0}(D)$ and $\alpha A_{3,0}(D) + A_{2,1}(D)$ are minimized. For designs with the same value of $3A_{3,0}(D) + A_{2,1}(D)$, they must have the same values of $A_{3,0}(D)$ and $A_{2,1}(D)$. Then $A_{4,0}(D)$ is minimized among designs with the minimum of $3A_{3,0}(D)+A_{2,1}(D)$ because D has MA with respect to W_{scf}. Among designs with the same values of $3A_{3,0}(D)+A_{2,1}(D)$ and $A_{4,0}(D)$, $A_{5,0}(D)$ is minimized because D has MA with respect to W_2 and $A_{3,1}(D)$ is minimized because D has MA with respect to W_{scf}; therefore, $10A_{5,0}(D) + A_{3,1}(D)$ is also minimized. For designs with the same values of $3A_{3,0}(D) + A_{2,1}(D)$, $A_{4,0}(D)$ and $10A_{5,0}(D) + A_{3,1}(D)$, they must have the same $A_{5,0}(D)$ and $A_{3,1}(D)$ values. Continuing this type of argument shows that D has MA with respect to W_{cc}. □

Proof of Theorem 3. Note that D achieves the lower bound in Theorem 2 if and only if D_1 achieves the lower bound in Lemma 4(ii). When the latter lower bound is achieved, $K_2(D_1)$ is minimized and $K_t(D_1)$ is uniquely determined for $t \geq 3$. By Lemma 3, $K_{t,1}(D)$ is determined by $K_{t,0}(D)$ for $t \geq 3$. Because D_T has MA, by Lemma 2, $K_{3,0}(D), K_{4,0}(D), \ldots, K_{n,0}(D)$ are minimized sequentially. Then any combined sequence of $(K_{3,0}(D), K_{4,0}(D), \ldots, K_{n,0}(D))$ and $(K_{2,1}(D), K_{3,1}(D), \ldots, K_{n,1}(D))$ is also minimized sequentially as long as $K_{t,1}(D)$ is minimized after $K_{t,0}(D)$ for $t = 2, \ldots, n$. Hence, D has minimum moment aberration and MA with respect to W_{scf}, W_1 and W_2. Finally, because $A_{2,1}(D)$ is minimized among all possible designs, by Lemma 7, D has MA with respect to W_{cc}. □

Proof of Theorem 4. *Necessity.* When $p = n-k-1$, by Lemma 4(i), the principal block D_1 is an $OA(s,n,s,1)$. Then it must contain a row of all zeros and other $s-1$ rows without zeros.

Sufficiency. Let $u = (u_1, \ldots, u_n)$ be a row vector of D_T without zeros. Because none of u_i is zero, the linear equation $\sum_{i=1}^{n-k} x_i u_i = 0$ has s^{n-k-1} solutions over $GF(s)$. Let F be an $(n-k) \times L_{n-k-1}$ matrix, where the columns correspond to the solutions with the first nonzero element being unity. Clearly, F has rank $n-k-1$, and it is an $(n-k-2)$-flat in $PG(n-k-1,s)$. On the other hand, D_T is an $[n,n-k]$ linear code over $GF(s)$. Let

$T=(t_{ij})$ be the $(n-k)\times n$ generator matrix of D_T. We need to show that T and F have no columns in common so that the resulting blocked design $D=(D_T,D_F)$ is an RME design.

Without loss of generality, let $T=[I_{n-k},E]$, where I_{n-k} is the $n-k$ identity matrix and E is an $(n-k)\times k$ matrix. Because the row vectors of T form a basis for D_T, u can be uniquely represented as a linear combination of the row vectors of T. Then it is clear that $\sum_{i=1}^{n-k} t_{ij}u_i=u_j\neq 0$ for $j=1,\ldots,n$. This proves that T and F have no columns in common. □

Proof of Theorem 5. We only need to prove the MA optimality. Following the argument preceding the theorem, we can write an $(s^{n-k}:s^{n-k-1})$ RME design as $D=(D_T,D_F)$, where D_T is a projection design of $\tilde{H}_{n-k}$. Recall that the principal block D_1 is an $OA(s,n,s,1)$; therefore, each level appears exactly once in each column. It is evident that $\delta_{ij}(D_1)=0$ when $i\neq j$ and $\delta_{ij}(D_1)=n$ when $i=j$; hence, $K_t(D_1)=s^{-1}n^t$ for $t>0$. By Lemma 3, $K_{t,1}(D)$ is determined by $K_{t,0}(D)$ for $t>0$. By (11) and (12), $A_{t,1}(D)$ is determined by $A_{1,0}(D),\ldots,A_{t,0}(D)$ uniquely. Thus, to sequentially minimize the sequences in (1), (2), (3) and (4), it is sufficient to sequentially minimize $A_{1,0}(D),\ldots,A_{n,0}(D)$. Then the result follows from the condition that D_T has MA among all projection designs of $\tilde{H}_{n-k}$. □

Proof of Theorem 6. Given $K_{3,0}(D)$, $K_{4,0}(D),\ldots,K_{t,0}(D)$, $3\leq t\leq n$, by Lemma 5, sequentially minimizing $K_{2,1}(D),K_{3,1}(D),\ldots,K_{t-1,1}(D)$ is equivalent to sequentially minimizing $K_3(D_{\text{un}}),K_4(D_{\text{un}}),\ldots,K_t(D_{\text{un}})$. Because D_T has MA, by Lemma 2, $K_{3,0}(D),K_{4,0}(D),\ldots,K_{n,0}(D)$ are minimized sequentially. Because D_{un} has MA, $K_3(D_{\text{un}}),K_4(D_{\text{un}}),\ldots,K_n(D_{\text{un}})$ are minimized sequentially. Then any combined sequence of $(K_{3,0}(D),K_{4,0}(D),\ldots,K_{n,0}(D))$ and $(K_{2,1}(D),K_{3,1}(D),\ldots,K_{n,1}(D))$ is also minimized sequentially as long as $K_{t-1,1}(D)$ is minimized after $K_{t,0}(D)$ for $t=3,\ldots,n$. Hence, D has minimum moment aberration and MA with respect to W_{scf}, W_1 and W_2. By Lemma 5, $K_3(D_{\text{un}})=K_{3,0}(D)+3K_{2,1}(D)+\textit{constant}$; therefore, $K_{3,0}(D)+3K_{2,1}(D)$ is minimized and, by (13) and (14), $A_{3,0}(D)+A_{2,1}(D)$ is minimized. Then, by Lemma 7, D has MA with respect to W_{cc}. □

Proof of Theorem 7. Because D_T has MA, $A_{3,0}(D),A_{4,0}(D),\ldots,A_{n,0}(D)$ are minimized sequentially. Note that $A_{2,1}(D),A_{3,1}(D)$, and so on are minimized in W_1 or W_2 no sooner than in W_{scf}. Therefore, if D has MA with respect to W_{scf}, it must have MA with respect to W_1 and W_2. The MA W_{cc} optimality follows from Lemma 7. □

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