

# Extra-Dimensional Approach to Option Pricing and Stochastic Volatility

M. Q. Truong\*

Department of Biological and Physical Sciences

Fontbonne University

St. Louis, MO 63105

February 5, 2010

## Abstract

The generalized 5D Black-Scholes differential equation with stochastic volatility is derived. The projections of the stochastic evolutions associated with the random variables from an enlarged space or super-space onto an ordinary space can be achieved via higher-dimensional operators. The stochastic nature of the securities and volatility associated with the 3D Merton-Garman equation can then be interpreted as the effects of the extra dimensions. We showed that the Merton-Garman equation is the first excited state, i.e.  $n = m = 1$ , within a family which contain an infinite numbers of Merton-Garman-like equations.

---

\*mtruong@fontbonne.edu

# 1 Introduction

The time-evolution of the option pricing has been well-studied starting with the work of Black and Scholes[1]. This pioneering result of Black and Scholes was then generalized to include stochastic volatility by Merton [2] and Garman[3].

The methodology of high energy physics have been used in the analysis of the option pricing problem [4]. The analysis of the problem of option pricing by the methods of theoretical physics provides additional computational power to the field of mathematical finance. The Black-Scholes partial differential equation and its generalization have been reinterpreted under quantum mechanical formalism by [5] and [6], where the Hamilton for the Merton-Garman equation was derived. The dynamics of the option price of a security derivative can then be given in terms of path integrals. This paper used methods of theoretical physics, specifically, extra-dimensional formalism [7], as applied to the analysis of problems associated with option pricing.

The general outline of this paper is divided as follows. In section 2, the generalized Black-Scholes equation is derived from the Langevin stochastic differential equations. In section 3, we review extra-dimensional theories and their applications. In section 4, we derive the Merton-Garman equation via extra-dimensional approach. And in section 5, conclusions and future outlook are drawn.

## 2 Derivative with Stochastic Volatility

A security is any financial instrument that can be traded on the markets. A security derivative is also a financial instrument that can be derived from an underlying security and can also be traded on the markets. Some of these security derivatives are futures, forwards and options. Suppose the value of

the option  $f$  at time  $t$  is given

$$f = f(t, S(t), V(t), K, T), \quad (1)$$

where  $S(t)$  is the value of the security at  $t$ ,  $V(t)$  is the variance of the stochastic volatility,  $K$  is the strike price, and  $T$  is the time of maturity. At time  $t = T$ , the value of the option can be characterized by  $f(T, S(T))$ . The stochastic nature of the security and the volatility are governed by the following coupled stochastic Langevin equations

$$\frac{dS(t)}{dt} = \phi S(t) + \sigma(t)S(t)R(t), \quad (2)$$

$$\frac{dV(t)}{dt} = \mu V(t) + \xi V(t)Q(t), \quad (3)$$

where  $\phi$  and  $\mu$  are the drift rates associated with the security  $S(t)$  and  $V(t)$  respectively,  $\sigma(t)$  is the stochastic volatility,  $V(t) = \sigma(t)^2$  is the variance. The terms  $R(t)$  and  $Q(t)$  are the correlated Gaussian white noise terms with zero means and the following relations

$$\langle R(t)R(t') \rangle = \langle Q(t)Q(t') \rangle = \frac{1}{\epsilon} \delta(t - t'), \quad (4)$$

$$\langle Q(t)R(t') \rangle = \langle R(t)Q(t') \rangle = \frac{\rho}{\epsilon} \delta(t - t'), \quad (5)$$

where  $-1 \leq \rho \leq 1$  is the correlation coefficient between  $S(t)$  and  $V(t)$ . We now write down the second-order Taylor series expansion for the total time derivative of the value of the option  $f$ . The series expansion yields

$$\begin{aligned} \frac{df}{dt} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{f(t + \epsilon, S(t + \epsilon), V(t + \epsilon)) - f(t, S(t), V(t))\} \quad (6) \\ \frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} \frac{dS}{dt} + \frac{\partial f}{\partial V} \frac{dV}{dt} \\ &\quad + \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2!} \left\{ \frac{\partial^2 f}{\partial S^2} \left( \frac{dS}{dt} \right)^2 + \frac{\partial^2 f}{\partial V^2} \left( \frac{dV}{dt} \right)^2 + \frac{\partial^2 f}{\partial S^2} \left( \frac{dS}{dt} \right)^2 \right\}. \end{aligned}$$

Using the coupled stochastic Langevin equations, the series yields

$$\begin{aligned}\frac{df}{dt} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{f(t + \epsilon, S(t + \epsilon), V(t + \epsilon)) - f(t, S(t), V(t))\} \quad (7) \\ \frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} (\phi S + \sigma S R) + \frac{\partial f}{\partial V} (\mu V + \xi V Q) \\ &\quad + \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2!} \left\{ \begin{aligned} &\frac{\partial^2 f}{\partial S^2} (\phi S + \sigma S R)^2 + \frac{\partial^2 f}{\partial V^2} (\mu V + \xi V Q)^2 \\ &+ 2 \frac{\partial^2 f}{\partial S \partial V} (\phi S + \sigma S R) (\mu V + \xi V Q) \end{aligned} \right\}.\end{aligned}$$

We expand equation (7) and use equations (4) and (5). Taking the limit  $\epsilon \rightarrow 0$ , and separating the Taylor expansion into the deterministic and stochastic parts, we have

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \phi S \frac{\partial f}{\partial S} + \mu V \frac{\partial f}{\partial V} + \frac{1}{2!} \left\{ \begin{aligned} &\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \xi^2 V^2 \frac{\partial^2 f}{\partial V^2} \\ &+ 2 \sigma S \xi V \rho \frac{\partial^2 f}{\partial S \partial V} \end{aligned} \right\} \quad (8) \\ &\quad + \sigma S \frac{\partial f}{\partial S} R + \xi V \frac{\partial f}{\partial V} Q.\end{aligned}$$

Recall that  $V = \sigma^2$ , we have

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \phi S \frac{\partial f}{\partial S} + \mu V \frac{\partial f}{\partial V} + \frac{1}{2!} \left\{ \begin{aligned} &\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \xi^2 V^2 \frac{\partial^2 f}{\partial V^2} \\ &+ 2 \sigma^3 S \xi \rho \frac{\partial^2 f}{\partial S \partial V} \end{aligned} \right\} \quad (9) \\ &\quad + \sigma S \frac{\partial f}{\partial S} R + \xi V \frac{\partial f}{\partial V} Q.\end{aligned}$$

We simplify the equation by the following substitutions

$$\omega = \frac{\partial}{\partial t} + \phi S \frac{\partial}{\partial S} + \mu V \frac{\partial}{\partial V} + \frac{1}{2!} \left\{ \begin{aligned} &\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \xi^2 V^2 \frac{\partial^2}{\partial V^2} \\ &+ 2 \sigma S \xi V \rho \frac{\partial^2}{\partial S \partial V} \end{aligned} \right\} \quad (10)$$

$$\alpha_1 = \sigma S \frac{\partial f}{\partial S} \quad (11)$$

$$\alpha_2 = \xi V \frac{\partial}{\partial V}. \quad (12)$$

The equation reduces to

$$\frac{df}{dt} = (\omega + \alpha_1 R + \alpha_2 Q) f, \quad (13)$$

where  $\omega, \alpha_1, \alpha_2$  are linear operators. We consider the following self-replicating portfolio

$$\pi(t) = \theta_1 f(t) + \theta_2 S(t), \quad (14)$$

where  $\theta_1$  and  $\theta_2$  are the amounts for the associated option  $f$  and security  $S$ . The total time derivative of the portfolio yields

$$\begin{aligned} \frac{d\pi}{dt} &= \theta_1 \frac{df}{dt} + \theta_2 \frac{dS}{dt} \\ \frac{d\pi}{dt} &= \theta_1 f \omega + \phi \theta_2 S + (\alpha_1 R + \alpha_1 Q) \theta_1 f + \theta_2 \sigma S R. \end{aligned} \quad (15)$$

By inspection, the total time derivative contains both deterministic and random terms. To remove the stochastic terms in equation (15), we use the following matrix relation

$$\begin{bmatrix} \alpha_1 & \sigma \\ \alpha_2 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 f \\ \theta_2 S \end{bmatrix} = 0. \quad (16)$$

Equation (15) reduces

$$\frac{d\pi}{dt} = \theta_1 f \omega + \phi \theta_2 S, \quad (17)$$

which is pure deterministic since the stochastic terms containing  $R(t)$  and  $Q(t)$  are removed. The elimination method of the Gaussian noise terms in equation (15) by using the matrix relation (16) is analogous to the hedging technique used in the case of constant volatility  $\sigma \neq \sigma(t)$ , i.e. where the fluctuations of one security is cancelled by another security. The absence of arbitrage forces the time-derivative of the portfolio to be directly proportional

to the risk-neutral rate  $r$ , hence

$$\frac{d\pi}{dt} = r\pi. \quad (18)$$

Equating equations (15) and (18) above yields

$$(\omega - r)\theta_1 f + (\phi - r)\theta_2 S = 0. \quad (19)$$

The equation (19) can be satisfied by requiring the following constraint relation

$$\begin{bmatrix} (\omega - r) & (\phi - r) \end{bmatrix} = \begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix} \begin{bmatrix} \alpha_1 & \sigma \\ \alpha_2 & 0 \end{bmatrix}, \quad (20)$$

where  $\lambda_1(t)$  and  $\lambda_2(t)$  are arbitrary. Writing equation (20) in component forms

$$\omega - r = \lambda_1(t)\alpha_1 + \lambda_2(t)\alpha_2 \quad (21)$$

$$\phi - r = \lambda_1(t)\sigma. \quad (22)$$

Substituting for  $\lambda_1(t)$ , equation (19) becomes

$$(\omega - r)f = \left\{ \left( \frac{\phi - r}{\sigma} \right) \alpha_1 + \lambda_2(t)\alpha_2 \right\} f. \quad (23)$$

Upon re-substitution, we obtain the Merton-Garman equation

$$\begin{aligned} & \frac{\partial f}{\partial t} + \frac{1}{2!} \left\{ \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \xi^2 V^2 \frac{\partial^2 f}{\partial V^2} + 2\sigma^3 S \xi \rho \frac{\partial^2 f}{\partial S \partial V} \right\} - r f \\ & = -rS \frac{\partial f}{\partial S} + (\lambda_2(t)\xi - \mu) V \frac{\partial f}{\partial V}. \end{aligned} \quad (24)$$

For a large class of problems, Hull and White [8] argued that we can redefine

$$\lambda_2(t)\xi - \mu = -\bar{\mu}. \quad (25)$$

The Merton-Garman equation becomes

$$\begin{aligned} & \frac{\partial f}{\partial t} + \frac{1}{2!} \left\{ \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \xi^2 V^2 \frac{\partial^2 f}{\partial V^2} + 2\sigma^3 S \xi \rho \frac{\partial^2 f}{\partial S \partial V} \right\} - r f \quad (26) \\ & = -rS \frac{\partial f}{\partial S} - \bar{\mu} \sigma^2 \frac{\partial f}{\partial S}. \end{aligned}$$

This equation is valid for any security derivative  $f$  with stochastic volatility.

### 3 Extra-Dimensional Theories and Application

This section illustrates one of the many tools of theoretical physics in analyzing problems associated with option trading. The idea of extra dimensions dated back to Kaluza [9] at the early 1920's. In this era, Maxwell's beautiful unified electromagnetic theory had inspired Einstein to unify space and time into spacetime. The spacetime unification and other postulates had allowed Einstein to formulate the general theory of relativity. Follow the same guiding principle, Kaluza was successfully able to unify gravity and electromagnetism by postulating an extra spacetime dimension. The electromagnetic theory was further enhanced by Klein [10] in the mid 1920's. The enhanced unified theory of gravity and electromagnetism is known today as Kaluza-Klein theory.

To illustrate the method, we consider a five dimensional  $5D$  spacetime with the following flat metric:  $\eta_{AB} = (\eta_{\mu\nu}, \zeta)$ , where  $\eta_{\mu\nu} = (-1, \delta_{ik})$  is four dimensional (4D) flat Minkowski metric. The Capital Latin indices run in 5D spacetime as  $A, B, .. = 0, 1, 2, 3, 5$ , Greek indices run in 4D space-time  $\mu, \nu, .. = 0, 1, 2, 3$ , and small Latin indices in 3D Euclidean space  $i, j, .. = 1, 2, 3$ . The symbol  $\epsilon = \pm 1$  (-1 for a time-like extra dimension, +1 for a space-like extra dimension) represents the signature of the extra-dimensions. To obtain the proper 5D Klein-Gordon equation, we have to reanalyze the

5D energy-momentum relation. Let us for a moment define the 5-position vector of the particle as  $x^A = (x^0, x^1, x^2, x^3, x^5) = (ct, x^1, x^2, x^3, x^5)$ , with the extra dimension being uncompactified and time-dependent  $x^5 = x^5(t)$ . The 5-velocity is defined by

$$U^A = \frac{dx^A}{d\tau} = \gamma \frac{dx^A}{dt} = \gamma \left( c, \vec{v}, \frac{dx^5}{dt} \right), \quad (27)$$

where  $\gamma$  is the usual Lorentz factor from the relation from special relativity,  $t = \gamma\tau$ , and  $\tau$  is the proper time. The 5D-momentum is defined by

$$P^A = mU^A = \gamma m \left( c, \vec{v}, \frac{dx^5}{dt} \right). \quad (28)$$

Analogous to 4D, the 5D energy-momentum relation can be obtained by

$$\begin{aligned} P^A P_A &= \gamma^2 m^2 \left( -c^2 + |\vec{v}|^2 + \epsilon \left( \frac{dx^5}{dt} \right)^2 \right) \\ &= \frac{-m^2 c^2}{1 - \frac{v^2}{c^2}} + \frac{m^2 v^2}{1 - \frac{v^2}{c^2}} + \epsilon \gamma^2 m^2 \left( \frac{dx^5}{dt} \right)^2 \\ &= -m^2 c^2 + \epsilon \gamma^2 m^2 \left( \frac{dx^5}{dt} \right)^2. \end{aligned} \quad (29)$$

If  $x^5$  is time-dependent, led to a non-invariant expression. Therefore  $x^5$  must be time-independent, so that we can have a vanishing term

$$\epsilon \gamma^2 m^2 \left( \frac{dx^5}{dt} \right)^2 = 0. \quad (30)$$

To obtain a proper 5D Klein-Gordon equation, we must rewrite the above expression by absorbing the Lorentz factor with the extra component

$$\epsilon m^2 \left( \frac{d(\gamma x^5)}{dt} \right)^2 = \epsilon m^2 \left( \frac{d(\bar{x}^5)}{dt} \right)^2 = 0, \quad (31)$$



where  $\bar{x}^5 = \gamma x^5$ , the relativistic extra component of the 5D position vector  $\bar{x}^5$ . The reason for redefining the extra dimension this way will be apparent when we examine the energy from the higher dimensional energy-momentum relation. The 5D Klein-Gordon equation:

$$(\square_5 - m^2 c^2)\Phi = 0, \quad (32)$$

where  $\square_5 = \eta^{AB}\partial_B\partial_A = \eta^{\mu\nu}\partial_\nu\partial_\mu + \epsilon\partial_5^2$ . We then compactified the extra space-like dimension and let it be spanned by the relativistic component,  $\bar{x}^5$ . The momentum along the extra dimension,  $\bar{x}^5$ , is quantized by  $\frac{n}{L}$ , where  $L$  is the radius of the compactified extra dimension, and  $n \in \mathbb{Z}$ . This choice of quantization led to the following periodic condition along the relativistic component  $\bar{x}^5$

$$\Phi(x^\mu, \bar{x}^5) = \Phi(x^\mu, \bar{x}^5 + 2\pi L). \quad (33)$$

The proper 5D Klein-Gordon equation can be written manifestly in 4D by considering the following field decomposition:

$$\Phi(x^\mu, \bar{x}^5) = \sum_n \Phi_n(x^\mu) \exp\left(\frac{in\bar{x}^5}{L}\right), n = 0, \pm 1, \pm 2, \dots \quad (34)$$

Using the above field decomposition and the 5D Klein-Gordon equation becomes

$$\left(\square - \epsilon \frac{\gamma^2 n^2}{(2\pi L)^2} - m^2 c^2\right)\Phi_n(x^\mu) = 0. \quad (35)$$

For space-like extra dimension,  $\epsilon = 1$ , and letting the effective mass be  $m_{eff} = \frac{\gamma^2 n^2}{(2\pi L)^2} + m^2 c^2$ . The 5D Klein-Gordon equation reduces to the 4D Klein-Gordon equation

$$(\square - m_{eff})\Phi_n(x^\mu) = 0. \quad (36)$$

The effect of the compactified extra dimensions and the associated periodic

boundary conditions is equivalent to an increase in the mass of the propagating particles, i.e. the modified Klein-Gordon equation (36).

## 4 Merton-Garman Equation via Extra Dimensional Approach

In this section, with the appropriate constraints, the Merton-Garman equation can be derived directly from a higher dimensional space or superspace. Furthermore, the stochastic or random nature of the security  $S(t)$  and volatility  $V(t)$  are seen as the effects of the extra dimensions. Mathematically,  $S(t)$  and  $V(t)$  are being projected from the higher manifold or supermanifold by a higher dimensional operators. The time evolution  $S(t)$  and  $V(t)$  are governed by super stochastic differential equation.

Consider an option of a security to assume the following form

$$f = f_5 = f(t, S(t), V(t), x^5(t), x^6(t)), \quad (37)$$

where  $t$ ,  $S(t)$ , and  $V(t)$  are the usual time, security and volatility respectively and the coordinate functions,  $x^5(t)$  and  $x^6(t)$ , are referred to simply as the coordinates of the extra dimensions. The function  $f_5$  is a real-valued function which maps an element of  $R^5 \subset E^5$  into the real  $R$ , where  $E^5$  is  $5D$  Euclidean space

$$f_5 : R^5 \rightarrow R. \quad (38)$$

Consider a supercoordinate or  $5D$  vector to represent a point in  $R^5$

$$\mathbf{z}^A = (z^2, z^3, z^4, z^5, z^6) = (t, S, V, x^5, x^6) \quad (39)$$

$$\mathbf{z}^A = t\hat{t} + S\hat{s} + V\hat{v} + x^5\hat{x}_5 + x^6\hat{x}_6, \quad (40)$$

where  $A = 2, 3, 4, 5, 6$ . The unit vectors

$$\hat{t} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \hat{s} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \hat{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \hat{x}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \hat{x}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (41)$$

We could also write the supercoordinate  $\mathbf{z}^A$  in matrix notation as

$$\mathbf{z}^A = t\hat{t} + S\hat{s} + V\hat{v} + x^5\hat{x}_5 + x^6\hat{x}_6 \quad (42)$$

$$\mathbf{z}^A = t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + S \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + V \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x^5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x^6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (43)$$

$$\mathbf{z}^A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ S \\ V \\ x^5 \\ x^6 \end{pmatrix} = I_5 z^A. \quad (44)$$

The magnitude of the  $5D$  vector is

$$|z^A| = |I_5 z^A| = \sqrt{t^2 + S^2 + V^2 + (x^5)^2 + (x^6)^2}. \quad (45)$$

We can write  $z^A$  in term of arbitrary unit vector by first finding the unit

vector in the  $z$  direction to be

$$\widehat{z} = \frac{\mathbf{z}^A}{|z^A|} = \frac{1}{\sqrt{t^2 + S^2 + V^2 + (x^5)^2 + (x^6)^2}} \begin{pmatrix} t \\ S \\ V \\ x^5 \\ x^6 \end{pmatrix}. \quad (46)$$

Taking the total differential of  $\mathbf{z}^A$  to first order, we have

$$\begin{aligned} d(\mathbf{z}^A) &= \sum_{B=2}^6 \frac{\partial \mathbf{z}^A}{\partial z^B} dz^B \\ d(\mathbf{z}^A) &= \frac{\partial \mathbf{z}^A}{\partial t} dt + \frac{\partial \mathbf{z}^A}{\partial S} dS + \frac{\partial \mathbf{z}^A}{\partial V} dV + \frac{\partial \mathbf{z}^A}{\partial x^5} dx^5 + \frac{\partial \mathbf{z}^A}{\partial x^6} dx^6 \\ d(\mathbf{z}^A) &= dt \widehat{t} + dS \widehat{s} + dV \widehat{v} + dx^5 \widehat{x}_5 + dx^6 \widehat{x}_6. \end{aligned} \quad (47)$$

Recall that  $f_5 : R^5 \rightarrow R$ , and define by  $y = f_5(z^A)$ . By inspection, taking a total differential of  $f_5(\mathbf{z}^A)$  is equivalent to a mapping from

$$df_5(\mathbf{z}^A) : R \rightarrow R. \quad (48)$$

The mapping gives

$$\begin{aligned} df_5(\mathbf{z}^A) &= \sum_{B=2}^6 \frac{\partial f_5(\mathbf{z}^A)}{\partial z^B} dz^B \\ df_5(\mathbf{z}^A) &= \frac{\partial f_5}{\partial t} dt + \frac{\partial f_5}{\partial S} dS + \frac{\partial f_5}{\partial V} dV + \frac{\partial f_5}{\partial x^5} dx^5 + \frac{\partial f_5}{\partial x^6} dx^6 \\ df_5(\mathbf{z}^A) &= \frac{\partial f_5}{\partial t} \widehat{t} \cdot \widehat{t} dt + \frac{\partial f_5}{\partial S} \widehat{s} \cdot \widehat{s} dS + \frac{\partial f_5}{\partial V} \widehat{v} \cdot \widehat{v} dV \\ &\quad + \frac{\partial f_5}{\partial x^5} \widehat{x}_5 \cdot \widehat{x}_5 dx^5 + \frac{\partial f_5}{\partial x^6} \widehat{x}_6 \cdot \widehat{x}_6 dx^6 \\ df_5(\mathbf{z}^A) &= \text{grad}_5(f_5) \cdot d\mathbf{z}^A, \end{aligned} \quad (49)$$

where

$$grad_5 () = \frac{\partial}{\partial t} \hat{t} + \frac{\partial}{\partial S} \hat{s} + \frac{\partial}{\partial V} \hat{v} + \frac{\partial}{\partial x^5} \hat{x}_5 + \frac{\partial}{\partial x^6} \hat{x}_6.$$

In matrix notation, we have

$$df_5 (\mathbf{z}^A) = \begin{pmatrix} \frac{\partial f_5}{\partial t} & \frac{\partial f_5}{\partial S} & \frac{\partial f_5}{\partial V} & \frac{\partial f_5}{\partial x^5} & \frac{\partial f_5}{\partial x^6} \end{pmatrix} \cdot \begin{pmatrix} dt \\ dS \\ dV \\ dx^5 \\ dx^6 \end{pmatrix} \quad (50)$$

$$df_5 (\mathbf{z}^A) = grad_5 (f_5) \cdot d\mathbf{z}^A.$$

In order to project pertinent information from superspace onto ordinary space, we define a mapping or projection by  $\Pi : R^5 \rightarrow R^3$ , such that

$$\begin{aligned} \Pi df_5 (\mathbf{z}^A) &= \Pi \left[ \frac{\partial f_5}{\partial t} dt + \frac{\partial f_5}{\partial S} dS + \frac{\partial f_5}{\partial V} dV + \frac{\partial f_5}{\partial x^5} dx^5 + \frac{\partial f_5}{\partial x^6} dx^6 \right] \\ df_5 (\Pi \mathbf{z}^A) &= \frac{\partial f_5}{\partial t} dt + \frac{\partial f_5}{\partial S} \left( dS + \frac{\partial S}{\partial x^5} dx^5 \right) + \frac{\partial f_5}{\partial V} \left( dV + \frac{\partial V}{\partial x^6} dx^6 \right) \quad (51) \\ df_5 (\mathbf{z}^\mu) &= \frac{\partial f_5}{\partial t} dt + \frac{\partial f_5}{\partial S} \left( dS + \frac{\partial S}{\partial x^5} dx^5 \right) + \frac{\partial f_5}{\partial V} \left( dV + \frac{\partial V}{\partial x^6} dx^6 \right), \end{aligned}$$

and requiring the following constrained equations

$$\frac{\partial f_5}{\partial S} \left( dS + \frac{\partial S}{\partial x^5} dx^5 \right) = \left( -i \frac{\partial f_5}{\partial x^5} \right) \frac{\partial}{\partial S}, \quad (52)$$

$$\frac{\partial f_5}{\partial V} dV + \frac{\partial f_5}{\partial x^6} dx^6 = \left( -i \frac{\partial f_5}{\partial x^6} \right) \frac{\partial}{\partial V}. \quad (53)$$

Writing equation (51) in matrix form allows us to see that important infor-

mation has been projected from superspace by the mapping  $\Pi$ ,

$$df_5(\mathbf{z}^\mu) = \begin{pmatrix} \frac{\partial f_5}{\partial t} & \frac{\partial f_5}{\partial s} & \frac{\partial f_5}{\partial V} \end{pmatrix} \cdot \begin{pmatrix} dt \\ dS + \frac{\partial S}{\partial x^5} dx^5 \\ dV + \frac{\partial V}{\partial x^6} dx^6 \end{pmatrix} \quad (54)$$

$$df_5(\mathbf{z}^\mu) = \begin{pmatrix} \frac{\partial f_5}{\partial t} & \frac{\partial}{\partial s} & \frac{\partial}{\partial V} \end{pmatrix} \cdot \begin{pmatrix} dt \\ (-i \frac{\partial}{\partial x^5}) f_5 \\ (-i \frac{\partial}{\partial x^6}) f_5 \end{pmatrix},$$

where  $\mu = 2, 3, 4$  is the index of the ordinary coordinates. We require that the coordinate functions of the extra dimensions to be ordinary-coordinate-independent,  $x^5 \neq x^5(S, V)$  and  $x^6 \neq x^6(S, V)$ .

Algebraically, we can demonstrate the projections from superspace onto ordinary space by writing down the second-order Taylor series expansion for the total differential of the option  $\Pi df_5(\mathbf{z}^A)$

$$\begin{aligned} \Pi df_5(\mathbf{z}^A) &= df_5(\Pi \mathbf{z}^A) = \lim_{\epsilon \rightarrow 0} \frac{\Pi}{\epsilon} \left\{ \begin{array}{l} f_5(t + \epsilon, S(t + \epsilon), V(t + \epsilon), x^5(t + \epsilon), x^6(t + \epsilon)) \\ - f_5(t, S(t), V(t), x^5(t), x^6(t)) \end{array} \right\} dt \\ df_5(\mathbf{z}^\mu) &= \Pi \left\{ \frac{\partial f_5}{\partial t} dt + \frac{\partial f_5}{\partial S} dS + \frac{\partial f_5}{\partial V} dV + \frac{\partial f_5}{\partial x^5} dx^5 + \frac{\partial f_5}{\partial x^6} dx^6 \right. \\ &\quad \left. + \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2!} \left\{ \begin{array}{l} \frac{\partial}{\partial S} \left( \frac{\partial f_5}{\partial S} dS \right) dS + \frac{\partial}{\partial V} \left( \frac{\partial f_5}{\partial S} dS \right) dV + \frac{\partial}{\partial x^5} \left( \frac{\partial f_5}{\partial S} dS \right) dx^5 \\ + \frac{\partial}{\partial x^6} \left( \frac{\partial f_5}{\partial S} dS \right) dx^6 + \frac{\partial}{\partial S} \left( \frac{\partial f_5}{\partial V} dV \right) dS + \frac{\partial}{\partial V} \left( \frac{\partial f_5}{\partial V} dV \right) dV \\ + \frac{\partial}{\partial x^5} \left( \frac{\partial f_5}{\partial V} dV \right) dx^5 + \frac{\partial}{\partial x^6} \left( \frac{\partial f_5}{\partial V} dV \right) dx^6 + \frac{\partial}{\partial S} \left( \frac{\partial f_5}{\partial x^5} dx^5 \right) dS \\ + \frac{\partial}{\partial V} \left( \frac{\partial f_5}{\partial x^5} dx^5 \right) dV + \frac{\partial}{\partial x^5} \left( \frac{\partial f_5}{\partial x^5} dx^5 \right) dx^5 + \frac{\partial}{\partial x^6} \left( \frac{\partial f_5}{\partial x^5} dx^5 \right) dx^6 \\ \frac{\partial}{\partial S} \left( \frac{\partial f_5}{\partial x^6} dx^6 \right) dS + \frac{\partial}{\partial V} \left( \frac{\partial f_5}{\partial x^6} dx^6 \right) dV + \frac{\partial}{\partial x^5} \left( \frac{\partial f_5}{\partial x^6} dx^6 \right) dx^5 \\ + \frac{\partial}{\partial x^6} \left( \frac{\partial f_5}{\partial x^6} dx^6 \right) dx^6 \end{array} \right\} \right\}. \end{aligned} \quad (55)$$

Recall from section 3, we can write a field decomposition of the option value  $f_5$  as

$$f_5(t, S(t), V(t), x^5, x^6) = \sum_{n,m=0}^{\infty} \widehat{f}_3(t, S(t), V(t))_{nm} \exp i \left( \begin{array}{l} \frac{n}{R_5} \cdot x^5 \\ + \frac{m}{R_5} \cdot x^6 \end{array} \right), \quad (56)$$

and is subjected to the periodic condition

$$f_5(t, S(t), V(t), x^5, x^6) = f_5(t, S(t), V(t), x^5(t) + 2n\pi R_5, x^6(t) + 2m\pi R_6),$$

where  $\widehat{f}_3(t, S(t), V(t))_{nm}$  is the Fourier expansion coefficients, and  $(n, m) \in Z$  are integers with radii  $R_5$  and  $R_6$ , are the compactified extra dimensions in the,  $x^5$  and  $x^6$  directions, respectively. The associated momenta in the extra dimensional directions are quantized as follows

$$p_5 = \frac{n}{R_5}, \quad (57)$$

$$p_6 = \frac{m}{R_6}. \quad (58)$$

In order to simplify the truncated Taylor series, we assume the following constrained relations by demanding that

$$\frac{\partial f_5}{\partial S} dS + \frac{\partial f_5}{\partial x^5} dx^5 = \left( -i \frac{\partial f_5}{\partial x^5} \right) \frac{\partial}{\partial S} \quad (59)$$

and

$$\frac{\partial f_5}{\partial V} dV + \frac{\partial f_5}{\partial x^6} dx^6 = \left( -i \frac{\partial f_5}{\partial x^6} \right) \frac{\partial}{\partial V}, \quad (60)$$

where we define the superprojection operators  $P_5 = -i \frac{\partial}{\partial x^5}$  and  $P_6 = -i \frac{\partial}{\partial x^6}$ . Applying the projection operators on the option  $f_5$ , we have the following equations

$$\begin{aligned} P_5 f_5 &= \left( -i \frac{\partial}{\partial x^5} \right) f_5 \quad (61) \\ &= \left( -i \frac{\partial}{\partial x^5} \right) \left\{ \sum_{n,m=0}^{\infty} \widehat{f}_3(t, S(t), V(t))_{nm} \exp i \left( \frac{n}{R_5} \cdot x^5 + \frac{m}{R_6} \cdot x^6 \right) \right\} \\ &= \frac{n}{R_5} f_5, \end{aligned}$$

similarly

$$\begin{aligned}
P_6 f_5 &= \left( -i \frac{\partial}{\partial x^6} \right) f_5 & (62) \\
&= -i \frac{\partial}{\partial x^6} \left\{ \sum_{n,m=0}^{\infty} \widehat{f}_3(t, S(t), V(t))_{nm} \exp i \left( \begin{array}{l} \frac{n}{R_5} \cdot x^5 \\ + \frac{m}{R_6} \cdot x^6 \end{array} \right) \right\} \\
&= \frac{m}{R_6} f_5.
\end{aligned}$$

The equations (61) and (62) give us the explicit quantized forms of the higher dimensional operators as

$$P_5 = \frac{n}{R_5}, \quad (63)$$

and

$$P_6 = \frac{m}{R_6}, \quad (64)$$

where  $m, n \in \mathbb{Z}$  are integers. More importantly, equations (63) and (64) control the stochastic contributions coming from the higher dimensional subspace of superspace. The constrained equations (59) and (60) become

$$\frac{\partial f_5}{\partial S} dS + \frac{\partial f_5}{\partial x^5} dx^5 = \left( -i \frac{\partial f_5}{\partial x^5} \right) \frac{\partial}{\partial S} = \frac{n}{R_5} \frac{\partial f_5}{\partial S} \quad (65)$$

$$\frac{\partial f_5}{\partial V} dV + \frac{\partial f_5}{\partial x^6} dx^6 = \left( -i \frac{\partial f_5}{\partial x^6} \right) \frac{\partial}{\partial V} = \frac{m}{R_6} \frac{\partial f_5}{\partial V}. \quad (66)$$

Recall that our assumed supermanifold such that coordinate functions of the extra dimensions to be ordinary-coordinate-independent,  $x^5 \neq x^5(S, V)$  and  $x^6 \neq x^6(S, V)$ , then we have

$$\begin{aligned}
\frac{\partial f_5}{\partial S} dS + \frac{\partial f_5}{\partial x^5} dx^5 &= \frac{n}{R_5} \frac{\partial f_5}{\partial S} \\
\frac{\partial f_5}{\partial S} \left( dS + \frac{\partial S}{\partial f} \cdot \frac{\partial f}{\partial x^5} dx^5 \right) &= \frac{n}{R_5} \frac{\partial f_5}{\partial S} \\
\frac{\partial f_5}{\partial S} \left( dS + \frac{\partial S}{\partial x^5} dx^5 \right) &= \frac{n}{R_5} \frac{\partial f_5}{\partial S}. & (67)
\end{aligned}$$



$$\frac{\partial S}{\partial x^5} = 0. \quad (68)$$

Equation (67) reduces to

$$\frac{\partial f_5}{\partial S} dS = \frac{n}{R_5} \frac{\partial f_5}{\partial S}.$$

This tells us the radius of the compactified radius must be  $R_5 = dS^{-1} = [(\phi S + \sigma SR) dt]^{-1}$ . Similarly, the compactified radius in the  $x^6$  direction is  $R_6 = dV^{-1} = [(\mu V + \xi VQ) dt]^{-1}$ . The truncated Taylor series simplifies to

$$\begin{aligned} df_5(\mathbf{z}^\mu) &= \lim_{\epsilon \rightarrow 0} \frac{\Pi}{\epsilon} \left\{ \begin{aligned} & f_5(t + \epsilon, S(t + \epsilon), V(t + \epsilon), x^5(t + \epsilon), x^6(t + \epsilon)) \\ & - f_5(t, S(t), V(t), x^5(t), x^6(t)) \end{aligned} \right\} dt \\ df_5(\mathbf{z}^\mu) &= \frac{\partial f_5}{\partial t} dt + \left( -i \frac{\partial f_5}{\partial x^5} \right) \frac{\partial}{\partial S} + \left( -i \frac{\partial f_5}{\partial x^6} \right) \frac{\partial}{\partial V} \\ &+ \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2!} \left\{ \begin{aligned} & \frac{\partial^2 f_5}{\partial S^2} dS^2 + 2 \frac{\partial^2 f_5}{\partial S \partial V} dS dV + \frac{\partial^2 f_5}{\partial V^2} dV^2 \\ & + \frac{\partial^2 f_5}{(\partial x^5)^2} (dx^5)^2 + \frac{\partial^2 f_5}{(\partial x^6)^2} (dx^6)^2 \\ & + 2 \frac{\partial^2 f_5}{\partial x^5 \partial S} dS dx^5 + 2 \frac{\partial^2 f_5}{\partial x^6 \partial S} dS dx^6 \\ & + 2 \frac{\partial^2 f_5}{\partial x^5 \partial V} dV dx^5 + 2 \frac{\partial^2 f_5}{\partial x^6 \partial V} dV dx^6 \\ & + 2 \frac{\partial^2 f_5}{\partial x^5 \partial x^6} dx^5 dx^6 \end{aligned} \right\}. \quad (69) \end{aligned}$$

Upon further simplification, we have the following relations

$$\begin{aligned} \frac{\partial^2 f_5}{\partial S^2} dS^2 + 2 \frac{\partial^2 f_5}{\partial x^6 \partial S} dS dx^6 &= \frac{\partial}{\partial S} dS \left( \frac{\partial f_5}{\partial S} dS + 2 \frac{\partial f_5}{\partial x^6} dx^6 \right) \quad (70) \\ &= \frac{\partial}{\partial S} dS \frac{\partial f_5}{\partial S} \left( dS + \frac{\partial S}{\partial x^6} dx^6 \right) \\ &= \frac{\partial^2 f_5}{\partial S^2} dS^2, \end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f_5}{\partial S^2} dS^2 + 2 \frac{\partial^2 f_5}{\partial x^5 \partial S} dS dx^5 \tag{71} \\
&= \frac{\partial}{\partial S} dS \left( \frac{\partial f_5}{\partial S} dS + 2 \frac{\partial f_5}{\partial x^5} dx^5 \right) \\
&= \frac{\partial}{\partial S} dS \left( \frac{\partial f_5}{\partial S} dS + \frac{\partial f_5}{\partial x^5} dx^5 \right) + \frac{\partial^2 f_5}{\partial S \partial x^5} dS dx^5 \\
&= \frac{\partial}{\partial S} dS \frac{\partial f_5}{\partial S} \left( dS + \frac{\partial S}{\partial x^5} dx^5 \right) + \frac{\partial^2 f_5}{\partial S \partial x^5} dS dx^5 \\
&= \frac{\partial^2}{\partial S^2} dS \left( -i \frac{\partial f_5}{\partial x^5} \right) + \frac{\partial^2 f_5}{\partial S \partial x^5} dS dx^5,
\end{aligned}$$

similarly,

$$\begin{aligned}
& \frac{\partial^2 f_5}{\partial V^2} dV^2 + 2 \frac{\partial^2 f_5}{\partial x^5 \partial V} dV dx^5 + 2 \frac{\partial^2 f_5}{\partial x^6 \partial V} dV dx^6 \tag{72} \\
&= \frac{\partial^2 f_5}{\partial V^2} dV^2 + 2 \frac{\partial^2 f_5}{\partial x^6 \partial V} dV dx^6 \\
&= \frac{\partial}{\partial S} dS \frac{\partial f_5}{\partial S} \left( dS + \frac{\partial S}{\partial x^6} dx^6 \right) \\
&= \frac{\partial^2}{\partial V^2} dV \left( -i \frac{\partial f_5}{\partial x^6} \right) + \frac{\partial^2 f_5}{\partial V \partial x^6} dV dx^6.
\end{aligned}$$

In terms of the superprojection operators, the truncated Taylor series becomes

$$\begin{aligned}
df_5(\mathbf{z}^\mu) &= \lim_{\epsilon \rightarrow 0} \frac{\Pi}{\epsilon} \left\{ \begin{array}{l} f_5(t + \epsilon, S(t + \epsilon), V(t + \epsilon), x^5(t + \epsilon), x^6(t + \epsilon)) \\ -f_5(t, S(t), V(t), x^5(t), x^6(t)) \end{array} \right\} dt \tag{73} \\
df_5(\mathbf{z}^\mu) &= \Pi \left\{ \frac{\partial f_5}{\partial t} dt + \frac{\partial}{\partial S} \left( -i \frac{\partial}{\partial x^5} f_5 \right) + \frac{\partial}{\partial V} \left( -i \frac{\partial}{\partial x^6} f_5 \right) \right. \\
&\quad \left. + \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2!} \left\{ \begin{array}{l} \left( -i \frac{\partial}{\partial x^5} \right) \left( -i \frac{\partial}{\partial x^5} f_5 \right) \frac{\partial^2}{\partial S^2} + 2 \left( -i \frac{\partial}{\partial x^5} \right) \left( -i \frac{\partial}{\partial x^6} f_5 \right) \frac{\partial^2}{\partial S \partial V} \right. \\ \left. + \left( -i \frac{\partial}{\partial x^6} \right) \left( -i \frac{\partial}{\partial x^6} f_5 \right) \frac{\partial^2}{\partial V^2} + \frac{\partial^2 f_5}{(\partial x^5)^2} (dx^5)^2 + \frac{\partial^2 f_5}{(\partial x^6)^2} (dx^6)^2 \right. \\ \left. + \left( -i \frac{\partial}{\partial x^5} f_5 \right) dx^5 \frac{\partial^2}{\partial x^5 \partial S} + \left( -i \frac{\partial}{\partial x^6} f_5 \right) dx^6 \frac{\partial^2}{\partial x^6 \partial V} \right. \\ \left. + 2 \frac{\partial^2 f_5}{\partial x^5 \partial x^6} dx^5 dx^6 \right\} \right\}.
\end{aligned}$$

Squaring the two constrained equations, we have

$$\begin{aligned}
\left(\frac{\partial f_5}{\partial S}dS + \frac{\partial f_5}{\partial x^5}dx^5\right)^2 &= \left(\frac{\partial f_5}{\partial S}dS\right)^2 \quad (74) \\
\left(\frac{\partial f_5}{\partial S}dS\right)^2 + 2\frac{\partial f_5}{\partial S}dS\frac{\partial f_5}{\partial x^5}dx^5 + \left(\frac{\partial f_5}{\partial x^5}dx^5\right)^2 &= \left(\frac{\partial f_5}{\partial S}dS\right)^2 \\
2\left(\frac{\partial f_5}{\partial S} \cdot \frac{\partial f_5}{\partial x^5}\right)dx^5dS + \left(\frac{\partial f_5}{\partial x^5}dx^5\right)^2 &= 0.
\end{aligned}$$

Since  $\left(\frac{\partial f_5}{\partial S} \cdot \frac{\partial f_5}{\partial x^5}\right) \neq 0$ , and using (68)

$$\begin{aligned}
(dx^5)^2 &= -2\left(\frac{\partial f_5}{\partial S} \cdot \frac{\partial f_5}{\partial x^5}\right)dx^5dS \\
(dx^5)^2 &= -2\left(\frac{\partial f_5}{\partial x^5}\right)^{-2}\left(\frac{\partial f_5}{\partial S} \cdot \frac{\partial f_5}{\partial x^5}\right)dx^5dS \\
(dx^5)^2 &= -2\left(\frac{\partial x^5}{\partial f_5}\right)^2\left(\frac{\partial f_5}{\partial S} \cdot \frac{\partial f_5}{\partial x^5}\right)dx^5dS \\
(dx^5)^2 &= -2\frac{\partial x^5}{\partial S}dx^5dS = 0 \quad (75)
\end{aligned}$$

The truncated Taylor series is simplified further and yields

$$\begin{aligned}
df_5(\mathbf{z}^\mu) &= \lim_{\epsilon \rightarrow 0} \frac{\Pi}{\epsilon} \left\{ \begin{array}{l} f_5(t + \epsilon, S(t + \epsilon), V(t + \epsilon), x^5(t + \epsilon), x^6(t + \epsilon)) \\ -f_5(t, S(t), V(t), x^5(t), x^6(t)) \end{array} \right\} dt \\
df_5(\mathbf{z}^\mu) &= \Pi \left\{ \frac{\partial f_5}{\partial t} dt + \frac{\partial}{\partial S} \left( -i \frac{\partial}{\partial x^5} f_5 \right) + \frac{\partial}{\partial V} \left( -i \frac{\partial}{\partial x^6} f_5 \right) \right. \quad (76) \\
&\quad \left. + \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2!} \left[ \begin{array}{l} \left( -i \frac{\partial}{\partial x^5} \right) \left( -i \frac{\partial}{\partial x^5} f_5 \right)^2 \frac{\partial^2}{\partial S^2} + 2 \left( -i \frac{\partial}{\partial x^5} \right) \left( -i \frac{\partial}{\partial x^6} f_5 \right) \frac{\partial^2}{\partial S \partial V} \\ + \left( -i \frac{\partial}{\partial x^6} \right) \left( -i \frac{\partial}{\partial x^6} f_5 \right)^2 \frac{\partial^2}{\partial V^2} + \left( -i \frac{\partial}{\partial x^5} f_5 \right) dx^5 \frac{\partial^2}{\partial x^5 \partial S} \\ + \left( -i \frac{\partial}{\partial x^6} f_5 \right) dx^6 \frac{\partial^2}{\partial x^6 \partial V} + 2 \frac{\partial^2 f_5}{\partial x^5 \partial x^6} dx^5 dx^6 \end{array} \right] \right\}.
\end{aligned}$$

The derived series expansion (76), written in terms of superprojection operators is in fact equivalent to the series expansion used to derive the Merton-Garman equation (7) with stochastic volatility. Using equations (63) and

(64) to project the stochastic behavior onto the ordinary space, the series yields

$$\begin{aligned}
df_5(\mathbf{z}^\mu) &= \lim_{\epsilon \rightarrow 0} \frac{\Pi}{\epsilon} \left\{ \begin{aligned} &f_5(t + \epsilon, S(t + \epsilon), V(t + \epsilon), x^5(t + \epsilon), x^6(t + \epsilon)) \\ &- f_5(t, S(t), V(t), x^5(t), x^6(t)) \end{aligned} \right\} dt \\
df_5(\mathbf{z}^\mu) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \begin{aligned} &\frac{\partial f_5}{\partial t} dt + n \frac{\partial f_5}{\partial S} (\phi S + \sigma SR) dt \\ &+ m \frac{\partial f_5}{\partial V} (\mu V + \xi V Q) dt \end{aligned} \right. \quad (77) \\
&\quad \left. + \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2!} \left[ \begin{aligned} &n^2 \frac{\partial^2}{\partial S^2} f_5 [(\phi S + \sigma SR) dt]^2 \\ &+ 2nm \frac{\partial^2}{\partial S \partial V} f_5 [(\phi S + \sigma SR) dt] [(\mu V + \xi V Q) dt] \\ &+ m^2 \frac{\partial^2}{\partial V^2} f_5 [(\mu V + \xi V Q) dt]^2 \\ &+ n \frac{\partial^2}{\partial x^5 \partial S} f_5 [(\phi S + \sigma SR) dt] dx^5 + 2 \frac{\partial^2}{\partial x^5 \partial x^6} f_5 dx^5 dx^6 \\ &+ m \frac{\partial^2}{\partial x^6 \partial V} f_5 [(\mu V + \xi V Q) dt] dx^6 + \end{aligned} \right] \right\}.
\end{aligned}$$

Taking the limit  $\epsilon \rightarrow 0$ , dividing both sides of equation (77) by  $dt$  and separating the equation into the deterministic and stochastic parts, the series reduces

$$\frac{df_5}{dt} = \frac{\partial f_5}{\partial t} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ + \frac{1}{2!} \left[ \begin{aligned} &n\phi S \frac{\partial f_5}{\partial S} + m\mu V \frac{\partial f_5}{\partial V} \\ &n^2 \sigma^2 S^2 \frac{\partial^2 f_5}{\partial S^2} + m^2 \xi^2 V^2 \frac{\partial^2 f_5}{\partial V^2} \\ &+ 2nm\sigma^3 S \xi \rho \frac{\partial^2 f_5}{\partial S \partial V} \\ &+ n\sigma S \frac{\partial f_5}{\partial S} R + m\xi V \frac{\partial f_5}{\partial V} Q \end{aligned} \right] \right\}. \quad (78)$$

Let's simplify equation (78) by defining the following equation

$$\frac{df_5}{dt} = f_5 \omega_{nm} + f_5 \alpha_n R + f_5 \delta_m Q, \quad (79)$$

where

$$\omega_{nm} = \frac{\partial}{\partial t} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ +\frac{1}{2!} \left[ \begin{array}{c} n\phi S \frac{\partial}{\partial S} + m\mu V \frac{\partial}{\partial V} \\ n^2\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + m^2\xi^2 V^2 \frac{\partial^2}{\partial V^2} \\ + 2nm\sigma^3 S\xi\rho \frac{\partial^2}{\partial S\partial V} \end{array} \right] \right\}, \quad (80)$$

$$\alpha_n = n\sigma S \frac{\partial}{\partial S}, \quad (81)$$

$$\delta_m = m\xi V \frac{\partial}{\partial V}. \quad (82)$$

We consider a following self-replicating higher-dimensional portfolio

$$\pi_5(t) = \theta_1 f_5(t) + \theta_2 S(t), \quad (83)$$

where  $\theta_1$  and  $\theta_2$  are the amounts of option  $f_5$  and stock  $S$ , respectively. The total time derivative of our portfolio is given

$$\frac{d\pi_5}{dt} = \theta_1 \frac{df_5}{dt} + \theta_2 \frac{dS}{dt}. \quad (84)$$

Using equations (79), the total time-derivative for our portfolio yields

$$\begin{aligned} \frac{d\pi_5}{dt} &= \theta_1 (f_5 \omega_{nm} + f_5 \alpha_n R + f_5 \delta_m Q) + \theta_2 (\phi S + \sigma SR) \\ \frac{d\pi_5}{dt} &= \omega_{nm} \theta_1 f_5 + \phi \theta_2 S + (\alpha_n R + \delta_m Q) \theta_1 f_5 + \theta_2 \sigma SR. \end{aligned} \quad (85)$$

We choose our portfolio in such a way that it satisfies the following hedging equations

$$\alpha_n R \theta_1 f_5 + \theta_2 \sigma SR = 0, \quad (86)$$

$$\delta_m Q \theta_1 f_5 = 0. \quad (87)$$

The random quantities  $R(t)$  and  $Q(t)$  can be automatically eliminated from (85) with the aid of equations (86) and (87). Thus, written in matrix nota-

tion, equations (86) and (87) yields the following matrix equation

$$\begin{bmatrix} \alpha_n & \sigma \\ \delta_m & 0 \end{bmatrix} \begin{bmatrix} \theta_1 f_5 \\ \theta_2 S \end{bmatrix} = 0. \quad (88)$$

Subsequently, the time-derivative of the portfolio (85) reduces to a pure deterministic equation

$$\frac{d\pi_5}{dt} = \omega_{nm}\theta_1 f_5 + \phi\theta_2 S. \quad (89)$$

The absence of arbitrage forces the time-derivative of the portfolio to be directly proportional to the risk-neutral rate  $r$ , hence

$$\frac{d\pi_5}{dt} = r\pi_5. \quad (90)$$

Solving equation (90) for the higher-dimensional portfolio, we have

$$\begin{aligned} \frac{d\pi_5}{\pi_5} &= r dt \\ \ln |\pi_5| &= \int r dt = rt + C \\ \pi_5(t) &= \pi_0 \exp(rt), \end{aligned} \quad (91)$$

where  $C$  is a integration constant and  $\pi_0 = \exp(C)$ . Continue with the derivation of the Merton-Garman equation, we note that equality of equations (89) and (90) yields

$$(\omega_{nm} - r)\theta_1 f_5 + (\phi - r)\theta_2 S = 0, \quad (92)$$

and in matrix notation

$$\begin{bmatrix} \omega_{nm} - r & \phi - r \end{bmatrix} \begin{bmatrix} \theta_1 f_5 \\ \theta_2 S \end{bmatrix} = 0. \quad (93)$$

Equation (93) can only be satisfied if

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} \begin{bmatrix} \omega_{nm} - r & \phi - r \end{bmatrix} = \begin{bmatrix} \alpha_n & \sigma \\ \delta_m & 0 \end{bmatrix}. \quad (94)$$

Solving equation (94) for  $\begin{bmatrix} \omega_{nm} - r & \phi - r \end{bmatrix}$ , we have

$$\begin{bmatrix} \omega_{nm} - r & \phi - r \end{bmatrix} = \begin{bmatrix} \lambda_1^*(t) & \lambda_2^*(t) \end{bmatrix} \begin{bmatrix} \alpha_n & \sigma \\ \delta_m & 0 \end{bmatrix}, \quad (95)$$

where we have  $\lambda_1^*(t)\lambda_1(t) + \lambda_2^*(t)\lambda_2(t) = \lambda_1^2(t) + \lambda_2^2(t) = 1$ . In component forms, the matrix equation (95) yields

$$\omega_{nm} - r = \lambda_1(t)\alpha_n + \lambda_2(t)\delta_m \quad (96)$$

$$\phi - r = \lambda_1(t)\sigma. \quad (97)$$

Solving equations (96) and (97) simultaneously, we have

$$\phi - r = \lambda_1\sigma = \left( \frac{\omega_{nm} - r}{\alpha_n} - \frac{\lambda_2(t)\delta_m}{\alpha_n} \right) \sigma, \quad (98)$$

where  $\lambda_1(t) = \frac{\omega_{nm} - r}{\alpha_n} - \frac{\lambda_2(t)\delta_m}{\alpha_n}$ . Solving for  $\omega_{nm} - r$ , we then have

$$\omega_{nm} - r = \left( \frac{\phi - r}{\sigma} \right) \alpha_n + \lambda_2(t)\delta_m. \quad (99)$$

We define a 5D Merton-Garman operator  $\Gamma_5$  by

$$\Gamma_{nm} = (\omega_{nm} - r) - \left( \frac{\phi - r}{\sigma} \right) \alpha_n - \lambda_2(t)\delta_m. \quad (100)$$

In term of 5D Merton-Garman operator  $\Gamma_5$ , the generalized 5D Merton-

Garman equation takes the compact form

$$\Gamma_{(n,m)}f_5 = 0. \quad (101)$$

The ordinary 3D Merton-Garman equation can now be obtained from equation (101) for  $n = m = 1$

$$\begin{aligned} \Gamma_{(1,1)}f_5 &= 0 \\ \left[ (\omega_{11} - r) - \left( \frac{\phi - r}{\sigma} \right) \alpha_1 - \lambda_2 \delta_1 \right] f_5 &= 0. \end{aligned} \quad (102)$$

To demonstrate the equivalency between equation (102) and equation (26), we use equations (80), (81), (82) and (25) in the dimensional reduction of the higher-dimensional Merton-Garman equation (101), which yields

$$\begin{aligned} 0 &= \Gamma_{(1,1)}f_5 = \left[ (\omega_{11} - r) - \left( \frac{\phi - r}{\sigma} \right) \alpha_1 - \lambda_2 \delta_1 \right] f_5 \\ 0 &= \frac{\partial}{\partial t} f_5 + \frac{1}{2!} \left\{ \sigma^2 S^2 \frac{\partial^2}{\partial S^2} f_5 + \xi^2 V^2 \frac{\partial^2}{\partial V^2} f_5 + 2\sigma^3 S \xi \rho \frac{\partial^2}{\partial S \partial V} f_5 \right\} \\ &\quad - r f_5 + r S \frac{\partial}{\partial S} f_5 + \bar{\mu} \sigma^2 \frac{\partial}{\partial S} f_5. \end{aligned} \quad (103)$$

Since the stochastic information had already been mapped onto the ordinary space, we can then safely replace  $f_5$  by  $f$ , hence

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} f + \frac{1}{2!} \left\{ \sigma^2 S^2 \frac{\partial^2}{\partial S^2} f + \xi^2 V^2 \frac{\partial^2}{\partial V^2} f + 2\sigma^3 S \xi \rho \frac{\partial^2}{\partial S \partial V} f \right\} \\ &\quad - r f + r S \frac{\partial}{\partial S} f + \bar{\mu} \sigma^2 \frac{\partial}{\partial S} f. \end{aligned} \quad (104)$$

By inspection, equation (104) is identical to equation (26) from section 2. Thus, we have shown that the generalized Black-Scholes equation or the Merton-Garman equation can be obtained by the higher-dimensional approach along with two constraints.



## 5 Summary

We showed that the celebrated Black-Scholes equation and its generalized version Merton-Garman equation can be obtained from an enlarged manifold. The Merton-Garman equation was shown to be an equation of the first excited state i.e.  $n = m = 1$ . In fact, there are an infinite number of the Merton-Garman-like equations contained in our superspace. These equations could be called excited states living in superspace. These excited states only manifest their presence in our ordinary space when acted upon by the superprojection operators  $P_5$  and  $P_6$ . For the ground state, i.e.  $n = m = 0$ , nothing is projected from superspace. In other words, the stochastic or random nature of the variables is confined in the extra dimensional subspace of superspace. In general, in order to extract forecasting aspects or predictive power of a financial theory, we need to recast the Merton-Garman equation into its quantum mechanical Schrodinger form. In quantum mechanical form, manipulation of pertinent information such as (predictive power, hedging, and arbitrage...) associated with a financial theory can be executed via the potential functions, i.e. the Hamiltonians or Lagrangians. Furthermore, the potential functions can also be influenced by the nature of the extra dimensions. The effects of the potential functions by the extra dimensions along with the Schrodinger interpretation of the classical theory, will then provide financial theorists or model builders with alternate research avenues and a larger theoretic framework in which financial theories are obtained.

## 6 References

### References

- [1] F. Black and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May 1973), 637-659
- [2] R. C. Merton. "The Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (Spring 1973), 141-83
- [3] M. Garman. "A General Theory of Asset Valuation under diffusion State Processes." Working Paper No 50, University of California, Berkeley, 1976.
- [4] J. -P. Bouchaud and D. Sornette. "The Black-Scholes Option Pricing Problem in Mathematical Finance." *Journal de Physique I* 4. 863-81 (1994).
- [5] B. E. J. Phys, I (France) 7, 1733-1753, (1997)
- [6] V Litnetsky. "The Path Integral Approach to Financial Modelling and Options Pricing." *Computational Economics* 11, 129. 1998.
- [7] M. Truong, *Physical Review* D74 (2006).
- [8] J. C. Hull and A. White. "The Pricing of Options on Assets with Stochastic Volatilities." *The Journal of Finance*, Vol XLII, No. 2 (June 1987),281-299.
- [9] T. Kaluza. *Sitzungdber. Berl. Akad.* (1921) 966.
- [10] O. Z. Klein. *Phys.* 37 (1926) 895.