FRACTIONAL PROCESSES AS MODELS IN STOCHASTIC FINANCE

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ABSTRACT. We survey some new progress on the pricing models driven by fractional Brownian motion or mixed fractional Brownian motion. In particular, we give results on arbitrage opportunities, hedging, and option pricing in these models. We summarize some recent results on fractional Black & Scholes pricing model with transaction costs. We end the paper by giving some approximation results and indicating some open problems related to the paper.

1. INTRODUCTION

The classical Black-Scholes pricing model is based on standard geometric Brownian motion. The log-returns of this model are independent and Gaussian. Various empirical studies on the statistical properties of log-returns show that the log-returns are not necessarily independent and also not Gaussian. One way to a more realistic modelling is to change the geometric Brownian motion to a geometric fractional Brownian motion: the dependence of the log-return increments can now be modelled with the Hurst parameter of the fractional Brownian motion. But then the pricing model admits arbitrage possibilities with continuous trading, and also with certain discrete type trading strategies.

The arbitrage possibilities with continuous trading depend on the notion of stochastic integration theory used in the definition of trading strategy. If these stochastic integrals are interpreted as Skorohod integrals, then the arbitrage possibilities with continuous trading disappear. We will not consider this approach in what follows. For a summary of the results obtained in this area we refer to two recent monographs on fractional Brownian motion [10] and [31]. If one uses Skorohod integration theory, then one has several problems with the financial interpretation of these continuous trading strategies. We refer to the above two monographs for more details on these issues; see also [11] and [41] for the critical remarks concerning the Skorohod approach from the finance point of view.

In this work we discuss the arbitrage possibilities in the fractional Black-Scholes pricing model and in the related mixed Brownian–fractional Brownian pricing model. Then we consider hedging of options in these models. The fractional Black-Scholes model admits strong arbitrage, and this implies that the initial wealth for the exact hedging strategy cannot be interpreted as a price of the option. But these replication results are interesting from the mathematical point of view. With proportional transaction costs the arbitrage possibilities disappear in the fractional Black-Scholes pricing model. Hence it is of some interest to know the hedging strategy without transaction costs. For the mixed Brownian-fractional Brownian pricing models the arbitrage possibilities are not that obvious, and the hedging price can be sometimes interpreted as the price of the option. We shall review some recent results related to these questions.

One possibility to study the properties of the fractional Black-Scholes pricing model is to approximate it with simpler pricing models. We will present some results on the approximation at the end of this work.

2. MODELS AND NOTIONS OF ARBITRAGE

2.1. **Definition.** The *fractional Brownian motion* (fBm) with *Hurst index* $H \in (0,1)$ is the centered Gaussian process $B = (B_t)_{t \in [0,T]}$ with $B_0 = 0$ and
$$\mathbf{Cov}\,[B_t, B_s] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

2.2. *Remark.* Some well-known properties of the fBm are:

   (i) The fBm has stationary increments.
   (ii) For $H = 1/2$ the fBm is the standard *Brownian motion* (Bm) W.
   (iii) If $H \neq 1/2$ the fBm is not a semimartingale (cf. [14, Theorem 3.2] or Example 3.9 later).
   (iv) If $H > 1/2$ the fBm has zero *quadratic variation* (QV) (cf. Definition 5.2 later). If $H < 1/2$ the QV is $+\infty$. For the Bm case $H = 1/2$ the QV is identity.
   (v) For $H > 1/2$ the fBm has *long range dependence* (LRD) in the sense that
$$\rho(n) = \mathbf{Cov}\,[B_k - B_{k-1}, B_{k+n} - B_{k+n-1}]$$
   satisfies
$$\sum_{n=1}^{\infty} |\rho(n)| = +\infty.$$
   (vi) The paths of the fBm are a.s. Hölder continuous with index $H - \varepsilon$, where H is the Hurst index and ε is any positive constant, but not Hölder continuous with index H. The first claim follows from the Kolmogorov–Chentsov criterion, and the second claim follows from the law of iterated logarithm of [2]:
$$\limsup_{t \downarrow 0} \frac{B_t}{t^H \sqrt{2 \ln \ln 1/t}} = 1 \quad \text{a.s..}$$

(vii) The fBm is *self-similar* with index H, i.e. for all $a > 0$,

$$\text{Law}\left((a^H B_{at})_{t\in[0,T/a]}\right) = \text{Law}\left((B_t)_{t\in[0,T]}\right).$$

Actually, the fBm is the (up to a multiplicative constant) unique centered Gaussian self-similar process with stationary increments.

In this survey we shall consider the following three discounted stock-price models in parallel:

2.3. **Definition.** Let $S = (S_t)_{t\in[0,T]}$ be the discounted stock-price.

(i) In the *Black–Scholes model* (BS model)

$$S_t = s_0 e^{\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t},$$

where W is a Bm, and $s_0, \sigma > 0$, $\mu \in \mathbb{R}$.

(ii) In the *fractional Black–Scholes model* (fBS model)

$$S_t = s_0 e^{\mu t + \nu B_t},$$

where B is a fBm with $H \neq 1/2$, and $s_0, \nu > 0$, $\mu \in \mathbb{R}$.

(iii) In the *mixed fractional Black–Scholes model* (mfBS model)

$$S_t = s_0 e^{\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t + \nu B_t},$$

where W is a Bm, B is a fBm with $H \neq 1/2$, W and B are independent, and $s_0, \sigma, \nu > 0$, $\mu \in \mathbb{R}$.

2.4. *Remark.* We shall often, for the sake of simplicity and without loss of any real generality, assume that $\mu = 0$ and $\sigma = \nu = s_0 = 1$.

2.5. *Remark.*   (i) The mfBS model is similar to the fBS model in the sense that they have essentially the same covariance structure. So, in particular, if $H > 1/2$, they both have LRD characterized by the Hurst index H.

(ii) The fBS model and the mfBS are different in the sense that the mfBS model has the same QV as the BS model (cf. Proposition 5.3) when $H > 1/2$. But the fBS model has 0 QV for $H > 1/2$. So, while the fBS model and the mfBS model have the same statistical LRD property, the pricing in these models is different, in the fBS model it might even be impossible.

We shall work, except in Section 7, in the canonical stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbf{P})$. So, $\Omega = C_{s_0}^+([0,T])$ the space of positive continuous functions over $[0,T]$ starting from s_0, and the stock-price is the co-ordinate process: $S_t(\omega) = \omega_t$. The filtration $(\mathscr{F}_t)_{t\in[0,T]}$ is generated by the stock-price S and augmented to satisfy the usual conditions of completeness and right-continuity. $\mathscr{F} = \mathscr{F}_T$, and the measure $\mathbf{P}$ is defined by the models in Definition 2.3.

2.6. **Definition.** A *portfolio*, or trading strategy, is an adapted process $\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t\in[0,T]} = (\Phi_t^0, \Phi_t)_{t\in[0,T]}$, where Φ_t^0 denotes the number of bonds and Φ_t

denotes the number of shares owned by the investor at time t. The *value* of the portfolio $\boldsymbol{\Phi}$ at time t is

$$V_t(\boldsymbol{\Phi}) = \Phi_t^0 + \Phi_t S_t,$$

since everything is discounted by the bond. The class of portfolios is denoted by $\mathscr{A}$.

There are some slightly different versions of the notion of free lunch, or arbitrage, that in discrete time would make little or no difference. However, in continuous time the issue of arbitrage is quite subtle as can be seen from the fundamental theorem of asset pricing by Delbaen and Schachermayer [18, Theorem 1.1]. We use the following definitions:

2.7. **Definition.**　　(i)　A portfolio $\boldsymbol{\Phi}$ is *arbitrage* if $V_0(\boldsymbol{\Phi}) = 0$, $V_T(\boldsymbol{\Phi}) \geq 0$ a.s., and $\mathbf{P}[V_T(\boldsymbol{\Phi}) > 0] > 0$.

　(ii)　A portfolio $\boldsymbol{\Phi}$ is *strong arbitrage* if $V_0(\boldsymbol{\Phi}) = 0$, and there exists a constant $c > 0$ such that $V_T(\boldsymbol{\Phi}) \geq c$ a.s..

　(iii)　A sequence of portfolios $(\boldsymbol{\Phi}^n)_{n \in \mathbb{N}}$ is *approximate arbitrage* if $V_0(\boldsymbol{\Phi}^n) = 0$ for all n and $V_T^\infty = \lim_{n \to \infty} V_T(\boldsymbol{\Phi}^n)$ exists in probability, $V_T^\infty \geq 0$ a.s., and $\mathbf{P}[V_T^\infty > 0] > 0$.

　(iv)　A sequence of portfolios is *strong approximate arbitrage* if it is approximate arbitrage and there exists a constant $c > 0$ such that $V_T^\infty \geq c$ a.s..

　(v)　A sequence of portfolios $(\boldsymbol{\Phi}^n)_{n \in \mathbb{N}}$ is *free lunch with vanishing risk* if it is approximate arbitrage, and

$$\lim_{n \to \infty} \operatorname*{esssup}_{\omega \in \Omega} \left| V_T(\boldsymbol{\Phi}^n)(\omega) \mathbf{1}_{\{V_T(\boldsymbol{\Phi}^n) < 0\}} \right| = 0.$$

3. Trading with (almost) simple strategies

In this section we consider non-continuous trading in continuous time. The basic classes of portfolios are:

3.1. **Definition.**　　(i)　A portfolio is *simple* if there exists a finite number of stopping times $0 \leq \tau_0 \leq \cdots \leq \tau_n \leq T$ such that the portfolio is constant on $(\tau_k, \tau_{k+1}]$, i.e.

$$\Phi_t = \sum_{k=0}^{n-1} \phi_{\tau_k} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where $\phi_{\tau_k} \in \mathscr{F}_{\tau_k}$, and an analogous expression holds for Φ^0. The class of simple portfolios is denoted by $\mathscr{A}^{\mathrm{si}}$.

　(ii)　A portfolio is *almost simple* if there exists a sequence $(\tau_k)_{k \in \mathbb{N}}$ of non-decreasing $[0, T]$-valued stopping times such that $\mathbf{P}[\exists_{k \in \mathbb{N}} \, \tau_k = T] = 1$ and the portfolio is constant on $(\tau_k, \tau_{k+1}]$, i.e.

$$\Phi_t = \sum_{k=0}^{N-1} \phi_{\tau_k} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where $\phi_{\tau_k} \in \mathscr{F}_{\tau_k}$, and N is an a.s. $\mathbb{N}$-valued random variable, and an analogous expression holds for Φ^0. The class of almost simple portfolios is denoted by $\mathscr{A}^{\mathrm{as}}$.

3.2. *Remark.* Obviously $\mathscr{A}^{\mathrm{si}} \subset \mathscr{A}^{\mathrm{as}}$, and the inclusion is proper. Also, note that for every ω the position Φ is changed only finitely many times. The difference between $\mathscr{A}^{\mathrm{si}}$ and $\mathscr{A}^{\mathrm{as}}$ is that in $\mathscr{A}^{\mathrm{si}}$ the number of readjustments is bounded in Ω, while in $\mathscr{A}^{\mathrm{as}}$ the number of readjustments is generally unbounded.

The notion of self-financing is obvious with (almost) simple strategies:

3.3. **Definition.** A portfolio $\Phi \in \mathscr{A}^{\mathrm{as}}$ is *self-financing* if, for all k, its value satisfies

$$V_{\tau_{k+1}}(\Phi) - V_{\tau_k}(\Phi) = \Phi_{\tau_{k+1}}\left(S_{\tau_{k+1}} - S_{\tau_k}\right),$$

or, equivalently, the *budget constraint*

$$\Phi^0_{\tau_{k+1}} + \Phi_{\tau_{k+1}}S_{\tau_k} = \Phi^0_{\tau_k} + \Phi_{\tau_k}S_{\tau_k}$$

holds for every readjustment time τ_k of the portfolio.

Henceforth, we shall always assume that the portfolios are self-financing.

3.4. **Theorem.** *In the BS model there is*

   (i) *no arbitrage in the class $\mathscr{A}^{\mathrm{si}}$,*
   (ii) *strong approximate arbitrage in the class $\mathscr{A}^{\mathrm{si}}$,*
   (iii) *strong arbitrage in the class $\mathscr{A}^{\mathrm{as}}$,*

Proof. The claim (i) follows from the fact that the geometric Bm remains a martingale in the sub-filtration $(\mathscr{F}_{\tau_k})_{k \leq n}$, and thus the claim reduces to discrete-time considerations. Claims (ii) and (iii) follow from the doubling strategy of Example 3.5 below. $\square$

3.5. *Example.* Consider, without loss of generality, the risk-neutral normalized BS model

$$S_t = s_0 e^{W_t - \frac{1}{2}t}.$$

Let $t_k = T(1 - 2^{-k})$, $c_k = e^{\sqrt{T2^{-k}} - \frac{1}{2}T2^{-k}} - 1$, and

$$\tau = \inf\left\{t_k; \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \geq c_k\right\} = \inf\left\{t_k; \frac{W_{t_k} - W_{t_{k-1}}}{\sqrt{t_k - t_{k-1}}} \geq 1\right\}.$$

Define a self-financing almost simple strategy by setting $V_0(\Phi) = 0$, and

$$\Phi_t = \sum_{k=0}^{\infty} \phi_{t_k}\mathbf{1}_{(t_k \wedge \tau, t_{k+1} \wedge \tau]}(t),$$

where, for $k = 0, 1, \ldots$,

$$\phi_{t_k} = \frac{1 - V_{t_k}(\Phi)}{S_{t_k}c_{k+1}}.$$

Now, the c_k's were chosen in such a way that $\mathbf{P}[\tau < T] = 1$. So, $\tau = t_N$ a.s. for some random $N \in \mathbb{N}$, and

$$
\begin{aligned}
V_\tau(\boldsymbol{\Phi}) &= V_{t_{N-1}}(\boldsymbol{\Phi}) + \phi_{t_{N-1}}\left(S_{t_N} - S_{t_{N-1}}\right) \\
&\geq V_{t_{N-1}}(\boldsymbol{\Phi}) + \frac{1 - V_{t_{N-1}}(\boldsymbol{\Phi})}{S_{t_{N-1}} c_N} S_{t_{N-1}} c_N \\
&= 1
\end{aligned}
$$

a.s.. So, we have strong arbitrage in the class $\mathscr{A}^{\mathrm{as}}$. Also, by setting

$$
\Phi_t^n = \sum_{k=0}^n \phi_{t_k} \mathbf{1}_{(t_k \wedge \tau, t_{k+1} \wedge \tau]}(t),
$$

we see that we have strong approximate arbitrage in the class $\mathscr{A}^{\mathrm{si}}$.

In order to exclude doubling-type arbitrage strategies like Example 3.5 one traditionally assumes that the value of the portfolio is bounded from below:

3.6. **Definition.** A portfolio is *nds-admissible* (*no doubling strategies*) if there exists a constant $a \geq 0$ such that

$$
\inf_{t \in [0,T]} V_t(\Phi) \geq -a \quad \text{a.s.}
$$

The class of nds-admissible portfolios is denoted by $\mathscr{A}^{\mathrm{nds}}$.

3.7. *Remark.* The sell-short–and–hold strategy $\Phi = -\mathbf{1}_{[0,T]} \in \mathscr{A}^{\mathrm{si}} \setminus \mathscr{A}^{\mathrm{nds}}$.

By Delbaen and Schachermayer [18, Theorem 1.1] the BS model has no free lunch with vanishing risk, and hence no arbitrage, in the class $\mathscr{A}^{\mathrm{nds}}$. The situation for fBS model is different:

3.8. **Theorem.** *For $H \neq \frac{1}{2}$, in the fBS model there is*

  (i) *free lunch with vanishing risk in the class $\mathscr{A}^{\mathrm{si}} \cap \mathscr{A}^{\mathrm{nds}}$,*
  (ii) *strong arbitrage in the class $\mathscr{A}^{\mathrm{as}} \cap \mathscr{A}^{\mathrm{nds}}$.*

Proof. The claims follow from Cheridito [14, Theorems 3.1 and 3.2]. □

Cheridito [14] constructed his arbitrage opportunities by using the trivial QV of the fBS model (0 for $H > 1/2$ and $+\infty$ for $H < 1/2$). So, his constructions do not work in the mfBS model. Also, Cheridito's arbitrage strategies are rather implicit in the sense that the stopping times they use are not constructed explicitly.

Let us also note that probably the first one to construct arbitrage in the fractional (Bachelier) model was Rogers [36]. His arbitrage was a doubling-type strategy similar to that of Example 3.5 with the twist that he avoided investing on "bad intervals" $(t_k, t_{k+1}]$ where the stock price was likely to fall. This was possible due to the memory of the fractional Brownian motion when $H \neq 1/2$. With this avoidance he was able to keep the value of his doubling strategy from falling below any predefined negative level, thus constructing an arbitrage opportunity in the class $\mathscr{A}^{\mathrm{as}} \cap \mathscr{A}^{\mathrm{nds}}$. Let us note

that Rogers [36] used a representation of the fBm starting from $-\infty$. So, he used memory from time $-\infty$, while Cheridito [14] and we use memory only from time 0.

The following very explicit Example 3.9, a variant of [9, Example 7], constructs approximate arbitrage in the fBS model for $H \neq 1/2$ and in the mfBS model for $H \in (1/2, 3/4)$, where the approximating strategies are from the class $\mathscr{A}^{\mathrm{si}}$. The construction follows an easy intuition: Due to the memory of the fBm the stock price tends to increase (decrease) in the future, if it already increased (decreased) in the past if $H > 1/2$, and the other way around if $H < 1/2$. Example 3.9 also shows that forward integrals with respect to fBm with $H \neq 1/2$ and mixed fBm with $H \in (1/2, 3/4)$ are not continuous in terms of the integrands. Thus, due to the Dellacherie–Meyer–Mokobodzky–Bichteler theorem, this proves that the fBm is not a semimartingale, and the mixed fBm is not a semimartingale when $H \in (1/2, 3/4)$.

3.9. *Example.* (i) Consider the fBS model

$$S_t = e^{B_t},$$

where $H \neq 1/2$. Let $t_k^n = Tk/n$, $\alpha_H = 1$, if $H > 1/2$, $\alpha_H = -1$, if $H < 1/2$, and

$$\Phi_t^n = \alpha_H n^{2H-1} \sum_{k=1}^{n-1} \frac{\log S_{t_k^n} - \log S_{t_{k-1}^n}}{S_{t_k^n}} \mathbf{1}_{\left(t_k^n, t_{k+1}^n\right]}(t).$$

Then, assuming $V_0(\mathbf{\Phi}^n) = 0$, and applying Taylor's theorem,

$$
\begin{aligned}
V_T(\mathbf{\Phi}^n) &= \alpha_H n^{2H-1} \sum_{k=1}^{n-1} \left(B_{t_k^n} - B_{t_{k-1}^n}\right) \left(\frac{S_{t_{k+1}^n}}{S_{t_k^n}} - 1\right) \\
&= \alpha_H n^{2H-1} \sum_{k=1}^{n-1} \left(B_{t_k^n} - B_{t_{k-1}^n}\right) \left(B_{t_{k+1}^n} - B_{t_k^n}\right) \\
&\quad + \alpha_H n^{2H-1} \sum_{k=1}^{n-1} \left(B_{t_k^n} - B_{t_{k-1}^n}\right) e^{\xi_k^n} \left(B_{t_{k+1}^n} - B_{t_k^n}\right)^2,
\end{aligned}
$$

where $|\xi_k^n| \in [0, |B_{t_{k+1}^n} - B_{t_k^n}|]$. Now the first term tends to $T^{2H} |2^{2H-1} - 1|$ in probability by [29, Theorem 9.5.2], and the second one vanishes as n goes to infinity using the Hölder continuity of fBm B.

(ii) Consider the mfBS model

$$S_t = e^{W_t - \frac{1}{2}t + B_t},$$

where $H \in (1/2, 3/4)$. The strategy of part (i) will still be strong approximate arbitrage. Indeed, after a Taylor expansion as above, we basically have to deal with the sum of the four terms

(3.10) $$\int_0^T K_t^n \, dW_t, \quad \int_0^T L_t^n \, dW_t, \quad \int_0^T K_t^n \, dB_t, \quad \int_0^T L_t^n \, dB_t,$$

where

$$K_t^n = n^{2H-1} \sum_{k=1}^{n-1} \mathbf{1}_{\left(t_k^n, t_{k+1}^n\right]}(t) \left(W_{t_k} - W_{t_{k-1}}\right),$$

$$L_t^n = n^{2H-1} \sum_{k=1}^{n-1} \mathbf{1}_{\left(t_k^n, t_{k+1}^n\right]}(t) \left(B_{t_k} - B_{t_{k-1}}\right),$$

and the integrals are just shorthand notation for the forward Riemann sums. Note that K^n and L^n converge to zero uniformly in probability by the Hölder continuity of (fractional) Brownian motion for $H < 3/4$. Therefore, the first two terms in (3.10) will tend to zero in probability by the Dellacherie–Meyer–Mokobodzky–Bichteler theorem [35, Theorem II.11]. The third term will tend to zero in probability because of the independence of W and B. The fourth term will go to $T^{2H}(2^{2H-1} - 1)$ in probability by part (i) of this example. We also note that $\Phi^n S$ inherits the unform convergence in probability to zero from $K^n + L^n$. Hence the amount of money invested in the stock converges to zero as n tend to infinity.

For the mfBS model the situation is the following:

3.11. **Theorem.** *For the mfBS model there is*

  (i) *strong approximate arbitrage in the class $\mathscr{A}^{\mathrm{si}}$ if $H \in (1/2, 3/4)$,*
  (ii) *no free lunch with vanishing risk in the class $\mathscr{A}^{\mathrm{nds}}$ if $H \in (3/4, 1)$.*

Proof. Claim (i) follows from Example 3.9(ii). Claim (ii) follows from Cheridito [13]. Indeed, in [13] it is shown that in this case the mixed fBm is actually equivalent in law to a Bm. □

Although the situation is bad arbitrage-wise for the fBS and the mfBS models in the class $\mathscr{A}^{\mathrm{si}} \cap \mathscr{A}^{\mathrm{nds}}$, Cheridito [14] showed that there is no arbitrage in the fBS model if there must be a fixed positive time between the readjustments of the portfolio (later arbitrage in this class was studied by Jarrow et al. [27]):

3.12. **Definition.** Let $\mathscr{T}$ be a class of finite sequences of non-decreasing stopping times $\boldsymbol{\tau} = (0 \le \tau_0 \le \cdots \le \tau_n \le T)$ satisfying some additional conditions, which can be specified as in Proposition 3.13 or Definition 4.1 below. A simple portfolio Φ is $\mathscr{T}$-*simple* if it is of the form

$$\Phi_t = \sum_{k=0}^{n-1} \phi_{\tau_k} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where $\phi_{\tau_k} \in \mathscr{F}_{\tau_k}$, $\boldsymbol{\tau} = (\tau_k)_{k=0}^{n} \in \mathscr{T}$. The class of $\mathscr{T}$-simple strategies is denoted by $\mathscr{A}^{\mathscr{T}-si}$.

3.13. **Proposition.** *Let $\mathscr{T}_h = \cup_{h>0} \{\boldsymbol{\tau}; \tau_{k+1} - \tau_k \ge h\}$. Then there is no arbitrage in the fBS model in the class $\mathscr{A}^{\mathscr{T}_h-\mathrm{si}}$.*

Proof. The claim is Cheridito's [14, Theorem 4.3]. □

4. Trading with delay-simple strategies

While Proposition 3.13 seems promising the class $\mathscr{A}^{\mathscr{T}_h-\mathrm{si}}$ is more restrictive than it may appear at a first sight. Indeed, e.g. the archetypical stopping time $\tau = \inf\{t \geq 0; S_t - S_0 \geq 1\}$ does not belong to $\mathscr{T}_h$ if S is the geometric Bm. To remedy this problem we propose the following more general class of stopping times and simple strategies:

4.1. Definition.   (i) For any stopping time τ, let $C_{S_\tau}^+([\tau, T])$ be the random space of continuous positive paths $\omega = (\omega_t)_{t \in [\tau(\omega), T]}$ with $\omega_{\tau(\omega)} = S_{\tau(\omega)}(\omega)$ fixed. A sequence of non-decreasing stopping times $\boldsymbol{\tau} = (\tau_k)_{k=0}^n$ satisfies the *delay* property if for all τ_k there is an $\mathscr{F}_{\tau_k}$-measurable open *delay set* $U_k \subset C_{S_{\tau_k}}^+([\tau_k, T])$ and an $\mathscr{F}_{\tau_k}$-measurable a.s. positive random variable ε_k such that $\tau_{k+1} - \tau_k \geq \varepsilon_k$ in the set $U_k \cap \{\tau_{k+1} > \tau_k\}$. The set of non-decreasing sequences of stopping times satisfying the delay property is denoted by $\mathscr{T}_{\mathrm{de}}$.
(ii) The class of *delay-simple* strategies is $\mathscr{A}^{\mathscr{T}_{\mathrm{de}}-\mathrm{si}}$.

4.2. Theorem. *All the models BS, fBS and mfBS are free of arbitrage in the class $\mathscr{A}^{\mathscr{T}_{\mathrm{de}}-\mathrm{si}}$.*

Before we prove Theorem 4.2 we discuss the class of delay-simple strategies.

4.3. Remark. The difference between the classes $\mathscr{T}_h$ and $\mathscr{T}_{\mathrm{de}}$ is that in $\mathscr{T}_h$ there is a fixed delay $h > 0$ between the stopping times, while in $\mathscr{T}_{\mathrm{de}}$ the delay between the stopping times depend on the path one is observing: If there is a delay on the path you are observing then there is also a delay on all the paths that are close enough of the path that one is observing.

Obviously $\mathscr{T}_h \subset \mathscr{T}_{\mathrm{de}}$, and the inclusion is proper.

4.4. Example. The following sequences of stopping times are in $\mathscr{T}_{\mathrm{de}}$:

(i)
$$\tau_{k+1} = \inf\left\{t > \tau_k; S_t - S_{\tau_k} \geq b_t^k\right\},$$
where b^k's are continuous function with $b_{\tau_k}^k > 0$. Indeed, take
$$U_k = \left\{\omega; S_t(\omega) < \omega_t^0 \text{ for all } t \in [\tau_k, T]\right\},$$
where ω^0 is some path for which $\tau_{k+1}(\omega^0) > \tau_k(\omega^0)$.

(ii)
$$\tau_{k+1} = \inf\left\{t > \tau_k; S_t - S_{\tau_k} \leq a_t^k\right\},$$
where a^k's are continuous function with $a_{\tau_k}^k < 0$. Indeed, take
$$U_k = \left\{\omega; S_t(\omega) > \omega_t^0 \text{ for all } t \in [\tau_k, T]\right\},$$
where ω^0 is some path for which $\tau_{k+1}(\omega^0) > \tau_k(\omega^0)$.

(iii) One can show that
$$\tau_{k+1} = \inf\left\{t > \tau_k; S_t - S_{\tau_k} \leq a_t^k \text{ or } S_t - S_{\tau_k} \geq b_t^k\right\},$$

a^k's and b^k's are continuous with $a^k_{\tau_k} < 0 < b^k_{\tau_k}$, is in $\mathscr{T}_{\mathrm{de}}$ (see [9, Example 6 (i)]).

4.5. *Example.* We construct a stopping time τ in the fractional Wiener space such that $(\tau_0, \tau_1) := (0, \tau)$ is not in $\mathscr{T}_{\mathrm{de}}$: $\tau = \inf\{t > 0; e^{B_t + t^a} = 1\}$. By the law of iterated logarithm $\tau > 0$ a.s. if $a < H$. However, any open set $U \subset C^+_{S_0}([0, T])$ contains sequences (ω^n) for which $\tau(\omega^n) \to 0$.

4.6. **Definition.** A process S satisfies the $\mathscr{T}$-*conditional up'n'down* property ($\mathscr{T}$-CUD) if, for all $\boldsymbol{\tau} \in \mathscr{T}$ and all k, either

$$\mathbf{P}\left[S_{\tau_{k+1}} > S_{\tau_k}\big|\mathscr{F}_{\tau_k}\right] > 0 \text{ and } \mathbf{P}\left[S_{\tau_{k+1}} < S_{\tau_k}\big|\mathscr{F}_{\tau_k}\right] > 0$$

or

$$\mathbf{P}\left[S_{\tau_{k+1}} = S_{\tau_k}\big|\mathscr{F}_{\tau_k}\right] = 1.$$

If there are no additional restrictions for $\mathscr{T}$ (except that it contains non-decreasing finite sequences of stopping times), we say simply that S satisfies CUD.

The following lemma can be proved analogously to [27, Lemma 1].

4.7. **Lemma.** *There is no arbitrage in the class $\mathscr{A}^{\mathscr{T}-\mathrm{si}}$ if and only if the model satisfies $\mathscr{T}$-CUD.*

CUD is related to the support of the stock-price model S. Another support-related condition we need is:

4.8. **Definition.** A continuous positive process S has *conditional full support* (CFS) if, for all stopping times τ,

$$\mathrm{supp}\, \mathbf{P}[S \in \cdot |\mathscr{F}_\tau] = C^+_{S_\tau}([\tau, T]) \quad \text{a.s..}$$

4.9. *Remark.*   (i) CFS is equivalent to the *conditional small-ball property*: For every stopping time τ, all the open balls contained in $C^+_{S_\tau}([\tau, T])$ have a.s. positive regular conditional probability, i.e.

$$\mathbf{P}\left[\sup_{t \in [\tau, T]} \left|S_t - S^0_t\right| \leq \varepsilon \Big|\mathscr{F}_\tau\right] > 0$$

a.s. for all $S^0 \in C^+_{S_\tau}([\tau, T])$ and $\mathscr{F}_\tau$-measurable a.s. positive random variables ε. For a proof of this see Pakkanen [34, Lemma 2.3].

  (ii) By Pakkanen [34, Lemma 2.10] a process X has CFS with respect to its own filtration $\mathscr{F}^X_t = \sigma(X_s,\ 0 \leq s \leq t)$ if and only if it has the CFS with respect to the augmentation of $\mathscr{F}^X_t$.

 (iii) By Guasoni et al. [25, Lemma 2.9] one can replace the stopping times with deterministic times in Definition 4.8.

 (iv) CFS is neither necessary nor sufficient for no-arbitrage in $\mathscr{A}^{\mathrm{si}}$. On the one hand, any bonded martingale satisfies no-arbitrage in $\mathscr{A}^{\mathrm{si}}$, but violates CFS. On the other hand $W_t + t^a$, $a < 1/2$, has arbitrage in $\mathscr{A}^{\mathrm{si}}$ by the law of the iterated logarithm, but satisfies CFS. However, CFS is sufficient for absence of arbitrage with the class $\mathscr{A}^{\mathscr{T}_{\mathrm{de}}-\mathrm{si}}$. This will be shown in the next lemma.

4.10. **Lemma.** *Suppose S has CFS. Then there is no arbitrage in the model S in the class $\mathscr{A}^{\mathscr{T}_{\mathrm{de}}-\mathrm{si}}$.*

Proof. By Lemma 4.7 we need to show that the $\mathscr{T}_{\mathrm{de}}$-CUD is satisfied. If $\tau_{k+1} = \tau_k$, this is certainly the case. So, we can assume that $\tau_{k+1} > \tau_k$.

We show that $\mathbf{P}[S_{\tau_{k+1}} > S_{\tau_k}|\mathscr{F}_{\tau_k}] > 0$ a.s.; the proof for $\mathbf{P}[S_{\tau_{k+1}} < S_{\tau_k}|\mathscr{F}_{\tau_k}] > 0$ a.s. follows analogously.

By the CFS it is enough to show that $\{S_{\tau_{k+1}} > S_{\tau_k}\} \subset C^+_{S_{\tau_k}}([\tau_k, T])$ contains an open set. Let U_k be an ε_k-delay set for τ_k, i.e. $U \subset C^+_{S_{\tau_k}}([\tau_k, T])$ is open and $\tau_{k+1} - \tau_k \geq \varepsilon_k$ on U_k. We first assume that U_k contains a strictly increasing path ω^0 on $[\tau_k, T]$. Denote by $B_{\omega^0}(\eta_k)$ the open η_k-ball around ω_0 in $C^+_{S_{\tau_k}}([\tau_k, T])$. Choosing η_k sufficiently small we have $B_{\omega^0}(\eta_k) \subset U_k$ (because U_k is open) and $\omega^0_{\tau_k+\varepsilon_k} > \omega^0_{\tau_k} + \eta_k$ (because ω^0 is strictly increasing). Hence, for every $\omega \in B_{\omega^0}(\eta_k)$,

$$
\begin{aligned}
\omega_{\tau_{k+1}(\omega)} - S_{\tau_k} \quad &> \quad \omega^0_{\tau_{k+1}(\omega)} - \eta_k - S_{\tau_k} \\
&\geq \quad \omega^0_{\tau_k+\varepsilon_k} - S_{\tau_k} - \eta_k \\
&= \quad \omega^0_{\tau_k+\varepsilon_k} - \omega^0_{\tau_k} - \eta_k \\
&> \quad 0,
\end{aligned}
$$

So, $B_{\omega^0}(\eta_k) \subset \{S_{\tau_{k+1}} > S_{\tau_k}\}$, and the claim follows, if U_k contains a strictly increasing path. If U_k does not contain a strictly increasing path, we proceed as follows. Being an open set in $C^+_{S_{\tau_k}}([\tau_k, T])$, U_k contains paths that are strictly increasing on a small enough interval $[\tau_k, \tau_k+2\eta_k]$. Hence, there is a strictly increasing path ω^0 and an open ball B_k around ω^0 in $C^+_{S_{\tau_k}}([\tau_k, T])$ such that any $\omega \in B_k$ coincides with some path $\bar\omega \in U_k$ on the segment $[\tau_k, \tau_k + \eta_k]$. Hence, $\tau_{k+1}(\omega) - \tau_k \geq (\tau_{k+1}(\bar\omega) - \tau_k) \wedge \eta_k \geq \varepsilon_k \wedge \eta_k =: \varepsilon^0_k$ for every $\omega \in B_k$. Therefore B_k is an ε^0_k-delay set which contains a strictly increasing path and so the first case applies. $\square$

Proof of Theorem 4.2. By [22, Theorem 2.1] the Bm, the fBm, and the mixed fBm all have CFS in the space $C_0([0, T])$ (with respect to the filtration generated by the respective process), since their spectral measures have heavy enough tails. For a nice proof that fBm has CFS see also [15]. So, the BS, the fBS, and the mfBS models all have CFS in $C^+_{s_0}([0, T])$, because with any homeomorphism η on $C_0([0, T])$ the mapping $\omega \mapsto s_0 e^{\omega+\eta}$ is a homeomorphism between $C_0([0, T])$ and $C^+_{s_0}([0, T])$. So, the claim follows from Lemma 4.10. $\square$

5. Continuous trading

While the previous sections were concerned with trading strategies which can be readjusted finitely many times only, we will now admit continuous readjustment of the portfolio. A natural generalization of the self-financing property in Definition 3.3 can be given in terms of forward integrals. Here we stick to the simplest possible definition of forward integrals due to [20], but refer to [37] for the general theory.

5.1. Definition. Let $t \leq T$ and let $X = (X_s)_{s \in [0,T]}$ be a continuous process. The *forward integral* of a process $Y = (Y_s)_{s \in [0,T]}$ with respect to X (along dyadic partitions) is

$$\int_0^t Y_s \, \mathrm{d}X_s := \lim_{n \to \infty} \sum_{\substack{i=0,\ldots,2^n-1, \\ Ti/2^n \leq t}} Y_{Ti/2^n} \left(X_{T(i+1)/2^n} - X_{Ti/2^n} \right),$$

if the limit exists $\mathbf{P}$-almost surely.

If necessary, we interpret the forward integral in an improper sense at $t = T$. Itô's formula for the forward integral depends on the quadratic variation of the integrator.

5.2. Definition. The pathwise *quadratic variation* (QV) of a stochastic process (along dyadic partitions) is

$$\langle X \rangle_t := \lim_{n \to \infty} \sum_{\substack{i=0,\ldots,2^n-1, \\ Ti/2^n \leq t}} \left(X_{T(i+1)/2^n} - X_{Ti/2^n} \right)^2,$$

if, for all $t \leq T$, the limit exists $\mathbf{P}$-almost surely.

5.3. Proposition.   (i) *For the fBS model and the mfBS model with $H < 1/2$ the limit in Definition 5.2 diverges to infinity.*
  (ii) *For the fBS model with $H > 1/2$, the QV is constant 0.*
 (iii) *The QV in the BS model and in the mfBS model with $H > 1/2$ is given by*

$$\mathrm{d} \langle S \rangle_t = \sigma^2 S_t^2 \, \mathrm{d}t.$$

Proof. It is well known that Bm has the identity map as QV. Moreover, fBm has zero quadratic variation for $H > 1/2$ and infinite quadratic variation for $H < 1/2$, see e.g. [10], Chapter 1.8. By independence, the QV of the mixed fBm is the sum of the QV of Bm and fBm. Finally, the stock models under consideration are $\mathscr{C}^1$-functions of these processes (up to a finite variation drift), and so a result by [20], p. 148, applies. $\square$

The following Itô formula for the forward integrals with continuous integrator can be derived by a Taylor expansion as usual, see [20].

5.4. Lemma. *Let X be a continuous process with continuous QV. Suppose $f \in \mathscr{C}^{1,2}([0,T] \times \mathbb{R})$. Then, for $0 \leq t \leq T$,*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial}{\partial t} f(u, X_u) \, \mathrm{d}u + \int_0^t \frac{\partial}{\partial x} f(u, X_u) \, \mathrm{d}X_u$$

$$+ \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f(u, X_u) \, \mathrm{d} \langle X \rangle_u$$

In particular, this formula implies that the forward integral on the right hand side exists and has a continuous modification.

After this short digression on forward integrals we can introduce several classes of portfolios.

5.5. **Definition.**   (i) A portfolio is *self-financing*, if, for all $0 \leq t \leq T$,

$$V_t(\mathbf{\Phi}) = V_0(\mathbf{\Phi}) + \int_0^t \Phi_t dS_t.$$

The class of self-financing portfolios (without any extra constraints) is denoted by $\mathscr{A}$.

(ii) A self-financing portfolio is called a *spot strategy*, if $\Phi_t = \varphi(t, S_t)$ for some deterministic function φ, i.e. the number of shares held in the stock depends on time and the spot only. We apply the notation $\mathscr{A}^{spot}$ for the class of spot strategies.

The following theorem discusses arbitrage with spot strategies in the BS model. It again illustrates some subtleties of arbitrage theory in continuous time, even for models which admit an equivalent martingale measure. As in the case of almost simple strategies, arbitrage is possible, if arbitrarily large losses are allowed prior to maturity.

5.6. **Theorem.**   (i) *In the BS model there is strong arbitrage in the class $\mathscr{A}^{spot}$.*

(ii) *In the BS model there is no free lunch with vanishing risk in the class $\mathscr{A} \cap \mathscr{A}^{nds}$.*

Proof. (i) We give a direct construction making use of Itô's formula (Lemma 5.4) and the QV of the Black-Scholes model. W.l.o.g. we assume $\sigma = 1$ and $\mu = 0$. Let

$$\Phi_t = -\frac{\frac{\partial}{\partial x} v(t, W_t)}{S_t},$$

where $v(t, x)$ is the heat kernel

$$v(t, x) = \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{1}{2} \frac{x^2}{T-t}}.$$

By Lemma 5.4, applied to the Bm W,

$$V_T(\Phi) = \int_0^T \Phi_t \, dS_t = -\int_0^T \frac{\partial}{\partial x} v(t, W_t) \, dW_t = v(0,0) - v(T, W_T) = \frac{1}{\sqrt{2\pi T}},$$

almost surely. So, we have constructed a strong arbitrage and it belongs to the class $\mathscr{A}^{spot}$, because the Bm W is a deterministic function of time and the Black-Scholes stock S.

(ii) The BS model has an equivalent martingale measure. Hence the fundamental theorem of asset pricing [17] ensures that there is no free lunch with vanishing risk with nds-admissible, self-financing strategies. $\square$

The construction of the 'doubling' arbitrage in the previous theorem only relied on the quadratic variation structure of the model. In the pure fractional BS model with $H > 1/2$ the QV is constant zero. This fact, combined with Itô's formula, can be exploited to construct an nds-admissible arbitrage in class $\mathscr{A}^{spot}$. The following simple example is due to Dasgupta and Kallianpur [16] and Shiryaev [39].

5.7. *Example.* Choosing $\Phi_t = S_t - S_0$, we obtain by Itô's formula (Lemma 5.4) and the zero QV property of the fBS model with $H > 1/2$,

$$(S_t - S_0)^2 = 2 \int_0^t \Phi_u \, dS_u.$$

Hence, $\boldsymbol{\Phi}$ is nds-admissible (it is bounded from below by 0) and an arbitrage. Again, this construction of an arbitrage applies to all models with zero QV and $P(S_T \neq S_0) > 0$.

We now consider hedging in the fBS model with Hurst parameter larger than a half. Although there exists strong arbitrage in the class $\mathscr{A}^{\mathrm{nds}} \cap \mathscr{A}^{\mathrm{as}}$ by Theorem 3.8, one can still consider the hedging problem in the fBS model. Indeed, in spite of arbitrage one may still be interested in hedging per se. But, it must be noted that hedging cannot be used as a pricing paradigm in the presence of strong arbitrage, since for any hedge one can find a super-hedge with smaller initial capital by combining the hedge with a strong arbitrage.

By a straightforward generalization of the previous example, we observe that a smooth European style option, i.e. with pay-off $f(S_t)$ for some $f \in \mathscr{C}^1$ can be hedged with initial endowment $f(S_0)$ and the strategy $\Phi_t = f'(S_t)$. In reality many options, like vanilla options, have a convex pay-off function, which does not belong to class $\mathscr{C}^1$. A generalization to this situation is possible with some extra effort as outlined next.

5.8. **Definition.** Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function and $H > 1/2$. If we can find a self-financing strategy $\boldsymbol{\Phi}$ and a constant c^f such that

$$(5.9) \qquad\qquad f(S_T) = c^f + \int_0^T \Phi_s \, dS_s,$$

then $\boldsymbol{\Phi}$ is a *hedging strategy* and c^f is a *hedging cost* of the option $f(S_T)$.

5.10. *Remark.*   (i) Because of the strong arbitrage possibilities in the fBS model one cannot interpret the hedging cost c^f as a minimal super-replication price.

(ii) The strong arbitrage possibility of the fBS model does not imply that one can take $c^f = 0$ in (5.9): One can super-hedge with zero capital, but the hedge may not be exact. While from the purely monetary point of view this does not matter, there may be situations where one is penalized for not hedging exactly.

If f is a convex function, then f_x^+ (f_x^-) is the right (left) derivative of f. The following theorem can be regarded as a generalization of the Itô formula in Lemma 5.4 for non-smooth convex functions in the pure fractional Brownian motion setting.

5.11. **Theorem.** *Suppose S is the fBS model with $H > 1/2$ and f is a convex function. Then*

$$(5.12) \qquad\qquad f(S_T) = f(S_0) + \int_0^T f_x^+(S_u) \, dS_u.$$

In particular, the European option $f(S_T)$ can be perfectly hedged with cost $f(S_0)$ and the hedging strategy given by $\Phi_t = f_x^+(S_t)$.

Proof. One proves Theorem 5.11 by showing that the integral exists as a generalized Lebesgue–Stieltjes integral. This is done with the help of some fractional Besov space techniques. Finally, one proves that the integral exists as a forward integral and actually even as a Riemann-Stieltjes integral. For the rigorous proof see [4]. Note that one can replace the right derivative f_x^+ by the left derivative f_x^-, as both derivatives differ on a countable set only. □

5.13. *Example.* If the convex function f corresponds to the call option, i.e. $f(x) = (x - K)^+$, then we observe that the stop–loss–start–gain portfolio replicates the call option:

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbf{1}_{\{S_t \geq K\}} \, \mathrm{d}S_t.$$

Note that this again gives an arbitrage strategy, if the option is at–the–money or out–of–the–money.

If $H < 1/2$ stochastic integrals for typical spot strategies with respect to the fBS model fail to exist. So it makes little sense to consider continuous trading in this situation. This unfortunate property is related to the infinite QV of the fBS model for small Hurst parameter and thus applies for the mixed model with $H < 1/2$ as well.

For the remainder of the section we shall therefore discuss the mfBS model with $H > 1/2$. In the case $H > 3/4$, the mfBS model is equivalent in law to the BS model, see [13]. Therefore, all constructions of arbitrages with doubling strategies and all results on no arbitrage with nds-strategies directly transfer from the BS model to the mfBS model with $H > 3/4$. Moreover, the latter model inherits the completeness of the BS model. We now discuss to what extent the mixed model with $1/2 < H \leq 3/4$ differs from the BS model. The argumentation below only makes use of the fact that the mixed model has the same QV as the BS model and has conditional full support.

5.14. **Theorem.** *Suppose S is the mfBS with $H > 1/2$. Then,*

    (i) *There is strong arbitrage in the class $\mathscr{A}^{spot}$.*
    (ii) *There is no nds-admissible arbitrage $\mathbf{\Phi}$ of the form*

$$\Phi_t = \varphi\left(t, \max_{0 \leq u \leq t} S_u, \min_{0 \leq u \leq t} S_u, \int_0^t S_u \, \mathrm{d}u, S_t\right)$$

       with $\varphi \in \mathscr{C}^1([0, T] \times \mathbb{R}_+^4)$. A strategy of this form will be called smooth *from now on.*

Proof. (i) Here the same constructive example as in Theorem 5.6 applies, because the mfBS model has the same QV as the BS model.
(ii) We fix some smooth strategy $\mathbf{\Phi}$. By a slightly more general Itô formula

than the one in Lemma 5.4 one can conclude that there is a continuous functional $v : [0, T] \times C_{s_0}^+([0,T]) \to \mathbb{R}$ such that $V_t(\mathbf{\Phi}) = v(t, S)$. By the full support property, the paths of the mfBS model can be approximated by paths of the BS model and vice versa. In this way, absence of arbitrage can be transferred from the BS model to the mfBS model. The details are spelled out in [9], Theorem 4.4.

We point out, that in the special case $\mathbf{\Phi} = (\Phi^0, \Phi) \in \mathscr{A}$ with $\Phi_t = \varphi(t, S_t)$ and $\Phi_t^0 = \varphi^0(t, S_t)$ for some sufficiently smooth functions (φ, φ^0), the value process $V_t(\mathbf{\Phi})$ can be linked to a PDE. This was exploited in [1] in order to prove absence of arbitrage in this special case. $\qquad\square$

5.15. *Remark.*   (i) In Theorem 5.14, (ii), the differentiability of φ at $t = T$ can be relaxed to some extent and absence of arbitrage still holds. The resulting class of strategies contains hedges for many relevant European, Asian, and lookback options. These hedges (as functionals on the paths) and the corresponding option prices (deduced by hedging and no-arbitrage relative to this class of portfolios) are the same as in the BS model. For the details we refer to [9]. We note that this robustness of hedging strategies was already shown by Schoenmakers and Kloeden [38] in the case of European options.

(ii) The no-arbitrage result in Theorem 5.14, (ii), can be extended in several directions. Additionally to the running maximum, minimum and average, the strategy can depend on other factors, which are supposed to be of finite variation and satisfy some continuity condition as functionals on the paths. The investor also is allowed to switch between different smooth strategies at a large class of stopping times and still absence of arbitrage holds true for these stopping-smooth strategies. For the exact conditions on the stopping times we refer to Section 6 in [9], but we note that many typical ones such as the first level crossing of the stock are included.

6. Trading under transaction costs

Recently Guasoni [23] and Guasoni et. al. [25] have shown, that allowing transaction costs in the fBS model the arbitrage possibilities disappear. First they introduce, following Jouini and Kallal [28], the notion of ε- consistent price system.

6.1. **Definition.** Let S be a continuous process with paths in $C_{S_0}^+([0,T])$. An ε-consistent price system is a pair $(\widetilde{S}, \mathbf{Q})$ of a probability $\mathbf{Q}$ equivalent to $\mathbf{P}$, and a $\mathbf{Q}$-martingale $\widetilde{S} = (\widetilde{S}_t)_{0 \le t \le T}$, such that $S_0 = \widetilde{S}_0$, and for $0 \le t \le T$, $\varepsilon > 0$

$$1 - \varepsilon \le \frac{\widetilde{S}_t}{S_t} \le 1 + \varepsilon, \text{ a.s.}$$

With proportional transaction costs one can not use continuous trading. Denote by $\mathscr{V}(\Phi)$ the total variation of the process Φ. In this section a

trading strategy Φ is predictable finite-variation $\mathbb{R}$-valued process such that $\Phi_0 = \Phi_T = 0$. The value of Φ with ε- costs $V^\varepsilon(\Phi)$ is

$$V^\varepsilon(\Phi) = \int_0^T \Phi_s dS_s - \varepsilon \int_0^T S_s d\mathcal{V}(\Phi)_s.$$

Define $V_t^\varepsilon(\Phi)$ by

$$V_t^\varepsilon(\Phi) = V^\varepsilon(\Phi 1_{(0,t)}),$$

and so $V^\varepsilon(\Phi) = V_T^\varepsilon(\Phi)$.

Next, we define the set of admissible strategies in this context, following [25]: given $M > 0$, the strategy Φ is M-admissible, if for all $t \in [0,T]$ we have that

$$V_t^\varepsilon(\Phi) \geq -M(1 + S_t) \text{ a.s.}$$

The set of M-admissible strategies is denoted by $\mathscr{A}_M^{\mathrm{adm}}(\varepsilon)$. Define also

$$\mathscr{A}^{\mathrm{adm}}(\varepsilon) = \cup_{M>0} \mathscr{A}_M^{\mathrm{adm}}(\varepsilon).$$

Finally we say that S admits arbitrage with ε-transaction costs if there is $\Phi \in \mathscr{A}^{\mathrm{adm}}(\varepsilon)$ such that $V^\varepsilon(\Phi) \geq 0$ and $\mathbf{P}(V^\varepsilon(\Phi) > 0) > 0$.

We can now state the fundamental theorem of asset pricing with ε-transaction costs given in [25, Theorem 1.11]:

6.2. **Theorem.** *Let $S \in C_{s_0}^+([0,T])$. Then the following two conditions are equivalent:*

  (i) *For each $\varepsilon > 0$ there exists an ε-consistent price system.*
  (ii) *For each $\varepsilon > 0$, there is no arbitrage for ε-transaction costs.*

It is shown by Guasoni et al. [24] that conditional full support implies the existence of an ε-consistent price system for every $\varepsilon > 0$. Therefore, the fBS models and the mfBS models do not adimit arbitrage under transaction cost with the classes of strategies $\mathscr{A}^{\mathrm{adm}}(\varepsilon)$ for $\varepsilon > 0$.

We will study a concrete hedging problem with proportional transaction costs.

In Theorem 5.11 it was shown that the European option $f(S_T)$ can be perfectly hedged with cost $f(S_0)$ and hedging strategy $\Phi_t = f_x^-(S_t)$. Take $T = 1$, put $t_i^n = \frac{i}{n}$, $i = 0,\dots,n$, and consider the discretized hedging strategy Φ^n

$$(6.3) \qquad \Phi_t^n = \sum_{i=1}^n f_x^-(S_{t_{i-1}^n}) 1_{(t_{i-1}^n, t_i^n]}(t).$$

Consider now discrete hedging with proportional transaction costs $k_n = k_0 n^{-\alpha}$ with $\alpha > 0, k_0 > 0$. The value of the strategy Φ^n at time $T = 1$ is

$$(6.4) \quad V_1(\Phi^n; k_n) = f(S_0) + \int_0^1 \Phi_t^n dS_t - k_n \sum_{i=1}^n S_{t_{i-1}^n} |f_x^-(S_{t_i^n}) - f_x^-(S_{t_{i-1}^n})|.$$

Note that there is no transaction costs at time $t = 0$.

In the next theorem μ^f is the second derivative f_{xx} of the convex function f. The derivative exists in a distributional sense, and μ^f is a Radon measure. The occupation measure Γ_{B^H} of fractional Brownian motion B^H is defined by $\Gamma_{B^H}([0,t] \times A) = \lambda\{s \in [0,t] : B_s^H \in A\}$; here λ is the Lebesgue measure and A is a Borel set. Denote by $l^H(x,t)$ the local time of fractional Brownian motion B^H; recall that local time l^H is the density of the occupation measure with respect the Lebesgue measure.

The following theorem is proved in [3]:

6.5. Theorem. *Let $V_1(\Phi; k_n)$ be the value of the discrete hedging strategy Φ^n with proportional transaction costs $k_n = k_0 n^{-\alpha}$.*

(i) *If $\alpha > 1 - H$, then as $n \to \infty$*

$$V_1(\Phi^n; k_n) \to f(S_1) \text{ in probability.}$$

(ii) *If $\alpha = 1 - H$, then as $n \to \infty$*

$$(6.6) \qquad V_1(\Phi^n; k_n) \to f(S_1) - \sqrt{\frac{2}{\pi}} k_0 \int_{\mathbb{R}} \int_0^1 S_t dl^H(\ln(a), t) \mu^f(da).$$

6.7. *Remark.* Note that one can write the limit result in (6.6) as

$$f(S_1) = f(S_0) + \int_0^1 f_x^-(S_u) dS_u + \sqrt{\frac{2}{\pi}} k_0 \int_{\mathbb{R}} \int_0^1 S_t dl^H(\ln(a), t) \mu^f(da);$$

if l^W is the local time for Brownian motion, then the Itô-Tanaka formula gives

$$f(W_1) = f(0) + \int_0^1 f_x^-(W_u) dW_u + \frac{1}{2} \int_0^1 \int_{\mathbb{R}} dl^W(a, u) \mu^f(da).$$

Hence asymptotical transaction costs with $\alpha = 1 - H$ have a similar effect as the existence of a non-trivial quadratic variation.

7. Approximations

Binary tree approximations. The famous Donsker's invariance principle links random walks to the Bm. By using this principle one can approximate the BS model with Cox–Ross–Rubinstein (CRR) binomial trees. To be more precise, let for all $n \in \mathbb{N}$, $(\xi_k^n)_{k \in \mathbb{N}}$ be i.i.d. random variables with $\mathbf{P}[\xi_k^n = 1] = 1/2 = \mathbf{P}[\xi_k^n = -1]$. Set

$$W_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k^n.$$

Then the Donsker's invariance principle states that the processes W^n, $n \in \mathbb{N}$, converge in the Skorohod space $D([0,T])$ to the Bm. Let S^n to be the binomial model defined by

$$S_t^n = \prod_{s \le t} (1 + \Delta W_s^n).$$

Then the processes $S^n, n \in \mathbb{N}$, converge weakly in $D([0,T])$ to the geometric Bm $S_t = e^{W_t - t/2}$, i.e. the binomial models S^n, $n \in \mathbb{N}$, approximate the BS model.

In [40] a fractional CRR model was constructed that approximates the fBS model when $H > 1/2$, and later this approximation was extended in different directions by Nieminen [33] and Mishura and Rode [32]. We give here a brief overview of the construction in [40]:

Let $(\xi_k^n)_{k \in \mathbb{N}}$ be as before, and let $k(t, s)$ be the kernel that transforms the Bm into a fBm:

$$k(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{1}{2}} (u - s)^{H - \frac{3}{2}} \, du,$$

where

$$c_H = (H - \frac{1}{2}) \sqrt{\frac{(2H + \frac{1}{2}) \Gamma(\frac{1}{2} - H)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)}},$$

and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$ is the Gamma function. Then

$$B_t = \int_0^t k(t, s) \, dW_s.$$

To get a piece-wise constant process in $D([0, T])$ one must regularize the kernel:

$$k^n(t, s) = n \int_{s - 1/n}^s k \left(\frac{\lfloor nt \rfloor}{n}, u \right) \, du.$$

Set

$$B_t^n = \int_0^t k^n(t, s) \, dW_s^n$$

and

$$S_t^n = \prod_{s \leq t} (1 + \Delta B_s^n).$$

7.1. Theorem. *Let $H > 1/2$.*

  (i) *The random walks B^n, $n \in \mathbb{N}$, converge weakly in $D([0, T])$ to the fBm B.*
  (ii) *The binary models S^n, $n \in \mathbb{N}$, converge weakly in $D([0, T])$ to the fBS model $S = e^B$.*
  (iii) *The fractional CRR binary models $S^n, n \in \mathbb{N}$, are complete, but exhibit arbitrage opportunities if n is sufficiently large.*

Proof. (i) is the "fractional invariance principle" [40, Theorem 1], (ii) follows basically from (i), the continuous mapping theorem, and a Taylor expansion of $\log(S^n)$, cf. the proof of [40, Theorem 3] for details. The completeness claim of (iii) is obvious, since the market models are binary. The arbitrage claim of (iii) follows from the fact that if we have only gone up in the binary tree for long enough, the stock-price will increase in the next step no matter which branch the process takes in the tree. We refer to the proof of [40, Theorem 5] for details. □

A main motivation for considering the approximation S^n is that the continuous time process $S_t = e^{B_t}$ solves the SDE

$$dS_t = S_t dB_t, \quad S_0 = 1$$

in the sense of forward integration. Alternatively, one can build an integral on Wick-Riemann sums [6, 10, 19, 31] and examine the SDE

$$dX_t = X_t d^\diamond B_t, \quad X_0 = 1.$$

Here, $X_t = \exp\{B_t - t^{2H}/2\}$. Thus, the processes S and X only differ by a deterministic factor. Without going into any details here, we note that the *Wick product* can be defined by

$$e^{\Phi - E[\Phi^2]/2} \diamond e^{\Psi - E[\Psi^2]/2} = e^{(\Phi+\Psi) - E[(\Phi+\Psi)^2]/2}$$

for centered Gaussian random variables Φ and Ψ and can be extended to larger classes of random variables by bilinearity and denseness arguments, see e.g. [6, 19]. Somewhat surprisingly, there is a very simple analogue of the Wick product for the binary random variables ξ_k^n, $k = 1, \ldots, n$, see [26], which gives rise to a natural binary discretization of X_t suggested by Bender and Elliott [7].

The *discrete Wick product* can be defined as $(A, B \subset \{1, \ldots, n\})$

$$\prod_{i \in A} \xi_i^n \diamond_n \prod_{i \in B} \xi_i^n := \left\{ \begin{array}{cl} \prod_{i \in A \cup B} \xi_i^n & \text{if } A \cap B = \emptyset \\ 0 & \text{otherwise} \end{array} \right.,$$

and extends by bilinearity to $L^2(\mathscr{F}_n)$, where $\mathscr{F}_n$ denotes the σ-field generated by $(\xi_1^n, \ldots, \xi_n^n)$. A discrete version of the Wick-fractional Black-Scholes model is then defined by

$$X_t^n = \mathop{\diamond}_{s \le t} (1 + \Delta B_s^n).$$

Bender and Elliott [7] argue in favor of this discretization that the discrete Wick product separates influences of the drift and volatility.

7.2. **Theorem.** *Let $H > 1/2$.*

(i) *The binary models X^n, $n \in \mathbb{N}$, converge weakly in $D([0, T])$ to the Wick-fractional Black-Scholes model X.*

(ii) *The Wick-fractional CRR binary models $X^n, n \in \mathbb{N}$, are complete, but exhibit arbitrage opportunities if n is sufficiently large.*

The proof of (ii) is similar to the one of Theorem 7.1, (iii), and can be found in [7]. As is pointed out there, the use of the discrete Wick products kills a part of the memory as compared to the discrete-time model S^n. It turns out, however, that the remaining part of the memory is still sufficient to construct an arbitrage. Completeness again follows from the fact that the model is binary. For the proof of (i), one cannot argue by the continuous mapping theorem, because the discrete Wick product is not a pointwise operation. Instead the relation of the Wick powers to Hermite polynomials and explicit computations of the Walsh decomposition (which can be considered a discrete analogue of the chaos decomposition to some extent) can be exploited, see [8].

Arbitrage-free approximation. The results in this section are motivated by [30], where the authors give an arbitrage-free approximation to fBS

model. The prelimit models in this approximation are not complete, however.

Recall the following classical result: Let $N = (N_t)_{t \in \mathbb{R}_+}$ be a Poisson process with intensity 1, and set

$$W_t^n = \frac{1}{\sqrt{n}} \left(N_{nt} - nt \right).$$

Then W^n converges to a Bm W in the Skorohod space $D([0, T])$, the process $dS_t^n = S_{t-}^n dW_t^n$, $S_0^n = S_0$ converges weakly to the BS model, and the approximation is complete and arbitrage-free.

We approximate the fractional Black–Scholes model $(S, (\mathscr{F}_t)_{t \in [0,T]}, \mathbf{P})$ with a sequence $(S^n, (\mathscr{F}^n)_{t \in [0,T]})$ of models driven by scaled renewal counting processes. The prelimit models are complete and arbitrage-free. The approximation is based on the limit theorem of Gaigalas and Kaj [21]. It goes as follows: let G be a continuous distribution function with heavy tails. i.e.

(7.3) $$1 - G(t) \sim t^{-(1+\beta)}$$

as $t \to \infty$ with $\beta \in (0, 1)$.

Take η_i to be the sojourn times of a renewal counting process N. Assume that $\eta_i \sim G$ for $i \geq 2$; for the first sojourn time η_1 assume that it has the distribution $G_0(t) = \frac{1}{\mu} \int_0^t (1 - G(s)) ds$ (here μ is the normalizing constant), so that the renewal counting process

$$N_t = \sum_{k=1}^{\infty} 1_{\{\tau_k \leq t\}}$$

is stationary, where $\tau_1 = \eta_1$ and $\tau_k := \eta_1 + \cdots + \eta_k$.

Take now independent copies $N^{(i)}$ of N, numbers $a_m \geq 0$, $a_m \to \infty$ such that

(7.4) $$\frac{m}{a_m^\beta} \to \infty;$$

using the terminology of Gaigalas and Kaj we can speak of *fast connection rate*.

Define the *workload* process $W(m, t)$ by

$$W(m, t) = \sum_{i=1}^{m} N_t^{(i)};$$

note that the process N^m is a counting process, since the sojourn distribution is continuous. We have that $EW(m, t) = \frac{mt}{\mu}$, since $W(m, t)$ is a stationary process.

7.5. **Proposition** (Gaigalas and Kaj [21]).   *Assume* (7.3) *and* (7.4)*. Let*

$$Y^m(t) := \mu^{\frac{3}{2}} \sqrt{\frac{\beta(1 - \beta)(2 - \beta)}{2}} \frac{W(m, a_m t) - m\mu^{-1} a_m t}{m^{\frac{1}{2}} a_m^{1 - \frac{\beta}{2}}}.$$

Then Y^m converges weakly [in the Skorohod space D] to a fractional Brownian motion B^H, where $H = 1 - \frac{\beta}{2}$.

Since the process Y^m is a semimartingale, it has a semimartingale decomposition

(7.6) $$Y^m = M^m + H^m;$$

here $H^m = B^m - A^m$ and B^m is the compensator of the normalized aggregated counting process W. Note that the process H^m is a continuous process with bounded variation.

Up to a constant we have that the square bracket of the martingale part M^m of the semimartingale Y^m is

$$[M^m, M^m]_t = C \frac{W(m, a_m t)}{m a_m^{2-\beta}}.$$

But our assumption imply that $[M^m, M^m]_t \overset{L^1(P)}{\to} 0$, as $m \to \infty$. With the Doob inequality we obtain that $\sup_{s \le t} |M_s^m| \overset{P}{\to} 0$, and fBm is the limit of a sequence of continuous processes with bounded variation.

It is not difficult to check that the solution to the linear equation

$$dS_t^m = S_{t-}^m \, dY_s^m$$

converges weakly in the Skorohod space to geometric fractional Brownian motion.

The driving process Y^m is a scaled counting process minus the expectation. It is well known that such models are complete and arbitrage-free. Hence we have a complete and arbitrage-free approximation to fractional Black–Scholes model. See [42] for more details.

7.7. *Remark.* If one computes the hedging price and the hedging strategy for the European call $(S_T^m - K)^+$ in the prelimit sequence, and lets $m \to \infty$ one gets in the limit the stop-loss-start-gain hedging given in Example 5.13.

Microeconomic approximation. So far there has been few economic justifications to use fractional models in option-pricing. E.g the LRD of the stock-price, measured by the Hurst index H, is usually given as an econometric fact (and even that is questionable). One attempt to build a microeconomic foundation for fractional models was that of Bayraktar et al. [5]. They showed how the fBS model can arise as a large time-scale many-agent limit when there are inert agents, i.e. investors who change their portfolios infrequently, and the log-price is given by the market imbalance. We will briefly explain their framework and their main result here.

Consider n agents. Each agent k has a *trading mood* $x^k = (x_t^k)_{t \in [0, \infty)}$ that takes values in a finite state-space $E \subset \mathbb{R}$ containing zero: $x_t^k > 0$ means buying, $x_t^k < 0$ means selling, and $x_t^k = 0$ means inactivity at time t. The agents are homogeneous and independent. The trading mood x^k is a semi-Markov process defined as

$$x_t^k = \sum_{m=0}^{\infty} \xi_m^k \mathbf{1}_{\left[\tau_m^k, \tau_{m+1}^k\right)}(t),$$

where the E-valued random variables ξ_m^k and the stopping times τ_m^k satisfy

$$\mathbf{P}\left[\xi_{m+1}^k = j, \tau_{m+1}^k - \tau_m^k \le t \,\middle|\, \xi_1^k, \ldots, \xi_m^k, \tau_1^k, \ldots, \tau_m^k\right]$$

$$= \mathbf{P}\left[\xi_{m+1}^k = j, \tau_{m+1}^k - \tau_m^k \le t \,\middle|\, \xi_m^k\right]$$

$$= Q(\xi_m^k, j, t).$$

So, $(\xi_m^k)_{m\in\mathbb{N}}$ is a homogeneous Markov chain on E with transition probabilities $p_{ij} = \lim_{t\to\infty} Q(i, j, t)$. It is assumed that $p_{ij} > 0$ for all $i \ne j$ so that (p_{ij}) admits a unique stationary measure $\mathbf{P}^*$. On the sojourn times $\tau_{m+1}^k - \tau_m^k$ given ξ_m^k it is assumed that:

(i) The average sojourn times are finite.

(ii) The sojourn time at the inactive state is heavy-tailed, i.e. there exist a constant $\alpha \in (1, 2)$ and a locally bounded slowly varying at infinity function L such that

$$\mathbf{P}\left[\tau_{m+1}^k - \tau_m^k \ge t \,\middle|\, \xi_m^k = 0\right] \sim t^{-\alpha} L(t).$$

(L is slowly varying at infinity if, for all $x > 0$, $L(xt)/L(t) \to 1$, when $t \to \infty$).

(iii) The sojourn times at the active states $i \ne 0$ are lighter-tailed than the sojourn time at the inactive state:

$$\lim_{t\to\infty} \frac{\mathbf{P}[\tau_{m+1}^k - \tau_m^k \ge t | \xi_m^k = i]}{t^{-(\alpha+1)} L(t)} = 0.$$

(iv) The distribution of the sojourn times have continuous and bounded densities with respect to the Lebesgue measure.

An agent-independent process $\Psi = (\Psi_t)_{t\in[0,\infty)}$ describes the sizes of typical trades: Agent k accumulates the asset S at the rate $\Psi_t x_t^k$ at time t. The process Ψ is assumed to be a continuous semimartingale with Doob–Meyer decomposition $\Psi = M + A$ such that $\mathbf{E}[\langle M \rangle_T] < \infty$ and $\mathbf{E}[\mathscr{V}(A)] < \infty$, and Ψ and the x^k's are independent. As before, $\mathscr{V}(A)$ denotes the total variation of A on $[0, T]$.

The log-price X^n for the asset with n agents is assumed to be given by the *market imbalance*:

$$X_t^n = X_0 + \sum_{k=1}^n \int_0^t \Psi_s x_s^k \, \mathrm{d}s.$$

The *aggregate order rate* is

$$Y_t^{\varepsilon,n} = \sum_{k=1}^n \Psi_t x_{t/\varepsilon}^k.$$

Let $\mu \ne 0$ be the expected trading mood under the stationary measure $\mathbf{P}^*$, and define the centered aggregate order process

$$X_t^{\varepsilon,n} = \int_0^t Y_s^{\varepsilon,n} \, \mathrm{d}s - \mu n \int_0^t \Psi_s \, \mathrm{d}s.$$

Then, the main result [5, Theorem 2.1] states that in the limit the centered log-prices are given by a stochastic integral with respect to a fBm:

7.8. Theorem. *There exists a constant $c > 0$ such that*

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{\varepsilon^{1-H} \sqrt{nL(1/\varepsilon)}} X^{\varepsilon,n} = c \int_0^{\cdot} \Psi_t \, dB_t,$$

where B is a fBm with Hurst index $H = (3 - \alpha)/2 > 1/2$. The limits are weak limits in the Skorohod space $D([0,T])$.

7.9. Remark. Assume that $\Psi \equiv 1$, i.e. the trades, and consequently the log-prices, are completely determined by the agents' intrinsic trading moods. Then

$$X^{\varepsilon,n} = \varepsilon X_{t/\varepsilon}^n - \mu n t.$$

(i)   The limit in Theorem 7.8 is the fBS model.
(ii) Bayraktar et al. [5] also considered a model where there are both active and inert investors (active investors have light-tailed sojourn times at the inactive state 0). Then they get, in the limit, the mfBS model.

8. CONCLUSIONS

We have given some recent results on the arbitrage and hedging in some fractional pricing models. If one wants to understand the pricing of options in the fBS model, then it is not clear to what extent the hedging capital given in (5.12) can be interpreted as the price of the option. On the other hand, these exact hedging results may have some value, if one studies the hedging problem in the presence of transaction costs. The mixed Brownian-fractional Brownian pricing model has less arbitrage possibilities, but it is possible to model the dependency of the log-returns in this model family. One can also modify this model to include more 'stylized' properties of log-returns, but the hedging prices will be the same as without these 'stylized' features.

The mixed model seems to be a good candidate to include several of the observed 'stylized' facts of log-returns in the modelling of stock prices. Hence it is reasonable to study how the properties of the standard gBS model change in the mixed model. We have shown in [9] that the hedging is the same in all models having the same structural quadratic variation as a functional of the stock price path. For example, recently Bratyk and Mishura have considered quantile hedging problems in mixed models; see [12] for more details.

Open problems. We finish by giving some open problems related to the present survey.

Are fractional and mixed models free of simple arbitrage?

What kind of random variables have a Riemann-Stieltjes integral representations in the fBS model?

Can one verify statistically that option prices depend only on the quadratic variation of the underlying stock prices?

What is the best way to estimate quadratic variation?

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