Dividend problem with Parisian delay for a spectrally negative Lévy risk process

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Abstract. In this paper we consider dividend problem for an insurance company whose risk evolves as a spectrally negative Lévy process (in the absence of dividend payments) when Parisian delay is applied. The objective function is given by the cumulative discounted dividends received until the moment of ruin when so-called barrier strategy is applied. Additionally we will consider two possibilities of delay. In the first scenario ruin happens when the surplus process stays below zero longer than fixed amount of time $\zeta > 0$. In the second case there is a time lag d between decision of paying dividends and implementation.

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1 Introduction

In risk theory we usually consider classical Cramér-Lundberg risk process:

$$X_t = x + pt - S_t = \sum_{i=1}^{N_t} U_i, \tag{1}$$

where $x > 0$ denotes an initial reserve. We assume that $U_i, (i = 1, 2, ...)$ are i.i.d distributed claims with the distribution function F. The arrival process is a homogeneous Poisson process N_t with intensity λ. The premium income is modeled by a constant premium density p and the net profit condition is then $\lambda\chi/p < 1$, where $\mathbb{E}(U_1) = \chi < \infty$. Lately there has been considered more general setting of a spectrally negative Lévy process. That is, $X = \{X_t\}_{t\geq 0}$ is a process with stationary and independent increments with only negative jumps. We will assume that process starts from $X_0 = x$ and later we will use convention $\mathbb{P}(\cdot|X_0 = x) = \mathbb{P}_x(\cdot)$ and $\mathbb{P}_0 = \mathbb{P}$. Such process takes into account not

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only large claims compensated by a steady income at rate $p > 0$, but also small perturbations coming from the Gaussian component and possibly additionally (when $\Pi_X(-\infty, 0) = \infty$ for the jump measure Π_X of X) compensated countable infinite number of small claims arriving over each finite time horizon. Working under this class of models, it became apparent that, despite of the diversity of possible probabilistic behaviors it allows, typically all results may be elegantly expressed in a unifying manner via the c-harmonic scale function $W^{(c)}(x)$ defined via its Laplace transform. This paper further illustrates this aspect, by unveiling the way the scale functions intervenes in a quite complicated control problem.

The classic research of the scandinavian school had focused on determining the "ruin probability" of the process (1) ever becoming negative, under the assumption that X has positive profits. Since however in this case the surplus has the unrealistic property that it converges to infinity with probability one, De Finetti [16] introduced the dividend barrier model, in which all surpluses above a given level are transferred (subject to a discount rate) to a beneficiary, and raised the question of optimizing this barrier. Formally, we consider the risk process controlled by the dividend policy π given by

$$U_t^\pi = X_t - L_t^\pi, \tag{2}$$

where $X_0 = x > 0$ is an initial reserves and L_t^π is an increasing adapted process representing the cumulative dividends paid out by the company up till time t. The optimization objective function is given by the average cumulative discounted dividends received until the moment of ruin:

$$v^\pi(x) = \mathbb{E}_x \int_0^{\sigma^\pi} e^{-qt} \mathrm{d}L_t^\pi, \tag{3}$$

where σ^π is a ruin probability that we specify later depending on the considered scenario and q is a discounting rate.

The objective of beneficiaries of an insurance company is to maximize $v^\pi(x)$ over all admissible strategies π:

$$v_*(x) = \sup_{\pi \in \Pi} v^\pi(x), \tag{4}$$

where Π is a set of all admissible strategies.

An intricate "bands strategy" solution was discovered by Gerber [17], [18], as well as the fact that for exponential claims, this reduces to a simple barrier strategy: "pay all you can above a fixed constant barrier a".

There has been a great deal of work on De Finetti's objective, usually concerning barrier strategies. Gerber and Shiu [19] and Jeanblanc and Shiryaev [23] consider the optimal dividend problem in a Brownian setting. Irbäck [22] and Zhou [31] study the constant barrier under the Cramér-Lundberg model (1). Hallin [21] formulated time dependent integro-differential equations describing the payoff associated to a $2n$ bands policy. The optimality of the "bands strategy" was recently established by Albrecher and Thonhauser [2] in the presence of fixed interest rates as well. For related work considering both excess-of-loss reinsurance and dividend distribution policies (e.g. in a diffusion setting), see Asmussen et al. [4], and for work including also a utility function, see Grandits et al. [20].

Dassios and Wu [13] for classical risk process (1) consider a Parisian type delay between a decision to pay a dividend and its implementation. The decision to pay is taken when the surplus reaches the fixed barrier a but it is implemented only when the surplus stays above barrier longer than fixed $d > 0$. The dividend is paid at the end of this period. This strategy we will denote by π_a. In this paper we generalize this result into the general spectrally negative Lévy risk process. In this case the ruin time equals: $\sigma^\pi = \sigma_a = \inf\{t \geq 0 : U_t^{\pi_a} < 0\}$. Since the ruin time is classical one we know that optimal strategy is a band strategy and we know also the necessary condition when an optimal strategy is the barrier strategy. We still believe that this new Parisian strategy (although not optimal within all strategies) could be very useful for the insurance companies giving possibility of natural delay between decision and it implementation.

In this paper we also consider Parisian delay at the ruin. We denote this strategy by π^a. That is ruin occurs if process U^π stays below zero for longer period than a fixed $\zeta > 0$. Formally, we define last moment before time t that process U_t^π was above zero:

$$\varsigma_t^U = \sup\{s < t : \mathbf{1}_{(U_s^\pi \geq 0)} \mathbf{1}_{(U_t^\pi < 0)} = 1\}. \tag{5}$$

Parisian time of ruin is given by

$$\sigma^\pi = \sigma^\zeta = \inf\{t > 0 : t - \varsigma_t^U \geq \zeta\}. \tag{6}$$

We first analyze the strategy π^a according to which the dividends are paid according to classical barrier dividend strategy transferring all surpluses above a given level a to dividends. We also prove the verification theorem for this type of ruin. In particular we find sufficient condition for the barrier strategy to be optimal.

In fact combination of both scenarios is also available. The name for this delay comes from Parisian option that prices are activated or canceled depending on the type of option if the underlying asset stays above or below the barrier long enough in a row (see [13] and [3]).

We believe that giving possibility of Parisian delay could describe better many situations of insurance company giving possibility of checking if indeed company's reserves increase and we can pay dividends (in the first scenario) or giving possibility for the insurance company to get solvency (in the second scenario).

The paper is organized as follows. In Section 2 we introduce basic notions and notations. In Section 3 we find the discounted cumulative dividends payments until Parisian ruin time. In Section 4 we prove the verification theorem and find necessary conditions for the barrier strategy to be optimal. In Section 5 we analyze the case when there is a time lag between decision to pay dividends and its implementation.

2 Preliminaries

We first review some fluctuation theory of spectrally negative Lévy processes and refer the reader for more background to Kyprianou [25], Sato [30] and Bertoin [8] and references therein.

In this paper we consider a spectrally negative Lévy process $X = \{X_t\}_{t\geq 0}$, that is a Lévy process with the Lévy measure ν satisfying $\nu(0,\infty) = 0$ (for simplicity we exclude the case of a compound Poisson process with negative jumps). Since jumps of a spectrally negative Lévy process X are all non-positive, moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geq 0$ and is given by $\mathbb{E}[\mathrm{e}^{\theta X_t}] = \mathrm{e}^{t\psi(\theta)}$ for some function $\psi(\theta)$ that is well defined at least on the positive half-axes where it is strictly convex with the property that $\lim_{\theta\to\infty}\psi(\theta) = +\infty$. Moreover, ψ is strictly increasing on $[\Phi(0),\infty)$, where $\Phi(0)$ is the largest root of $\psi(\theta) = 0$. We shall denote the right-inverse function of ψ by $\Phi : [0,\infty) \to [\Phi(0),\infty)$. We will consider also the dual process $\widehat{X}_t = -X_t$ which is a spectrally positive Lévy process with the jump measure $\widehat{\nu}\,(0,y) = \nu\,(-y,0)$. Characteristics of $\widehat{X}$ will be indicated by using a hat over the existing notation for characteristics of X.

For any θ for which $\psi(\theta) = \log \mathbb{E}[\exp\theta X_1]$ is finite we denote by $\mathbb{P}^\theta$ an exponential tilting of measure $\mathbb{P}$ with Radon-Nikodym derivative with respect to $\mathbb{P}$ given by

$$\left.\frac{\mathrm{d}\mathbb{P}^\theta}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\theta X_t - \psi(\theta)t\right), \tag{7}$$

where $\mathcal{F}_t$ is a right-continuous filtration natural filtration of X. Under the measure $\mathbb{P}^\theta$ the process X is still a spectrally negative Lévy process with characteristic function ψ_θ given by

$$\psi_\theta(s) = \psi(s+\theta) - \psi(\theta). \tag{8}$$

Throughout the paper we assume that the following (regularity) condition is satisfied:

$$\sigma > 0 \qquad \text{or} \qquad \int_{-1}^{0} x\nu(\mathrm{d}x) = \infty \qquad \text{or} \qquad \nu(\mathrm{d}x) << \mathrm{d}x, \tag{9}$$

where σ a Gaussian coefficient of X.

2.1 Scale functions

For $p \geq 0$, there exists a function $W^{(p)} : [0,\infty) \to [0,\infty)$, called the *p-scale function*, that is continuous and increasing with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(c)}(y)\mathrm{d}y = (\psi(\theta) - c)^{-1}, \qquad \theta > \Phi(c). \tag{10}$$

The domain of $W^{(c)}$ is extended to the entire real axis by setting $W^{(c)}(y) = 0$ for $y < 0$. We denote $W^{(0)}(x) = W(x)$. For later use we mention some properties of the function $W^{(c)}$ that have been obtained in literature. On $(0,\infty)$ the function

$y \mapsto W^{(c)}(y)$ is right- and left-differentiable and under the condition (9), it holds that $y \mapsto W^{(c)}(y)$ is continuously differentiable for $y > 0$. Moreover, if $\sigma > 0$ it holds that $W^{(c)} \in C^\infty(0, \infty)$ with $W^{(c)\prime}(0^+) = 2/\sigma^2$; if X has unbounded variation with $\sigma = 0$, it holds that $W^{(c)\prime}(0^+) = \infty$ (see [32, Lemma 4]).

The function $W^{(c)}$ plays a key role in the solution of the two-sided exit problem as shown by the following classical identity. Letting τ_a^+, τ_a^- be the entrance times of X into (a, ∞) and $(-\infty, -a)$ respectively,

$$\tau_a^+ = \inf\{t \geq 0 : X_t \geq a\}, \qquad \tau_a^- = \inf\{t \geq 0 : X_t < -a\}$$

it holds for $y \in [0, a]$ that

$$\mathbb{E}_y \left[e^{-\alpha \tau_a^+}, \tau_0^- > \tau_a^+ \right] = W^{(\alpha)}(y)/W^{(\alpha)}(a). \tag{11}$$

Closely related to $W^{(c)}$ is function $Z^{(c)}$ given by

$$Z^{(c)}(y) = 1 + q\overline{W}^{(c)}(y), \tag{12}$$

where $\overline{W}^{(c)}(y) = \int_0^y W^{(c)}(z)\mathrm{d}z$ is the anti-derivative of $W^{(c)}$. Moreover, the scale functions appear also in so-called two-sided downward exit problem:

$$E_x \left[e^{-\alpha \tau_0^-}, \tau_0^- < \tau_a^+ \right] = Z^{(\alpha)}(x) - Z^{(\alpha)}(a) \frac{W^{(\alpha)}(x)}{W^{(\alpha)}(a)}. \tag{13}$$

and in one-sided downward exit problem that for any β with $\psi(\beta) < \infty$, $\alpha \geq \psi(\beta) \vee 0$ and $x \geq 0$ gives:

$$\mathbb{E}_x \left[e^{-\alpha \tau_0^- + \beta X_{\tau_0^-}}, \tau_0^- < \infty \right] = e^{\beta x} \left(Z_\beta^{(u)}(x) - \frac{u}{\Phi(u)} W_\beta^{(u)}(x) \right), \tag{14}$$

where $W_\beta^{(u)}$ and $Z_\beta^{(u)}$ are scale functions with respect to the measure P^β, $u = \alpha - \psi(\beta)$ and $u/\Phi(u)$ is understood in the limiting sense if $u = 0$. In fact for each $x \in \mathbb{R}$, $W^{(c)}(x)$ is analytically extendable, as a function in c, to the whole complex plane; and hence the same is true of $Z^{(c)}(x)$. In which case arguing again by analytic extension one may weaken the requirement that $\alpha \geq \psi(\beta) \vee 0$ to simply $\alpha \geq 0$.

The 'tilted' scale functions can be linked to non-tilted scale functions via the relation $\mathrm{e}^{\beta x} W_\beta^{(\alpha - \psi(\beta))}(x) = W^{(\alpha)}(x)$ from [6, Remark 4]. This relation implies that $\mathrm{e}^{\beta x}[W_\beta^{(c)\prime}(x) + \beta W_v^{(c)}(x)] = W^{(\alpha)\prime}(x)$ and

$$Z_\beta^{(u)}(x) = 1 + u \int_0^x \mathrm{e}^{-\beta z} W^{(\alpha)}(z)\mathrm{d}z.$$

It is also well-known the solution of one-sided exit problem:

$$\mathbb{E}_x \left[e^{-q\tau_a^+}, \tau_a^+ < \infty \right] = e^{-\Phi(q)(a-x)}. \tag{15}$$

2.2 Reflection at the supremum

Write $\overline{X}$ for the running supremum of X, that is,

$$\overline{X}_t = \sup_{0\le s\le t}(X_s \vee 0),$$

where we use notations $y \vee 0 = \max\{y, 0\}$. By $Y = \overline{X} - X$ we denote the Lévy process X reflected at its past supremum $\overline{X}$. It was shown in [6] and [32] that the Laplace transform of entrance time

$$\tau_a = \inf\{t \ge 0 : Y_t > a\}$$

of the reflected process Y into (a, ∞) can be expressed in terms of the functions $Z^{(q)}$ and $W^{(q)}$ as follows:

$$\mathbb{E}_x[\mathrm{e}^{-\alpha\tau_a}, \tau_a < \infty] \quad = \quad Z^{(\alpha)}(a-x) - qW^{(\alpha)}(a-x)\frac{W^{(\alpha)}(a)}{W^{(\alpha)\prime}(a)}, \qquad (16)$$

where $x \in [0, a]$ and $\alpha \ge 0$. In fact, more general result holds true. We read off from [6, Theorem 1] that for $u = \alpha - \psi(\beta) \ge 0$, $x \in [0, a]$ it holds that

$$\mathbb{E}_x[\mathrm{e}^{-\alpha\tau_a - \beta(Y_{\tau_a}-a)}, \tau_a < \infty] \quad = \quad \mathrm{e}^{\beta x}\left[Z_\beta^{(u)}(x) - C_\beta W_\beta^{(u)}(x)\right], \qquad (17)$$

where $C_\beta = [uW_\beta^{(u)}(a) + \beta Z_\beta^{(u)}(a)]/[W_\beta^{(u)\prime}(a) + \beta W_\beta^{(u)}(a)]$.

The following result proved in [7, Propostion 1] concerns the value function associated to the dividend barrier policy π^a treated only up to regular (not Parisian ruin) time $\sigma^a = \inf\{t \ge 0 : U_t^{\pi^a} < 0\}$.

Proposition 1 *Let $a > 0$. For $x \in [0, a]$ it holds that*

$$\mathbb{E}_x\left[\int_0^{\tau_a} \mathrm{e}^{-qt}\mathrm{d}\overline{X}_t\right] = \frac{W^{(q)}(x)}{W^{(q)\prime}(a)}. \qquad (18)$$

2.3 Parisian ruin

One of most important characteristics in risk theory is a ruin probability defined by $\mathbb{P}(\tau_0^- < \infty)$ for $\tau_0^- = \inf\{t > 0 : X_t < 0\}$. Czarna and Palmowski [9] extended this notion to so-called Parisian ruin probability, that occurs if the process X stays below zero for period longer than a fixed $\zeta > 0$ (see also [10, 11] for the result concerning classical riks process). Formally, we define the excursion below zero:

$$\varsigma_t^X = \sup\{s < t : \mathbf{1}_{(X_s \ge 0)}\mathbf{1}_{(X_t<0)} = 1\}.$$

Parisian time of ruin is given by

$$\tau^\zeta = \inf\{t > 0 : t - \varsigma_t^X \ge \zeta\}$$

and Parisian ruin probability we define as:

$$\mathbb{P}(\tau^\zeta < \infty | X_0 = x) = \mathbb{P}_x(\tau^\zeta < \infty).$$

The following result summarize [9, Theorems 1 and 2].

Theorem 1 *Parisian ruin probability equals:*

$$\mathbb{P}_x(\tau^\zeta = \infty) = \mathbb{P}_x(\tau_0^- = \infty)\mathbb{P}(\tau^\zeta < \infty) + \left(1 - \mathbb{P}(\tau^\zeta < \infty)\right)\left(1 - \int_0^\infty \mathbb{P}(\tau_z^+ > \zeta)\mathbb{P}_x(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)\right) \tag{19}$$

and

$$\mathbb{P}_x(\tau_0^- = \infty) = \psi'(0+)W(x), \tag{20}$$

$$\int_0^\infty e^{-\theta s}\, ds \int_0^\infty \mathbb{P}(\tau_z^+ > s)\mathbb{P}_x(\tau_0^- < \infty, -X_{\tau_0^-} \in dz) \tag{21}$$

$$= \frac{1 - \psi'(0+)W(x)}{\theta} - \frac{1}{\theta} e^{\Phi(\theta)x}\left(Z_{\Phi(\theta)}^{(-\theta)}(x) + \frac{\theta}{\Phi(-\theta)} W_{\Phi(\theta)}^{(-\theta)}(x)\right). \tag{22}$$

Moreover,

(i) If X is a process of bounded variation, then

$$\mathbb{P}(\tau^\zeta < \infty) = \frac{\int_0^\infty \mathbb{P}(\tau_z^+ > \zeta)\mathbb{P}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)}{1 - \rho + \int_0^\infty \mathbb{P}(\tau_z^+ > \zeta)\mathbb{P}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)},$$

where

$$\int_0^\infty e^{-\theta s}\, ds \int_0^\infty \mathbb{P}(\tau_z^+ > s)\mathbb{P}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz) = \frac{1}{\theta p}\int_0^\infty \left(1 - e^{-\Phi(\theta) z}\right)\widehat{\nu}(z, \infty) dz. \tag{23}$$

(ii) If X is a process of unbounded variation, then

$$\mathbb{P}(\tau^\zeta < \infty) = \lim_{b \to \infty} \frac{q(b, \zeta) - q(b, \infty)}{q(b, \zeta)}, \tag{24}$$

where

$$\int_0^\infty \int_0^\infty q(s, t)\, dt\, ds = \frac{m(\omega)\Phi(\omega)(\beta - \omega)}{\beta\omega^2(\Phi(\beta) - \Phi(\omega))} \tag{25}$$

and we assume that there exists function $n(\epsilon)$ such that the limit

$$m(\omega) = \lim_{\epsilon \downarrow 0} \frac{P(-\underline{X}_{e_\omega} \leq \epsilon)}{n(\epsilon)} \tag{26}$$

is well-defined and finite.

3 Parisian delay at ruin

In section we will consider Parisian ruin time (6) and dividends paid according to barrier strategy that correspond to reducing the risk process U to the level a if $x > a$, by paying out the amount $(x-a)^+$, and subsequently paying out the minimal amount of dividends to keep the risk process below the level a. It is well known (see [7]) that for $0 < x \leq a$ the corresponding controlled risk process, say U^{π^a} under $\mathbb{P}_x$ is equal in law to the process $a - Y = \{a - Y_t : t \geq 0\}$ under $\mathbb{P}_x$ where

$$Y_t = (a \vee \overline{X}_t) - X_t$$

and $\overline{X}_t = \sup_{s \leq t} X_s$ is running supremum of risk process X. Moreover,

$$v_a(x) = v^{\pi^a}(x) = \mathbb{E}_x \left(\int_0^{\tau^\zeta} e^{-qt} dL_t \right),$$

where $L_t = a \vee \overline{X}_t - a$.

Note that for $x \leq a$,

$$v_a(x) = \mathbb{E}_x \left[e^{-q\tau_a^+}, \tau_a^+ < \tau^\zeta \right] v_a(a) \tag{27}$$

and

$$v_a(x) = x - a + v_a(a) \qquad \text{for } x > a. \tag{28}$$

Assume that $X \to \infty$ a.s. Then by Markov property and fact that X jumps only downwards we derive:

$$\mathbb{P}_x(\tau^\zeta = \infty) = \mathbb{P}_x(\tau_a^+ < \tau^\zeta)\mathbb{P}_a(\tau^\zeta = \infty). \tag{29}$$

Hence

$$\mathbb{P}_x(\tau_a^+ < \tau^\zeta) = \frac{\mathbb{P}_x(\tau^\zeta = \infty)}{\mathbb{P}_a(\tau^\zeta = \infty)}.$$

Using change of measure (7) with $\theta = \Phi(q)$, Optional Stopping Theorem and fact that on $\mathbb{P}^{\Phi(q)}$ process X tends to infinity a.s. (since $\psi'_{\Phi(q)}(0+) = \psi'(\Phi(q)+) > 0$), we have

$$\mathbb{E}_x \left[e^{-q\tau_a^+}, \tau_a^+ < \tau^\zeta \right] = \frac{V^{(q)}(x)}{V^{(q)}(a)}, \tag{30}$$

where

$$V^{(q)}(x) = e^{\Phi(q)x} \mathbb{P}_x^{\Phi(q)}(\tau^\zeta = \infty). \tag{31}$$

The probability $\mathbb{P}_x^{\Phi(q)}(\tau^\zeta = \infty) = 1 - \mathbb{P}_x^{\Phi(q)}(\tau^\zeta < \infty)$ hence also function $V^{(q)}$ could be found using Theorem 1.

It also follows from Theorem 1 that under the condition (9) function $V^{(q)}(y)$ (similarly like $W^{(c)}(y)$) is continuously differentiable for $y \in \mathbb{R}$.

Moreover, for $n \in \mathbb{N}$, by (28),

$$\begin{aligned} v_a(a) &\geq \mathbb{E}_a\left[e^{-q\tau^+_{a+1/n}}, \tau^+_{a+1/n} < \tau^\zeta\right]\left(v_a(a+\frac{1}{n})\right) \\ &= \mathbb{E}_a\left[e^{-q\tau^+_{a+1/n}}, \tau^+_{a+1/n} < \tau^\zeta\right]\left(v_a(a)+\frac{1}{n}\right) \end{aligned}$$

and

$$\begin{aligned} v_a(a) &\leq \mathbb{E}_a\left[e^{-q\tau^+_{a+1/n}}, \tau^+_{a+1/n} < \tau^\zeta\right]\left(v_a(a)+\frac{1}{n}\right) \\ &\quad +\frac{1}{nq}\left(1-\mathbb{E}_a\left[e^{-q\tau^+_{a+1/n}}, \tau^+_{a+1/n} < \tau^\zeta\right]\right). \end{aligned}$$

Last increment in above equation is $\mathrm{o}(1/n)$ since by strictly positive drift X is regular for $(0,\infty)$. Hence,

$$v_a(a) = \frac{V^{(q)}(a)}{V^{(q)}\left(a+\frac{1}{n}\right)}\left(v_a(a)+\frac{1}{n}\right) + \mathrm{o}\left(\frac{1}{n}\right)$$

and then

$$v_a(a) = \frac{V^{(q)}(a)}{V^{(q)\prime}(a)}.$$

Thus from (27), (28) and (30) it follows that v_a is continuously differentiable for all $x \in \mathbb{R}$ and

$$v_a(x) \quad = \quad v_{\pi^a}(x) = \begin{cases} \frac{V^{(q)}(x)}{V^{(q)\prime}(a)}, & x \leq a, \\ \\ x - a + \frac{V^{(q)}(a)}{V^{(q)\prime}(a)}, & x > a. \end{cases} \tag{32}$$

In particular,

$$v_a'(a) = 1. \tag{33}$$

Hence we get the following theorem.

Theorem 2 *The value function corresponding to the barrier strategy π^a is given by (32). The optimal barrier a^* is given by:*

$$a^* = \inf\{a > 0 : V^{(q)\prime}(a) \leq V^{(q)\prime}(x) \ \ \textit{for all } x\}. \tag{34}$$

In particular, if $V^{(q)} \in \mathcal{C}^2(\mathbb{R})$, then

$$V^{(q)\prime\prime}(a^*) = 0. \tag{35}$$

Remark 1 Note that $V^{(q)} \in \mathcal{C}^2(\mathbb{R})$ if $W^{(q)} \in \mathcal{C}^2(\mathbb{R})$. This is the case if e.g. the Gaussian component is present.

4 Verification Theorem

To prove the optimality of a particular strategy π across all admissible strategies Π for the dividend problem (4), where the ruin time σ^π is given by the Parisian ruin (6), we are led, by standard Markovian arguments, to consider the following variational inequalities:

$$\Gamma f(x) - qf(x) \leq 0, \qquad \text{if} \quad x \in \mathbb{R}, \tag{36}$$

$$f'(x) \geq 1, \qquad \text{if} \quad x \in \mathbb{R}, \tag{37}$$

for functions $f : \mathbb{R} \to \mathbb{R}$ in the domain of the extended generator Γ of the extended generator of the process X which acts on $C^2(0,\infty)$ functions f as

$$\Gamma f(x) = \frac{\sigma^2}{2} f''(x) + p_0 f'(x) + \int_{-\infty}^{0} \left[f(x+y) - f(x) + f'(x)y1_{\{|y|<1\}}\right] \nu(\mathrm{d}y), \tag{38}$$

where ν is the Lévy measure of X and σ^2 denotes the Gaussian coefficient and $p_0 = c + \int_{-1}^{0} y\nu(\mathrm{d}y)$ if the jump-part has bounded variation; see Sato [30, Ch. 6, Thm. 31.5]. In particular, if $E[|X|] < \infty$ and X has unbounded variation [bounded variation], a function f that is C^2 [C^1] on $[0,\infty)$ and that is ultimately linear lies in the domain of the extended generator.

Theorem 3 *Let $C \in (0,\infty]$ and suppose f is continuous and piecewise C^1 on $(-\infty, C)$ if X has bounded variation and that f is C^1 and piecewise C^2 on $(-\infty, C)$ if X has unbounded variation. Suppose that f satisfies (36) and (37). Then $f \geq \sup_{\pi\in\Pi_{\leq C}} v^\pi$, where $\Pi_{\leq C}$ is a set of all bounded strategies by C. In particular, if $C = \infty$, then $f \geq v_*$.*

Proof We will follow classical arguments. Let $\pi \in \Pi_{\leq C}$ be any admissible policy and denote by $L = L^\pi$ and $U = U^\pi$ the corresponding cumulative dividend process and risk process, respectively. By Sato [30, Ch. 6, Thm. 31.5] function $g(t,x,z) = e^{-qt} f(x)\mathbf{1}_{\{z\leq\zeta\}}$ is in a domain of extended generator of the three-dimensional Markov process $(t, U_t^\pi, t - \varsigma_t^U)$, with ς_t^U defined (5) and for simplicity notation we will assume that $\varsigma_t^U = t$ if $U_t^\pi \geq 0$. Note that finite number of discontinuities in f and hence also single discontinuity in $\mathbf{1}_{\{z\leq\zeta\}}$ are allowed here. Hence we are also allowed to apply Itô's lemma (e.g. [29, Thm. 32]) if X is of unbounded variation and the change of variable formula (e.g. [29, Thm. 31]) if X is of bounded variation:

$$\begin{aligned}\mathrm{e}^{-qt} f(U_t)\mathbf{1}_{\{t-\varsigma_t^U\leq\zeta\}} - f(U_0) \quad &= \quad J_f(t) - \int_0^t \mathrm{e}^{-qs} f'(U_{s^-})\mathrm{d}L_s^c \\ &\quad + \int_0^t \mathrm{e}^{-qs}(\Gamma f - qf)(U_{s-})\mathrm{d}s + M_t,\end{aligned} \tag{39}$$

where M_t is a local martingale with $M_0 = 0$, L^c is the pathwise continuous parts of L and for a function g the process J_g is given by

$$J_g(t) = \sum_{s\leq t} \mathrm{e}^{-qs} \left[g(A_s + B_s) - g(A_s)\right] \mathbf{1}_{\{B_s\neq 0\}}, \tag{40}$$

where $A_s = U_{s-} + \Delta X_s$ and $B_s = -\Delta L_s$ denotes the jump of $-L$ at time s. Let T_n be a localizing sequence of M. Applying Optional Stopping Theorem to the stopping times $T'_k = T_k \wedge \sigma^\pi$ and using Fatou's Theorem we derive:

$$\begin{aligned} f(x) \quad &\geq \quad \mathbb{E}_x \mathrm{e}^{-qT'_n} f(U_{T'_n}) \mathbf{1}_{\{t - \varsigma^U_{T'_n} \leq \zeta\}} - J_f(T'_n) + \mathbb{E}_x \int_0^{T'_n} \mathrm{e}^{-qs} f'(U_{s-}) \mathrm{d}L^c_s \\ &\quad -\mathbb{E}_x \int_0^{T'_n} \mathrm{e}^{-qs} (\Gamma f - qf)(U_{s-}) \mathrm{d}s. \end{aligned}$$

Invoking the variational inequalities $f'(x) \geq 1$ (hence $f(A_s + B_s) - f(A_s) \leq -\Delta L_s$ if $A_s > 0$) and $\Gamma f(x) - qf(x) \leq 0$ we have:

$$\begin{aligned} f(x) \quad &\geq \quad \mathbb{E}_x \mathrm{e}^{-qT'_n} f(U_{T'_n}) \mathbf{1}_{\{t - \varsigma^U_{T'_n} \leq \zeta\}} + \mathbb{E}_x \int_0^{T'_n} \mathrm{e}^{-qs} \mathrm{d}L_s \\ &\geq \quad \mathbb{E}_x \left[\mathrm{e}^{-q\sigma^\pi} w(U_{\sigma^\pi}); \sigma^\pi \leq T'_n \right] + \mathbb{E}_x \left[\int_0^{\sigma^\pi} \mathrm{e}^{-qs} \mathrm{d}L_s; \sigma^\pi \leq T'_n \right]. \end{aligned}$$

Letting $n \to \infty$ in conjunction with the monotone convergence theorem and using fact that $\mathbf{1}_{\{t - \varsigma^U_{\sigma^\pi} \leq \zeta\}} = 0$ complete the proof. □

Using verification theorem we find necessary conditions under which the optimal strategy takes the form of a barrier strategy.

Theorem 4 *Assume that $\sigma > 0$ or that X has bounded variation or, otherwise, suppose that $v_{a^*} \in C^2(0, \infty)$. If $q > 0$, then $a^* < \infty$ and the following hold true:*

(i) π^{a^*} is the optimal strategy in the set $\Pi_{\leq a^*}$ of all bounded strategies by a^* and $v_{a^*} = \sup_{\pi \in \Pi_{\leq a^*}} v^\pi$.

(ii) If $(\Gamma v_{a^*} - qv_{a^*})(x) \leq 0$ for $x > a^*$, the value function and optimal strategy of (4) is given by $v_* = v_{a^*}$, where the ruin time σ^π is given by the Parisian moment of ruin (6).

The proof of Theorem 4 is based on the verification theorem 3 and the following lemma.

Lemma 1 *(i) We have $a^* < \infty$.*

(ii) It holds that $(\Gamma v_{a^*} - qv_{a^*})(x) = 0$ for $x \leq a^*$.

(iii) For $x \leq a^*$

$$v'_{a^*}(x) \geq 1.$$

Proof Point (i) follows from the fact that $V^{(q)\prime}(y)$ is continuous and increasing. Indeed, note that by [24] we have $V^{(q)}(y) = e^{\Phi(q)y} \mathbb{P}_y(\tau^\zeta = \infty) \geq e^{\Phi(q)y} \mathbb{P}_y(\tau_0^- = \infty) = \frac{1}{\psi'(0+)} W^{(q)}(y)$ and $W^{(q)\prime}(y)$ tends to ∞ as $y \to \infty$. The proof of (ii) follows from (27) and the martingale property of

$$e^{-qt} \mathbb{E}_{X_t} \left[e^{-q\tau^+_{a^*}}, \tau^+_{a^*} < \tau^\zeta \right] = \mathbb{E} \left[\mathbb{E}_x \left[e^{-q\tau^+_{a^*}}, \tau^+_{a^*} < \tau^\zeta \right] | \mathcal{F}_t \right],$$

where $x \le a^*$. Point (iii) is a consequence of (33) and definition of a^*. □

Moreover, we can give other necessary condition for the barrier strategy to be optimal.

Corollary 1 *Suppose that*

$$V^{(q)\prime}(a) \le V^{(q)\prime}(b), \qquad \text{for all } a^* \le a \le b.$$

Then the barrier strategy at a^* is an optimal strategy.

Proof The proof is the same as the proof of [26, Theorem 2]. □

Corollary 2 *Suppose that, for $x > 0$, $\widehat{\nu}'(x)$ is monotone decreasing, then π_{a^*} is an optimal strategy of (4).*

Proof The proof is similar like the proof of [26, Theorem 3] using Theorem 1 and (31) together with known identity

$$\mathbb{P}_x^{\Phi(q)}(\tau_0^- < \infty) = \widehat{\kappa}_{\Phi(q)}(0,0)\widehat{U}_{\Phi(q)}(x,\infty), \tag{41}$$

where $\widehat{U}_{\Phi(q)}$ is a renewal function of the descending ladder height process $\widehat{H}_t$ under $\mathbb{P}^{\Phi(q)}$ and $\widehat{\kappa}_{\Phi(q)}(\alpha,\beta)$ is a Laplace exponent of bivariate descending ladder height process $(\widehat{L}_t^{-1}, \widehat{H}_t)$ under $\mathbb{P}^{\Phi(q)}$ with $\widehat{\kappa}_{\Phi(q)}(0,0) = \psi'(\Phi(q)) > 0$. In the proof one also has to use the following equality:

$$\begin{aligned}&\mathbb{P}_x^{\Phi(q)}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)\\ &\quad = \widehat{\kappa}_{\Phi(q)}(0,0)\int_0^x \widehat{U}_{\Phi(q)}(x-dy)\int_0^\infty e^{-\Phi(q)(z+v)}\widehat{\nu}(dz+v+y)\,dv,\end{aligned}$$

which follows from [25, (7.15), p. 195]. □

5 Parisian delay at the moment of dividend payments

In this section we analyze the case when we pay dividends only when surplus process stay above barrier a longer than a time lag $d > 0$. The dividends are paid at the end of that period and they are paid until regular ruin time $\sigma^{\pi_a} = \inf\{U_t^{\pi_a} < 0\}$. Then by (11) for $x \in [0,a]$,

$$v(x) = \mathbb{E}_x\left[e^{-q\tau_a^+}, \tau_a^+ < \tau_0^-\right]v(a) = \frac{W^{(q)}(x)}{W^{(q)}(a)}v(a); \tag{42}$$

and for $x \ge a$

$$\begin{aligned} v(x) &= \mathbb{E}_{x-a}\left[(X_d + v(a))e^{-q\tau_0^-}, \tau_0^- > d\right]\\ &\quad + v(a)\int_{[0,a)}\mathbb{E}_{a-y}\left[e^{-q\tau_a^+}, \tau_a^+ < \tau_0^-\right]\mathbb{E}_{x-a}\left[e^{-q\tau_0^-}, -X_{\tau_0^-} \in \mathrm{d}y, \tau_0^- \le d\right],\end{aligned} \tag{43}$$

where $\mathbb{E}_x\left[e^{-q\tau_a^+}, \tau_a^+ < \tau_0^-\right]$ is given in (11).

Double Laplace transform of $\mathbb{E}_z\left[e^{-q\tau_0^-}, -X_{\tau_0^-} \in \mathrm{d}y, \tau_0^- \le s\right]$ for $z \ge 0$ by (14) equals

$$\begin{aligned}
&\int_0^\infty \int_{[0,\infty)} e^{-\alpha s} e^{-\beta z} \mathbb{E}_z\left[e^{-q\tau_0^-}, -X_{\tau_0^-} \in \mathrm{d}y, \tau_0^- \le s\right] \mathrm{d}s\mathrm{d}y \\
&= \frac{1}{\alpha}\mathbb{E}_z\left[e^{-(\alpha+q)\tau_0^- + \beta X_{\tau_0^-}}, \tau_0^- < \infty\right] \\
&= \frac{1}{\alpha} e^{\beta z}\left(Z_\beta^{(u_q)}(z) - \frac{u_q}{\Phi(u_q)} W_\beta^{(u_q)}(z)\right) := H(\beta, z), \qquad (44)
\end{aligned}$$

where $u_q = \alpha + q - \psi(\beta)$. Moreover,

$$\begin{aligned}
&\int_0^\infty \int_{[0,\infty)} e^{-\alpha s} \mathbb{E}_z\left[X_s e^{-q\tau_0^-}, \tau_0^- > s\right] \mathrm{d}s = \frac{1}{\alpha}\left\{z\mathbb{E}_z\left[e^{-q\tau_0^-}\right] - \mathbb{E}_z\left[X_{\tau_0^-} e^{-(\alpha+q)\tau_0^-}\right]\right\} \\
&= \frac{1}{\alpha} z\left(Z^{(q)}(z) - \frac{q}{\Phi(q)} W^{(q)}(z)\right) - \frac{\partial}{\partial z} H(\beta, z)_{|\beta=0}. \qquad (45)
\end{aligned}$$

Further, the value $v(a)$ is determined by (43) if X has no Gaussian component ($\sigma = 0$) or by the smooth paste condition:

$$v'(a-) = v'(a+) \qquad (46)$$

otherwise.

Lemma 5.1 *If $\sigma > 0$ then (46) holds.*

Proof For $n \in \mathbb{N}$,

$$\begin{aligned}
v(a) \quad = \quad & v(a - 1/n)\mathbb{E}_a\left[e^{-q\tau_{a-1/n}^-}, \tau_{a-1/n}^- < \tau_{a+1/n}^+\right] \\
& + v(a + 1/n)\mathbb{E}_a\left[e^{-q\tau_{a+1/n}^+}, \tau_{a+1/n}^+ < \tau_{a-1/n}^-\right] + \mathrm{o}\left(\frac{1}{n}\right), \qquad (47)
\end{aligned}$$

where the last term is bounded above by $\frac{1}{n}\mathbb{P}(\tau_{1/n}^+ > d)$. Moreover, by (11),

$$\mathbb{E}_a\left[e^{-q\tau_{a+1/n}^+}, \tau_{a+1/n}^+ < \tau_{a-1/n}^-\right] = \frac{W^{(q)}(1/n)}{W^{(q)}(2/n)}$$

and by (13),

$$\mathbb{E}_a\left[e^{-q\tau_{a-1/n}^-}, \tau_{a-1/n}^- < \tau_{a+1/n}^+\right] = Z^{(q)}(1/n) - Z^{(q)}(2/n)\frac{W^{(q)}(1/n)}{W^{(q)}(2/n)}.$$

Multiplying both sides of (47) by two, subtracting $v(a-1/n)+v(a)$, dividing by $1/n$ gives:

$$\frac{v(a)-v(a-1/n)}{1/n} = \frac{v(a+1/n)-v(a)}{1/n} \tag{48}$$

$$+(v(a+1/n)-v(a-1/n))\left[\frac{2W^{(q)}(1/n)}{W^{(q)}(2/n)}-1\right] \tag{49}$$

$$+v(a-1/n)2q\frac{\int_0^{1/n} W^{(q)}(y)\,dy - \frac{W^{(q)}(1/n)}{W^{(q)}(2/n)}\int_0^{2/n} W^{(q)}(y)\,dy}{1/n} \tag{50}$$

where we use (12). Now, since $W^{(q)}(0)=0$, we have:

$$\lim_{n\to\infty}\frac{W^{(q)}(1/n)}{W^{(q)}(2/n)} = \frac{W^{(q)\prime}(0)}{2W^{(q)\prime}(0)} = \frac{1}{2}.$$

Hence increment (50) converges to $v(a)2q(W^{(q)}(0)-\frac{1}{4}W^{(q)}(0))=0$ as $n\to\infty$. Moreover,

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{2W^{(q)}(1/n)}{W^{(q)}(2/n)}-1\right] &= \lim_{n\to\infty} -\frac{1}{2}\frac{2/n}{W^{(q)}(2/n)}\frac{W^{(q)}(2/n)-2W^{(q)}(1/n)}{1/n^2}\\ &= -\frac{1}{2}\frac{1}{W^{(q)\prime}(0)}W^{(q)\prime\prime}(0)<\infty\end{aligned}$$

and $\lim_{n\to\infty}(v(a+1/n)-v(a-1/n))=0$ by the continuity of the value function. Hence increment (49) laos tends to 0 as $n\to\infty$. Taking limit as $n\to\infty$ in (48)-(50) completes the proof of (46). □

All results of this section could be summarized in the following theorem.

Theorem 5 *The value function $v(x)$ corresponding to the strategy π_a is given in (42) - (46).*

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