Transversality Conditions for Higher Order Infinite Horizon Discrete Time Optimization Problems

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Abstract

In this paper, we examine higher order difference problems: $\max_{\mathbf{c}} \sum_{t=0}^{\infty} U(\mathbf{c}(t), \mathbf{c}(t+1), \cdots, \mathbf{c}(t+N-1), t)$. Using the "squeezing" argument, we derive both Euler's condition and the transversality condition. In order to derive the two conditions, two needed assumptions are identified. A counterexample, in which the transversality condition is not satisfied without the two assumptions, is also presented.

Keywords: Keywords: Transversality condition; Dynamic optimization; Infinite horizon; Higher order difference problems

1. Introduction

In this paper, we consider the following reduced form model

$$(1)\quad \begin{cases} \max_{\mathbf{c}} \sum_{t=0}^{\infty} U\left(\mathbf{c}(t), \mathbf{c}(t+1), \cdots, \mathbf{c}(t+N-1), t\right) \\ \text{subject to } \mathbf{c}(0) = \mathbf{c}_0, \\ \forall t \geq 0,\ \left(\mathbf{c}(t), \mathbf{c}(t+1), \cdots, \mathbf{c}(t+N-1)\right) \in X(t) \subset \left(\mathbb{R}^n\right)^N, \end{cases}$$

where $N \in \mathbb{N}$, U is a real-valued Nth-order continuously differentiable function, and $\mathbf{c} \equiv (c_1, c_2, \cdots, c_n)$ is Nth-order continuously differentiable. [1] Notice that the objective functional of (1) can be infinite. [7] considers the continuous time first order differential problems: $v\left(\mathbf{x}\left(t\right), \dot{\mathbf{x}}\left(t\right), t\right)$. It generalizes the results of [5, 6, 10, 12]. So far, the most general form of the transversality conditions for continuous time version of problem (1) is presented in [11], which extends the first order case considered in [7](Theorem 3.2) to higher order cases. [7] was later extended to the discrete time stochastic case by [8]. In this paper, we aim to extend these results to deterministic higher order difference problems, using the "squeezing" argument.

The application of higher order difference problems can be widely found in economics. In particular, they appear in the discussion concerning the overlapping generations models. A satisfactory examination of the individuals' marriage and fertility decisions would necessitate the division of the representative agent's lifetime to multiple periods, instead of only two periods, young and old. However, as argued in [4], the properties of a model with two-period-lived agents cannot be readily extended to n-period-lived agents. To consider the n-period-lived agents case, transversality conditions for higher order difference problems would be imperative.

We first use the "squeezing" argument to derive both Euler's condition and the transversality condition for higher order difference problems, showing the argument needs two imperative assumptions. These two assumptions constitute the discrete time version of Assumption 1 and 2 in [11]. We then provide a counterexample, in which the transversality condition is not satisfied without the two assumptions. Because Assumption 1 and 2 are satisfied when a discounting factor is incorporated into the model, our transversality conditions also generalize the results obtained in the presence of discounting. For approaches on

[1] Normally, U is defined on $\left(\mathbb{R}^n\right)^N \times \mathbb{R}$. The domain of U is denoted by $X\left(t\right)$, in included in $\left(\mathbb{R}^n\right)^N$, for all t.

how to explicitly construct the optimal solutions to the undiscounted infinite horizon optimization problems, see [2, 3].

2. Derivation of the Transversality Conditions

Suppose that the optimal path to (1) exists and is given by $\mathbf{c}^*(t)$, optimal in the sense of an overtaking criterion to be defined below. We perturb it with Nth-order continuously differentiable curves $\mathbf{q}(t)$,

$$\mathbf{c}(t) = \mathbf{c}^*(t) + \varepsilon \cdot \mathbf{q}(t). \tag{2}$$

We define

$$\begin{aligned} V(\varepsilon, T) = \inf_{T \le T'} \sum_{t=0}^{T'} [U(&\mathbf{c}^*(t) + \varepsilon \cdot \mathbf{q}(t), \mathbf{c}^*(t+1) + \varepsilon \cdot \mathbf{q}(t+1), \\ &\cdots, \mathbf{c}^*(t+N-1) + \varepsilon \cdot \mathbf{q}(t+N-1), t) \\ &- U(\mathbf{c}^*(t), \mathbf{c}^*(t+1), \cdots, \mathbf{c}^*(t+N-1), t)]. \end{aligned} \tag{3}$$

In this paper, [1]'s notion of weak maximality is used as our optimality criterion. We assume that there exists an optimal path that satisfy the weak maximality criterion, which is defined as: an attainable path $(\mathbf{c}^*(t))$ is optimal if no other attainable path overtakes it[2]:

$$\begin{aligned} \lim_{T \to +\infty} \inf_{T \le T'} \sum_{t=0}^{T'} [U(&\mathbf{c}^*(t) + \varepsilon \cdot \mathbf{q}(t), \mathbf{c}^*(t+1) + \varepsilon \cdot \mathbf{q}(t+1), \\ &\cdots, \mathbf{c}^*(t+N-1) + \varepsilon \cdot \mathbf{q}(t+N-1), t) \\ &- U(\mathbf{c}^*(t), \mathbf{c}^*(t+1), \cdots, \mathbf{c}^*(t+N-1), t)] \le 0. \end{aligned} \tag{4}$$

Let $V(\varepsilon) = \lim_{T \to \infty} V(\varepsilon, T)$. Differentiating it with respect to ε, we have

$$\begin{aligned} &\lim_{\varepsilon \to {}^+0} \frac{V(\varepsilon)}{\varepsilon} \\ &= \lim_{\varepsilon \to {}^+0} \lim_{T \to \infty} \inf_{T \le T'} \sum_{t=0}^{T'} \frac{1}{\varepsilon} [U(\mathbf{c}^*(t) + \varepsilon \cdot \mathbf{q}(t), \mathbf{c}^*(t+1) + \varepsilon \cdot \mathbf{q}(t+1), \\ &\qquad \cdots, \mathbf{c}^*(t+N-1) + \varepsilon \cdot \mathbf{q}(t+N-1), t) \\ &\qquad - U(\mathbf{c}^*(t), \mathbf{c}^*(t+1), \cdots, \mathbf{c}^*(t+N-1), t)]. \end{aligned} \tag{5}$$

Let $\lim_{\varepsilon \to {}^+0} \frac{V(\varepsilon)}{\varepsilon} \equiv \Omega$. Generally, $\frac{d}{d\varepsilon} \lim_{T \to \infty} f(\varepsilon, T) = \lim_{T \to \infty} \frac{d}{d\varepsilon} f(\varepsilon, T)$ only when $\lim_{T \to \infty} \frac{d}{d\varepsilon} f(\varepsilon, T)$ converges uniformly for ε ([9]). We assume

[2] [1] shows that such a path exists once two assumptions are satisfied.

Assumption 1. Assume Ω converges uniformly for ε when $T \to \infty$
.

Assume Assumption 1, we can then restate (5) as

$$\Omega = \lim_{T\to\infty} \lim_{\varepsilon\to^{+}0} \inf_{T\leq T'} \frac{1}{\varepsilon} \times \sum_{t=0}^{T'} [U\left(\mathbf{c}^*(t) + \varepsilon\cdot\mathbf{q}(t), \cdots, \mathbf{c}^*(t+N-1) + \varepsilon\cdot\mathbf{q}(t+N-1), t\right) - U\left(\mathbf{c}^*(t), \cdots, \mathbf{c}^*(t+N-1), t\right)]. \tag{6}$$

We also assume

Assumption 2. We assume for any $T > 0$,

$$\inf_{T\leq T'} \sum_{t=0}^{T'} \frac{1}{\varepsilon} [U\left(\mathbf{c}^*(t) + \varepsilon\cdot\mathbf{q}(t), \cdots, \mathbf{c}^*(t+N-1) + \varepsilon\cdot\mathbf{q}(t+N-1), t\right) - U\left(\mathbf{c}^*(t), \cdots, \mathbf{c}^*(t+N-1), t\right)]$$

converges uniformly for ε.

As in [11], a precise interpretation of Assumption 2 can be given as follows: Let

$$A(T,\varepsilon) = \sum_{t=0}^{T'} \frac{1}{\varepsilon} [U\left(\mathbf{c}^*(t) + \varepsilon\cdot\mathbf{q}(t), \cdots, \mathbf{c}^*(t+N-1) + \varepsilon\cdot\mathbf{q}(t+N-1), t\right) - U\left(\mathbf{c}^*(t), \cdots, \mathbf{c}^*(t+N-1), t\right)].$$

Then there exists a sequence $A\left(T'_n, \varepsilon\right)$ for each $\varepsilon > 0$, so that $\lim_{n\to\infty} A\left(T'_n, \varepsilon\right) = \inf_{T\leq T'} A\left(T', \varepsilon\right)$, uniformly for ε, that is, the sequence is uniformly convergence for ε.

Assumptions 1 and 2 extend Assumption 3.1 in [7]. When Assumptions 1 and 2 are satisfied, then $\lim_{\varepsilon\to^{+}0}$ and $\inf_{T\leqslant T'}$ can be interchanged, and equality (6) can then restated as

$$\Omega = \lim_{T\to\infty} \inf_{T\leq T'} \lim_{\varepsilon\to^{+}0} \frac{1}{\varepsilon} \times \sum_{t=0}^{T'} [U\left(\mathbf{c}^*(t) + \varepsilon\cdot\mathbf{q}(t), \cdots, \mathbf{c}^*(t+N-1) + \varepsilon\cdot\mathbf{q}(t+N-1), t\right) - U\left(\mathbf{c}^*(t), \cdots, \mathbf{c}^*(t+N-1), t\right)]. \tag{7}$$

Because T' is finite uniformly for ε, if

$$\sum_{t=0}^{T'} \frac{1}{\varepsilon}[U(\mathbf{c}^*(t)+\varepsilon\cdot\mathbf{q}(t),\cdots,\mathbf{c}^*(t+N-1)+\varepsilon\cdot\mathbf{q}(t+N-1),t) \\ -U(\mathbf{c}^*(t),\cdots,\mathbf{c}^*(t+N-1)),t]$$

exists, (7) is then rewritten as

$$\begin{aligned}\Omega = \lim_{T\to\infty}\inf_{T\le T'}\sum_{t=0}^{T'}\frac{1}{\varepsilon} \\ \times[U(\mathbf{c}^*(t)+\varepsilon\cdot\mathbf{q}(t),\cdots,\mathbf{c}^*(t+N-1)+\varepsilon\cdot\mathbf{q}(t+N-1),t) \\ -U(\mathbf{c}^*(t),\cdots,\mathbf{c}^*(t+N-1),t)].\end{aligned} \tag{8}$$

From the differentiability of U, we have

$$\begin{aligned}&\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}[U(\mathbf{c}^*(t)+\varepsilon\cdot\mathbf{q}(t),\cdots,\mathbf{c}^*(t+N-1)+\varepsilon\cdot\mathbf{q}(t+N-1),t) \\ &\qquad -U(\mathbf{c}^*(t),\cdots,\mathbf{c}^*(t+N-1),t)] \\ &=\sum_{i=1}^{n}[\frac{\partial U(\mathbf{c}^*(t),\mathbf{c}^*(t+1),\cdots,\mathbf{c}^*(t+N-1),t)}{\partial c_i(t)}q_i(t) \\ &\qquad +\frac{\partial U(\mathbf{c}^*(t),\mathbf{c}^*(t+1),\cdots,\mathbf{c}^*(t+N-1),t)}{\partial c_i(t+1)}q_i(t+1) \\ &\qquad +\cdots+\frac{\partial U(\mathbf{c}^*(t),\mathbf{c}^*(t+1),\cdots,\mathbf{c}^*(t+N-1),t)}{\partial c_i(t+N-1)}q_i(t+N-1)].\end{aligned}$$

Hence,

$$\begin{aligned}\Omega = \lim_{T\to\infty}\inf_{T\le T'}\sum_{t=0}^{T'}\sum_{i=1}^{n}[&\frac{\partial U(\mathbf{c}^*(t),\mathbf{c}^*(t+1),\cdots,\mathbf{c}^*(t+N-1),t)}{\partial c_i(t)}q_i(t) \\ &+\frac{\partial U(\mathbf{c}^*(t),\mathbf{c}^*(t+1),\cdots,\mathbf{c}^*(t+N-1),t)}{\partial c_i(t+1)}q_i(t+1)+\cdots \\ &+\frac{\partial U(\mathbf{c}^*(t),\mathbf{c}^*(t+1),\cdots,\mathbf{c}^*(t+N-1),t)}{\partial c_i(t+N-1)}q_i(t+N-1)].\end{aligned}$$

We derive

$$\sum_{t=0}^{T'}\left(\sum_{i=1}^{n}\left(\frac{\partial U(t)}{\partial c_i(t)}q_i(t)+\cdots+\frac{\partial U(t)}{\partial c_i(t+N-1)}q_i(t+N-1)\right)\right)$$
$$=\sum_{i=1}^{n}\{\frac{\partial U(0)}{\partial c_i(0)}q_i(0)+\frac{\partial\left(U(0)+U(1)\right)}{\partial c_i(1)}q_i(1)+\cdots$$
$$+\frac{\partial\left(U(0)+\cdots+U(N-2)\right)}{\partial c_i(t)}q_i(N-2)$$
$$+\sum_{t=N-1}^{T'}\frac{\partial\left(U(t-N+1)+\cdots+U(t)\right)}{\partial c_i(t)}q_i(t)$$
$$+\left(\frac{\partial\left(U\left(T'-N+2\right)+\cdots+U\left(T'\right)\right)}{\partial c_i(T'+1)}\right)q_i\left(T'+1\right)+\cdots$$
$$+\frac{\partial U\left(T'\right)}{\partial c_i\left(T'+N-1\right)}q_i\left(T'+N-1\right)\}. \tag{9}$$

Hence, Euler's condition is

$$\frac{\partial U(0)}{\partial c_i(0)}=0,$$
$$\frac{\partial\left(U(0)+U(1)\right)}{\partial c_i(1)}=0,$$
$$\cdots\cdots,$$
$$\frac{\partial\left(U(0)+\cdots+U(N-2)\right)}{\partial c_i(t)}=0,$$
$$\frac{\partial\left(U(t-N+1)+\cdots+U(t)\right)}{\partial c_i(t)}=0,\ \text{for } N-1\leq t\leq T', \tag{10}$$

which extends the standard Euler's condition, and the transversality condition is given by

$$\lim_{T\to\infty}\inf_{T\leq T'}\sum_{i=1}^{n}[\frac{\partial\left(U\left(T'-N+2\right)+\cdots+U\left(T'\right)\right)}{\partial c_i\left(T'+1\right)}q_i\left(T'+1\right)$$
$$+\cdots+\frac{\partial\left(U\left(T'\right)\right)}{\partial c_i\left(T'+N-1\right)}q_i\left(T'+N-1\right)]\leq 0. \tag{11}$$

Note that when $\varepsilon \to {}^{-}0$, the argument is the same:

$$\lim_{T\to\infty}\sup_{T\le T'}\sum_{i=1}^{n}[\frac{\partial\left(U\left(T'-N+2\right)+\cdots+U\left(T'\right)\right)}{\partial c_i\left(T'+1\right)}q_i\left(T'+1\right)$$
$$+\cdots+\frac{\partial\left(U\left(T'\right)\right)}{\partial c_i\left(T'+N-1\right)}q_i\left(T'+N-1\right)]\ge 0. \tag{11'}$$

Next, we consider the linkage between our result and that in [7]. We fix $0<\bar{\alpha}<1$ and $\alpha:\mathbb{R}^+\to\mathbb{R}^+$, C^∞, $\alpha(0)=0$, $\cdots$, $\alpha^{(n-1)}(0)=0$, $\alpha(t)=\bar{\alpha}$, $t\ge 1$. We let $\varepsilon\to{}^{+}0$. Let $q(t)=\alpha c^*(t)$, then (11) is modified to

$$\lim_{T\to\infty}\inf_{T\le T'}\sum_{i=1}^{n}[\frac{\partial\left(U\left(T'-N+2\right)+\cdots+U\left(T'\right)\right)}{\partial c_i\left(T'+1\right)}\alpha c_i^*\left(T'+1\right)$$
$$+\cdots+\frac{\partial\left(U\left(T'\right)\right)}{\partial c_i\left(T'+N-1\right)}\alpha c_i^*\left(T'+N-1\right)]$$
$$=\bar{\alpha}\lim_{T\to\infty}\inf_{T\le T'}\sum_{i=1}^{n}[\frac{\partial\left(U\left(T'-N+2\right)+\cdots+U\left(T'\right)\right)}{\partial c_i\left(T'+1\right)}c_i^*\left(T'+1\right)$$
$$+\cdots+\frac{\partial\left(U\left(T'\right)\right)}{\partial c_i\left(T'+N-1\right)}c_i^*\left(T'+N-1\right)]$$
$$\le 0. \tag{12}$$

Because $\bar{\alpha}>0$, we then have

$$\lim_{T\to\infty}\inf_{T\le T'}\sum_{i=1}^{n}[\frac{\partial\left(U\left(T'-N+2\right)+\cdots+U\left(T'\right)\right)}{\partial c_i\left(T'+1\right)}c_i^*\left(T'+1\right)$$
$$+\cdots+\frac{\partial\left(U\left(T'\right)\right)}{\partial c_i\left(T'+N-1\right)}c_i^*\left(T'+N-1\right)]\le 0. \tag{13}$$

which is an extension of [7]'s transversality condition.

3. A Counterexample

We proceed to show that Assumption 1 and 2 are imperative in the sense that (11) becomes invalid if one of them is violated. We consider the following simple counterexample:

$$U\left(c(t),c(t+1),c(t+2),t\right)=\left(c(t)-\alpha\right)^2+\beta c(t+1)+\gamma c(t+2), \tag{14}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and the initial values $c(0) = c_0$, $c(1) = c_1$ are given. From (10), we see that Euler's condition is given by

$$\frac{\partial U(0)}{\partial c(0)} = 0,$$
$$\frac{\partial \left(U(0) + U(1)\right)}{\partial c(1)} = 0,$$
$$\frac{\partial \left(U(t-2) + U(t-1) + U(t)\right)}{\partial c(t)} = 0, 2 \leq t \leq T', \tag{15}$$

which implies

$$t = 0, \quad 2\left(c(0) - \alpha\right) = 0, \tag{15$'$}$$
$$t = 1, \quad 2\left(c(1) - \alpha\right) + \beta = 0, \tag{15$''$}$$
$$t = 2, \quad 2\left(c(2) - \alpha\right) + \beta + \gamma = 0, \tag{15$'''$}$$
$$t = 3, \quad 2\left(c(3) - \alpha\right) + \beta + \gamma = 0, \tag{15$''''$}$$
$$\cdots\cdots,$$
$$t = T', \quad 2\left(c\left(T'\right) - \alpha\right) + \beta + \gamma = 0. \tag{15$'''''$}$$

Thus, we have $c(2) = c(3) = \cdots = c\left(T'\right) = \alpha - \dfrac{\beta + \gamma}{2}$.

Choosing a p so that $p(0) = 0$ and $p(t) > 0$, there exists $T_0 > 0$, $p(t)$ is a constant $p_\infty > 0$ when $t \geq T_0$.

From (14), we see that

$$\frac{\partial \left(U\left(T' - 1\right) + U\left(T'\right)\right)}{\partial c(t+1)} q\left(T' + 1\right) = (\gamma + \beta) q\left(T' + 1\right) \leq 0, \tag{16}$$
$$\frac{\partial \left(U\left(T'\right)\right)}{\partial c(t+2)} q\left(T' + 2\right) = \gamma q\left(T' + 2\right) \leq 0. \tag{17}$$

Hence, we have arrived at a contradiction to (11).

Next, we show that Assumption 1 is violated, which causes this contradiction. We consider $U\left(c^*(t) + \varepsilon p(t), c^*(t+1) + \varepsilon p(t+1), c^*(t+2) + \varepsilon p(t+2)\right)$ $-U\left(c^*(t), c^*(t+1), c^*(t+2)\right)$. Substituting $c^*(t) = \alpha - \dfrac{\beta + \gamma}{2}$ into it, we have

$$U\left(c^*(t)+\varepsilon p(t), c^*(t+1)+\varepsilon p(t+1), c^*(t+2)+\varepsilon p(t+2)\right)$$
$$-U\left(c^*(t), c^*(t+1), c^*(t+2)\right)$$
$$=\left(c^*(t)+\varepsilon p(t)-\alpha\right)^2+\beta\left(c^*(t+1)+\varepsilon p(t+1)\right)+\gamma\left(c^*(t+2)+\varepsilon p(t+2)\right)$$
$$-\left(\left(c^*(t)-\alpha\right)^2+\beta c^*(t+1)+\gamma c^*(t+2)\right)$$

(18)

$$=\left(\varepsilon p(t)-\left(\frac{\beta+\gamma}{2}\right)\right)^2+\varepsilon\left(\beta p(t+1)+\gamma p(t+2)\right)-\left(\frac{\beta+\gamma}{2}\right)^2.$$

Hence,

$$\inf_{T\leq T'}\sum_{t=0}^{T'}\frac{\left(\varepsilon p(t)-\left(\frac{\beta+\gamma}{2}\right)\right)^2+\varepsilon\beta p(t+1)+\varepsilon\gamma p(t+2)-\left(\frac{\beta+\gamma}{2}\right)^2}{\varepsilon}$$
$$=\inf_{T\leq T'}\sum_{t=0}^{T'}\left(\varepsilon p(t)^2-(\beta+\gamma)p(t)+\beta p(t+1)+\gamma p(t+2)\right)$$
$$=\inf_{T\leq T'}\left(\varepsilon\sum_{t=0}^{T'}p(t)^2+\beta p\left(T'+1\right)+\gamma p\left(T'+2\right)\right)$$
$$=\inf_{T\leq T'}\left(\varepsilon\sum_{t=0}^{T'}\left(p(t)^2\right)\right)+\beta p_\infty+\gamma p_\infty$$

(19)

$$=\varepsilon\sum_{t=0}^{T}\left(p(t)^2\right)+\beta p_\infty+\gamma p_\infty.$$

Ω is the limit of (18) when $T\to\infty$, $\varepsilon\to 0$. However, because $\lim_{\varepsilon\to 0}\lim_{T\to\infty}\left(\varepsilon\sum_{t=0}^{T}\left(p(t)^2\right)+\beta p_\infty+\gamma p_\infty\right)=\infty$, whereas $\lim_{T\to\infty}\lim_{\varepsilon\to 0}\left(\varepsilon\sum_{t=0}^{T}\left(p(t)^2\right)+\beta p_\infty+\gamma p_\infty\right)=\beta p_\infty+\gamma p_\infty$, we see that Ω does not converge uniformly for ε when $T\to\infty$. Hence, Assumption 1 is violated and (11) is also not satisfied.

4. Conclusion

This paper gives the two assumptions that would be imperative when examining infinite horizon discrete time optimization problems in which the objective functions are unbounded. Our results generalizes the results of [5, 6, 7, 10, 12]($N = 1$) to higher order difference problems. Specifically, when $N = 1$, our transversality condition is the discrete time version of that of [7](Theorem 3.2). Moreover, our Assumption 1 and 2 also constitute the discrete time version of the Assumptions in [11]. As in [11], our Assumption 1 and 2 obviously hold when a discounting factor is incorporated into the model. In this sense, this paper also extends the transversality conditions examined in the presence of discounting.

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