Small time Edgeworth-type expansions for weakly convergent nonhomogeneous Markov chains.*

Valentin KONAKOV
Central Economics Mathematical Institute, Academy of Sciences
Nahimovskii av. 47, 117418 Moscow, Russia
E mail: kv24@mail.ru
Enno MAMMEN
Department of Economics, University of Mannheim
L 7, 3-5, 68131 Mannheim, Germany
E mail: emammen@rumms.uni-mannheim.de

February 1, 2008

Abstract

We consider triangular arrays of Markov chains that converge weakly to a diffusion process. Second order Edgeworth type expansions for transition densities are proved. The paper differs from recent results in two respects. We allow nonhomogeneous diffusion limits and we treat transition densities with time lag converging to zero. Small time asymptotics are motivated by statistical applications and by resulting approximations for the joint density of diffusion values at an increasing grid of points.

Keywords and phrases: Markov chains, diffusion processes, transition densities, Edgeworth expansions
Mathematics Subject Classifications: primary 62G07, secondary 60G60
Short title: Edgeworth-type expansions for Markov chains

*This research was supported by grant 436RUS113/467/81-2 from the Deutsche Forschungsgemeinschaft and by grants 05-01-04004 and 04-01-00700 from the Russian Foundation of Fundamental Researches. The first author worked on the paper during a visit at the Laboratory of Probability Theory and Random Models of the University Paris VI in 2006. He is grateful for the hospitality during his stay. This is the long version of a paper that has been submitted to Probability Theory and Related Fields for publication. We would like to thank Stephane Menozzi, two referees and the associate editor for helpful comments.

1 Introduction.

Recently, there was some activity on Edgeworth-type expansions for dependent data. In most approaches higher order expansions have been derived by application of classical Edgeworth expansions for independent data. The approaches differ in their main idea how the dependence structure can be reduced to the case of independent data. For sums of independent random variables and for functionals of such sums the theory of Edgeworth expansions is classical and well understood in a very general setting (see Bhattcharya and Rao (1976) and Götze (1989)). For models with dependent variables three approaches have been developed where the expansion is derived from models with sums of independent random variables. In the first approach mixing properties are used to approximate the Markov chain by a sum of independent random variables and it is shown that their Edgeworth expansion carries over to the Markov chain up to a certain accuracy. The mixing approach was first used by Götze and Hipp (1983) and it was further applied to continuous time processes in Kusuoka and Yoshida (2000) and Yoshida (2004). Under appropriate conditions Markov chains can be splitted at regeneration times into a sequence of i.i.d. variables. This fact has been used in Bolthausen (1980, 1982) to get Berry-Esseen bounds for Markov chains. For the statement of Edgeworth expansions the regenerative method has been used in Malinovskii (1987), Jensen (1989), Bertail and Clemencon (2004) and Fukasawa (2006a). The higher order Edgeworth expansions have been used to show higher order accuracy of different bootstrap schemes, see Mykland (1992),Bertail and Clemencon (2006) and Fukasawa (2006b).

Both approaches, the mixing method and the regenerative method only have been used for Markov chains with a Gaussian limit. In this paper we study Markov chains that converge weakly to a diffusion limit. For the treatment of this case we make use of the parametrix method. In this approach the transition density is represented as a nested sum of functionals of densities of sums of independent variables. Plugging Edgeworth expansions into this representation will result in an expansion for the transition density. Thus as in the mixing method and in the regenerative method the expansion is reduced to models with sums of independent random variables.

The parametrix method permits to obtain tractable representations of transition densities of diffusions and of Markov chains. For diffusions the parametrix expansion is based on Gaussian densities, see Lemma 1 below, and standard references for the parametrix method are the books of Friedman (1964) and Ladyzenskaja, Solonnikov and Ural'ceva (1968) on parabolic PDE [see also McKean and Singer (1967)]. For a short exposition of the parametrix method, see Section 3 and Konakov and Mammen (2000). Similar representations hold for discrete time Markov chains $X_{k,h}$, see Lemma 3 below. The parametrix method for Markov chains was developed in Konakov and Mammen (2000) and it is exposed in Section 3.2. In Konakov and Mammen (2002) the approach was used to state Edgeworth-type expansions for Euler schemes for stochastic differential equations. Related treatments of Euler schemes can be found in Bally and Talay (1996 a,b), Protter and Talay (1997), Jacod and Protter (1998), Jacod (2004), Jacod, Kurtz, Meleard and Protter (2005) and Guyon (2006).

In this paper we study triangular arrays of Markov chains $X_{k,h}$ $(k \geq 0)$ that converge weakly to a diffusion process Y_s $(s \geq 0)$ for $n \longrightarrow \infty$. We consider the Markov chains for the time interval $(0 \leq k \leq n)$. The corresponding time interval of the diffusion is $(0 \leq s \leq T)$. The term $h = T/n$ denotes the discretization step. We allow that T depends on n. In particular, we consider the case that $T \rightarrow 0$ for $n \rightarrow \infty$. Furthermore, we allow nonhomogeneous diffusion limits.

Weak convergence of the distribution of scaled discrete time Markov processes to diffusions has been extensively studied in the literature (see Skorohod (1965) and Stroock and Varadhan (1979)). Local limit theorems for Markov chains were given in Konakov and Molchanov (1984) and Konakov and Mammen (2000, 2002). In Konakov and Mammen (2000) it was shown that the transition density of a Markov chain converges with rate $O(n^{-1/2})$ to the transition density in the diffusion model. For the proof there an analytical approach was chosen that made essential use of the parametrix method.

The main result of this paper will give Edgeworth type expansions for the transition densities of the Markov chains $X_{k,h}$ $(0 \leq k \leq n)$. The first order term of the expansion is the transition density of the diffusion process Y_s $(0 \le s \le T)$. The order of the expansion is $o(h^{-1-\delta})$ with $\delta > 0$. Related results were shown in Konakov and Mammen (2005). The work of this paper generalizes the results in Konakov and Mammen (2005) in two directions. The time horizon T is allowed to converge to 0 and also cases are treated with nonhomogeneous diffusion limit. Small time asymptotics is done for two reasons. First of all it allows approximations for the joint density of values of the Markov chain at an increasing grid of points. Secondly, it is motivated by statistical applications. In statistics, diffusion models are used as an approximation to the truth. They can be motivated by a high frequency Markov chain that is assumed to run in the background on a very fine time grid and is only observed on a coarser grid. If the number of time steps between two observed values of the process converges to infinity this allows diffusion approximations (under appropriate conditions). This asymptotics reflects a set up occurring in the high frequency statistical analysis for financial data where diffusion approximations are used only for coarser time scales. For the finest scale discrete pattern in the price processes become transparent and do not allow diffusion approximations. The statistical implications of our result will be discussed elsewhere. The mathematical treatment of nonhomogeneous diffusion limits with time horizon T going to zero contributes some additional qualitatively new problems. In this case some additional terms appear that explode for $T \to 0$ and for this reason these terms need a qualitatively different treatment as in the case with fixed T. The nonhomogeneity adds an additional term in the Edgeworth expansion. See also below for more details.

The paper is organized as follows. In the next section we will present our model for the Markov chain and state our main result that gives an Edgeworth-type expansion for Markov chains. Connections with previously known results are also discussed in Section 2. In Section 3.1 we will give a short introduction into the parametrix method for diffusions. In Section 3.2 we will recall the parametrix approach developed in Konakov and Mammen (2000) for Markov chains. Technical discussions, auxiliary results and proofs are given in Sections 4 and 5.

2 The main result: an Edgeworth-type expansion for Markov chains converging to diffusions.

We consider a family of Markov processes in $\mathbb{R}^d$ that have the following form

$$X_{k+1,h} = X_{k,h} + m\left(kh, X_{k,h}\right)h + \sqrt{h}\xi_{k+1,h},\ X_{0,h} = x \in \mathbb{R}^d,\ k = 0, ..., n-1. \tag{1}$$

The innovation sequence $\left(\xi_{i,h}\right)_{i=1,\dots,n}$ is assumed to satisfy the Markov assumption: the conditional distribution of $\xi_{k+1,h}$ given the past $X_{k,h} = x_k, ..., X_{0,h} = x_0$ depends only on the last value $X_{k,h} = x_k$ and has a conditional density $q\left(kh, x_k, \cdot\right)$. The conditional covariance matrix corresponding to this density is denoted by $\sigma(kh, x_k)$ and the conditional $\nu - th$ cumulant by $\chi_\nu(kh, x_k)$. The transition densities of $\left(X_{i,h}\right)_{i=1,\dots,n}$ are denoted by $p_h\left(0, kh, x, \cdot\right)$. The time horizon $T = T(n) \le 1$ is allowed to depend on n and $h = T/n$ is the discretization step.

We make the following assumptions.

(A1) It holds that $\int_{\mathbb{R}^d} yq\left(t, x, y\right)dy = 0$ for $0 \le t \le 1,\ x \in \mathbb{R}^d$.

(A2) There exist positive constants $\sigma_\star$ and $\sigma^\star$ such that the covariance matrix $\sigma\left(t, x\right) = \int_{\mathbb{R}^d} yy^T q\left(t, x, y\right)dy$ satisfies

$$\sigma_\star \le \theta^T \sigma\left(t, x\right)\theta \le \sigma^\star$$

for all $\|\theta\| = 1$ and $t \in [0, 1]$ and $x \in \mathbb{R}^d$.

(A3) There exist a positive integer S' and a real nonnegative function $\psi(y)$, $y \in \mathbb{R}^d$ satisfying $\sup_{y\in\mathbb{R}^d} \psi(y) < \infty$ and $\int_{\mathbb{R}^d} \|y\|^S \psi(y)\, dy < \infty$ with $S = (S'+2)d + 4$ such that

$$\left|D_y^\nu q(t,x,y)\right| \le \psi(y), \; t \in [0,1], \; x, y \in \mathbb{R}^d \;\; |\nu| = 0,1,2,3,4$$

and

$$|D_x^\nu q(t,x,y)| \le \psi(y), \; t \in [0,1], \; x, y \in \mathbb{R}^d \;\; |\nu| = 0,1,2.$$

Moreover, for all $x, y \in R^d$, $h > 0, 0 \le t, t+jh \le 1, j \ge j_0$, with a bound j_0 that does not depend on x, t,

$$\left|D_x^\nu q^{(j)}(t,x,y)\right| \le Cj^{-d/2}\psi\left(j^{-1/2}y\right), |\nu| = 0,1,2,3$$

for a constant $C < \infty$. Here $q^{(j)}(t,x,y)$ denotes the j-fold convolution of q for fixed x as a function of y:

$$q^{(j)}(t,x,y) = \int q^{(j-1)}(t,x,u)q(t+(j-1)h, x, y-u)du,$$

$q^{(1)}(t,x,y) = q(t,x,y)$.

Note that the last condition is motivated by (A2) and the classical local limit theorem. Note also that for $1 \le j \le j_0$

$$\int \|y\|^S q^{(j)}(t,x,y)dy \le C(j_0\,, S).$$

(B1) The functions $m(t,x)$ and $\sigma(t,x)$ and their first and second derivatives w.r.t. t and their derivatives up to the order six w.r.t. x are continuous and bounded uniformly in t and x. All these functions are Lipschitz continuous with respect to x with a Lipschitz constant that does not depend on t. The functions $\chi_\nu(t,x)$, $|\nu| = 3,4$, are Lipschitz continuous with respect to t with a Lipschitz constant that does not depend on x. A sufficient condition for this is the following inequality

$$\int_{\mathbb{R}^d} (1+\|z\|^4)\, |q(t,x,z) - q(t',x,z)|\, dz \le C\, |t-t'|\,, 0 \le t, t' \le 1,$$

with a constant that does not depend on $x \in \mathbb{R}^d$. Furthemore, $D_x^\nu \sigma(t,x)$ exist for $|\nu| \le 6$ and are Holder continuous w.r.t. x with a positive exponent and a constant that does not depend on t.

(B2) There exists $\varkappa < \frac{1}{5}$ such that $\liminf_{n\to\infty} T(n)n^{\varkappa} > 0$.

The Markov chain $X_{k,h}$, see (1), is an approximation to the following stochastic differential equation in $\mathbb{R}^d$:

$$dY_s = m(s,Y_s)\, ds + \Lambda(s,Y_s)\, dW_s, \; Y_0 = x \in \mathbb{R}^d, \; s \in [0,T],$$

where $(W_s)_{s\ge 0}$ is the standard Wiener process and Λ is a symmetric positive definite $d \times d$ matrix such that $\Lambda(s,y)\Lambda(s,y)^T = \sigma(s,y)$. The conditional density of Y_t, given $Y_0 = x$ is denoted by $p(0,t,x,\cdot)$. We will use the following differential operators L and $\widetilde{L}$:

$$Lf(s,t,x,y) = \frac{1}{2}\sum_{i,j=1}^{d} \sigma_{ij}(s,x)\frac{\partial^2 f(s,t,x,y)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} m_i(s,x)\frac{\partial f(s,t,x,y)}{\partial x_i},$$

$$\tilde{L}f(s,t,x,y) = \frac{1}{2}\sum_{i,j=1}^{d} \sigma_{ij}(s,y)\frac{\partial^2 f(s,t,x,y)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} m_i(s,y)\frac{\partial f(s,t,x,y)}{\partial x_i}. \qquad (2)$$

To formulate our main result we need also the following operators

$$L'f(s,t,x,y)=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial\sigma_{ij}(s,x)}{\partial s}\frac{\partial^2 f(s,t,x,y)}{\partial x_i\partial x_j}+\sum_{i=1}^{d}\frac{\partial m_i(s,x)}{\partial s}\frac{\partial f(s,t,x,y)}{\partial x_i}$$

$$\widetilde{L}'f(s,t,v,z)=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial\sigma_{ij}(s,y)}{\partial s}\frac{\partial^2 f(s,t,x,y)}{\partial x_i\partial x_j}+\sum_{i=1}^{d}\frac{\partial m_i(s,y)}{\partial s}\frac{\partial f(s,t,x,y)}{\partial x_i}. \tag{3}$$

and the convolution type binary operation $\otimes$:

$$f\otimes g(s,t,x,y)=\int_s^t du\int_{R^d} f(s,u,x,z)\,g(u,t,z,y)\,dz.$$

Konakov and Mammen (2000) obtained a nonuniform rate of convergence for the difference $p_h(0,T,x,\cdot)-p(0,T,x,\cdot)$ as $n\to\infty$ in the case $T\asymp 1$. Edgeworth type expansions for the case $T\asymp 1$ and homogenous diffusions were obtained in Konakov and Mammen (2005). The goal of the present paper is to obtain an Edgeworth type expansion for nonhomegenous case which remains valid for the both cases $T\asymp 1$ or $T=o(1)$. The following theorem contains our main result. It gives Edgeworth type expansions for p_h. For the statement of the theorem we introduce the following differential operators

$$\mathcal{F}_1[f](s,t,x,y)=\sum_{|\nu|=3}\frac{\chi_\nu(s,x)}{\nu!}D_x^\nu f(s,t,x,y),$$

$$\mathcal{F}_2[f](s,t,x,y)=\sum_{|\nu|=4}\frac{\chi_\nu(s,y)}{\nu!}D_x^\nu f(s,t,x,y).$$

Furthermore, we introduce two terms corresponding to the classical Edgeworth expansion (see Bhattacharya and Rao (1976))

$$\widetilde{\pi}_1(s,t,x,y) = (t-s)\sum_{|\nu|=3}\frac{\overline{\chi}_\nu(s,t,y)}{\nu!}D_x^\nu\widetilde{p}(s,t,x,y), \tag{4}$$

$$\begin{aligned}\widetilde{\pi}_2(s,t,x,y) &= (t-s)\sum_{|\nu|=4}\frac{\overline{\chi}_\nu(s,t,y)}{\nu!}D_x^\nu\widetilde{p}(s,t,x,y)\\ &\quad+\frac{1}{2}(t-s)^2\left\{\sum_{|\nu|=3}\frac{\overline{\chi}_\nu(s,t,y)}{\nu!}D_x^\nu\right\}^2\widetilde{p}(s,t,x,y),\end{aligned} \tag{5}$$

where

$$\overline{\chi}_\nu(s,t,y)=\frac{1}{t-s}\int_s^t\chi_\nu(u,y)du$$

and $\chi_\nu(t,x)$ is the $\nu-th$ cumulant of the density of the innovations $q(t,x,\cdot)$. The gaussian transition densities $\widetilde{p}(s,t,x,y)$ are defined in (6). Note, that in the homogenous case $\chi_\nu(u,y)\equiv\chi_\nu(y)$ and $\overline{\chi}_\nu(s,t,y)\equiv\chi_\nu(y)$, where $\chi_\nu(y)$ is the $\nu-th$ cumulant of the density $q(y,\cdot)$.

Theorem 1. *Assume (A1)-(A3), (B1)-(B2). Then there exists a constant $\delta>0$ such that the following expansion holds:*

$$\sup_{x,y\in R^d} T^{d/2}\left(1+\left\|\frac{y-x}{\sqrt{T}}\right\|^{S'}\right)\times\Big|p_h(0,T,x,y)-p(0,T,x,y)$$

$$-h^{1/2}\pi_1(0,T,x,y) - h\pi_2(0,T,x,y)\Big| = O(h^{1+\delta}),$$

where S' is defined in Assumption (A3) and where

$$\begin{aligned}
\pi_1(0,T,x,y) &= (p \otimes \mathcal{F}_1[p])(0,T,x,y),\\
\pi_2(0,T,x,y) &= (p \otimes \mathcal{F}_2[p])(0,T,x,y) + p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p]](0,T,x,y)\\
&\quad + \frac{1}{2}p \otimes (L_\star^2 - L^2)p(0,T,x,y) - \frac{1}{2}p \otimes (L' - \widetilde{L}')p(0,T,x,y).
\end{aligned}$$

Here $p(s,t,x,y)$ is the transition density of the limiting diffusion Y_s and the operator $L_\star$ is defined as $\widetilde{L}$, but with the coefficients "frozen" at the point x. The norm $\|\cdot\|$ is the usual Euclidean norm.

Remark 1. The terms of the Edgeworth expansion have subgaussian tails and are of order $n^{-1/2}$ or n^{-1}, respectively:

$$\begin{aligned}
\left|h^{1/2}\pi_1(0,T,x,y)\right| &\le C_1 n^{-1/2} T^{-d/2} \exp\left[-C_2 \left\|\frac{y-x}{\sqrt{T}}\right\|^2\right],\\
|h\pi_2(0,T,x,y)| &\le C_1 n^{-1} T^{-d/2} \exp\left[-C_2 \left\|\frac{y-x}{\sqrt{T}}\right\|^2\right],
\end{aligned}$$

with some positive constants C_1 and C_2.

Remark 2. If the innovation density $q(t,x,\cdot)$ and the conditional mean $m(t,x)$ do not depend on x then we are in the classical case of independent non identically distributed random vectors. We now show that then the Edgeworth expansion of Theorem 1 coincides with the first two terms of the classical Edgeworth expansion $h^{1/2}\widetilde{\pi}_1(0,T,x,y) + h\widetilde{\pi}_2(0,T,x,y)$. Note first that in this case $L_\star = L, L' = \widetilde{L}'$ and $p(s,t,x,y) = \widetilde{p}(s,t,x,y)$ where $\widetilde{p}$ is defined in (6) with $\sigma(s,t,y) = \sigma(s,t) = \int_s^t \sigma(u)du$ and $m(s,t,y) =$

$m(s,t) = \int_s^t m(u)du$. This gives

$$\pi_1(0,T,x,y) = \int_0^T ds \int \widetilde{p}(0,s,x,v) \sum_{|\nu|=3} \frac{\chi_\nu(s)}{\nu!} D_v^\nu \widetilde{p}(s,T,v,y)dv$$

$$= -\sum_{|\nu|=3} \int_0^T \frac{\chi_\nu(s)}{\nu!} ds D_y^\nu \int \widetilde{p}(0,s,x,v)\widetilde{p}(s,T,v,y)dv$$

$$= -\sum_{|\nu|=3} \frac{T}{\nu!} \overline{\chi}_\nu(0,T) D_y^\nu \widetilde{p}(0,T,x,y)$$

$$= \sum_{|\nu|=3} \frac{T}{\nu!} \overline{\chi}_\nu(0,T) D_x^\nu \widetilde{p}(0,T,x,y) = \widetilde{\pi}_1(0,T,x,y),$$

$$\widetilde{p} \otimes \mathcal{F}_1[\widetilde{p}](s,T,z,y) = \int_s^T du \int \widetilde{p}(s,u,z,w) \sum_{|\nu|=3} \frac{\chi_\nu(u)}{\nu!} D_w^\nu \widetilde{p}(u,T,w,y)dw$$

$$= -\sum_{|\nu|=3} \int_s^T \frac{\chi_\nu(u)}{\nu!} du\; D_y^\nu \widetilde{p}(s,T,z,y) = (T-s) \sum_{|\nu|=3} \frac{\overline{\chi}_\nu(s,T)}{\nu!} D_z^\nu \widetilde{p}(s,T,z,y),$$

$$\mathcal{F}_1[\widetilde{p} \otimes \mathcal{F}_1[\widetilde{p}]](s,T,z,y) = (T-s) \sum_{|\nu|=3} \frac{\chi_\nu(s)}{\nu!} D_z^\nu \left[\sum_{|\nu'|=3} \frac{\overline{\chi}_{\nu'}(s,T)}{\nu'!} D_z^{\nu'} \widetilde{p}(s,T,z,y) \right]$$

$$= (T-s) \sum_{|\nu|=3, |\nu'|=3} \frac{\chi_\nu(s)}{\nu!} \frac{\overline{\chi}_{\nu'}(s,T)}{\nu'!} D_z^{\nu+\nu'} \widetilde{p}(s,T,z,y),$$

$$\widetilde{p} \otimes \mathcal{F}_2[\widetilde{p}](0,T,x,y) + \widetilde{p} \otimes \mathcal{F}_1[\widetilde{p} \otimes \mathcal{F}_1[\widetilde{p}]](0,T,x,y) = T \sum_{|\nu|=4} \frac{\overline{\chi}_\nu(0,T)}{\nu!} D_x^\nu \widetilde{p}(0,T,x,y)$$

$$+ \int_0^T ds \int \widetilde{p}(0,s,x,z)(T-s) \sum_{|\nu|=3, |\nu'|=3} \frac{\chi_\nu(s)}{\nu!} \frac{\overline{\chi}_{\nu'}(s,T)}{\nu'!} D_z^{\nu+\nu'} \widetilde{p}(s,T,z,y)dz$$

$$= T \sum_{|\nu|=4} \frac{\overline{\chi}_\nu(0,T)}{\nu!} D_x^\nu \widetilde{p}(0,T,x,y) + \sum_{|\nu|=3, |\nu'|=3} \frac{1}{\nu!} \frac{1}{\nu'!} \int_0^T \chi_\nu(s) \left(\int_s^T \chi_{\nu'}(u)du \right) ds D_x^{\nu+\nu'} \widetilde{p}(s,T,x,y).$$

For $\nu = \nu'$ we have

$$\int_0^T \chi_\nu(s) \left(\int_s^T \chi_{\nu'}(u)du \right) ds = \frac{1}{2} \int_0^T \int_0^T \chi_\nu(s)\chi_\nu(u) ds du = \frac{T^2}{2} \overline{\chi}_\nu(0,T) \overline{\chi}_\nu(0,T).$$

For $\nu \neq \nu'$ we get

$$\int_0^T \chi_\nu(s) \left(\int_s^T \chi_{\nu'}(u)du \right) ds + \int_0^T \chi_{\nu'}(s) \left(\int_s^T \chi_\nu(u)du \right) ds$$

$$= \int_0^T \int_s^T \left[\chi_\nu(s)\chi_{\nu'}(u) + \chi_{\nu'}(s)\chi_\nu(u) \right] ds du$$

$$= \frac{1}{2} \int_0^T \int_0^T \left[\chi_\nu(s)\chi_{\nu'}(u) + \chi_{\nu'}(s)\chi_\nu(u) \right] ds du$$

$$= \frac{T^2}{2} \overline{\chi}_\nu(0,T) \overline{\chi}_{\nu'}(0,T) + \frac{T^2}{2} \overline{\chi}_{\nu'}(0,T) \overline{\chi}_\nu(0,T).$$

From these equations we obtain

$$
\begin{aligned}
&\widetilde{p}\otimes\mathcal{F}_2[\widetilde{p}](0,T,x,y)+\widetilde{p}\otimes\mathcal{F}_1[\widetilde{p}\otimes\mathcal{F}_1[\widetilde{p}](0,T,x,y)\\
&=T\sum_{|\nu|=4}\frac{\overline{\chi}_\nu(0,T)}{\nu!}D_x^\nu\widetilde{p}(0,T,x,y)+\frac{T^2}{2}\left\{\sum_{|\nu|=3}\frac{\overline{\chi}_\nu(0,T)}{\nu!}D_x^\nu\right\}^2\widetilde{p}(0,T,x,y)\\
&=\widetilde{\pi}_2(0,T,x,y).
\end{aligned}
$$

This shows the claim that we get for this case the first two terms of the classical Edgeworth expansion.

Remark 3. If $\chi_\nu(t,x)=0$ for $|\nu|=3$ and for $t\in[0,T]\times R^d$ then it holds that $\mathcal{F}_1\equiv 0$. The Theorem 1 holds with

$$
\begin{aligned}
\pi_1(0,T,x,y) &= 0,\\
\pi_2(0,T,x,y) &= (p\otimes\mathcal{F}_2[p])(0,T,x,y)+\frac{1}{2}p\otimes(L_\star^2-L^2)p(0,T,x,y)-\frac{1}{2}p\otimes(L'-\widetilde{L}')p(0,T,x,y).
\end{aligned}
$$

If in addition $\chi_\nu(t,x)=0$ for $|\nu|=4$ then the first four moments of the innovations coincide with the first four moments of a normal distribution with zero mean and covariance matrix $\sigma(t,x)$. In this case we have $\mathcal{F}_2=0$ and we have

$$
\begin{aligned}
\pi_1(0,T,x,y) &= 0,\\
\pi_2(0,T,x,y) &= \frac{1}{2}p\otimes(L_\star^2-L^2)p(0,T,x,y)-\frac{1}{2}p\otimes(L'-\widetilde{L}')p(0,T,x,y)
\end{aligned}
$$

and the first two terms of the Edgeworth expansion do not depend on the innovation density. In particular, it holds that $\chi_\nu(t,x)=0$ for $|\nu|=3,4$ for Markov chains that are defined by Euler approximations to diffusions. Thus, an Edgeworth expansion for the Euler scheme holds with the same π_1 and π_2 as just defined. For the homogenous case we have that $L'=\widetilde{L}'=0$ and we obtain for the Euler scheme in this case

$$
\begin{aligned}
\pi_1(0,T,x,y) &= 0,\\
\pi_2(0,T,x,y) &= \frac{1}{2}p\otimes(L_\star^2-L^2)p(0,T,x,y).
\end{aligned}
$$

This result for $T=[0,1]$ under Hormander's condition on a diffusion matrix was obtained by Bally and Talay (1996).

Remark 4. We now shortly discuss an application of our result to statistics. Assume that one observes a Markov process $X_{1,h},...,X_{nk,h}$ at time points $k,2k,...,nk$. That means we assume that a high frequency Markov chain runs in the background on a very fine time grid but that it is only observed on a coarser grid. This asymptotics reflects a set up occurring in the high frequency statistical analysis for financial data where diffusion approximations are used only for coarser time scales. For the finest scale discrete pattern in the price processes become transparent that could not be modeled by diffusions. The joint distribution of the observed values of the Markov process is denoted by P_h. We assume that this joint distribution can be approximated by the distribution of $(Y_1,...,Y_n)$ where $Y_1,...,Y_n$ are the values of a diffusion on the equidistant grid $kh,2kh,...,nkh$. The joint distribution of $(Y_1,...,Y_n)$ is denoted by Q_h. According to our theorem the one-dimensional marginal distributions of P_h can be approximated by the one-dimensional marginal distributions of Q_h. Under appropriate conditions the L_1-norm of this difference is of order $k^{-1/2}$. This implies that the L_1-norm of the difference between the joint distributions P_h and Q_h is of order $nk^{-1/2}$. That means the diffusion approximation is only accurate if $k\gg n^2$, i.e. only if the grid of observed points is very coarse in comparison to the grid on which the Markov process lives. Only in this case it can be guarantied that a statistical inference that is based on the diffusion model is accurate. Or put it in another way, data that come from the Markov model could not be asymptotically statistically distinguished from diffusion observations. Our results help to analyze what may go wrong if $k \gg n^2$ does not hold. The (signed) transition densities $p + h^{1/2}\pi_1 + h\pi_2$ given in the statement of Theorem 1 define a joined (signed) measure R_h. According to Theorem 1, the marginal distributions of R_h approximate the one-dimensional marginal distributions of P_h with order $o(k^{-1-\delta})$. One may conjecture that under some regularity assumptions the exact order is $k^{-3/2}$. This implies that $\|P_h - R_h\|_1$ is of order $nk^{-3/2}$. Thus, this approximation is appropriate as long as $k \gg n^{2/3}$. This is a much more acceptable assumption. Now, one can check which statistical procedures behave differently under the models Q_h and R_h. These procedures may lead to erroneous conclusions for the Markov data.

3 The parametrix method.

3.1 The parametrix method for diffusions.

We now give a short overview on the parametrix method for diffusions. For any $s \in [0,T]$, $x, y \in \mathbb{R}^d$ we consider the following family of "frozen" diffusion processes

$$d\widetilde{Y}_t = m\,(t,y)\,dt + \Lambda\,(t,y)\,dW_t,\ \widetilde{Y}_s = x,\ s \le t \le T.$$

Let $\widetilde{p}^y\,(s,t,x,\cdot)$ be the conditional density of $\widetilde{Y}_t$, given $\widetilde{Y}_s = x$. In the sequel for any z we will denote $\widetilde{p}\,(s,t,x,z) = \widetilde{p}^z\,(s,t,x,z)\,,$ where the variable z acts here twice: as the argument of the density and as defining quantity of the process $\widetilde{Y}_t$.

The transition densities $\widetilde{p}$ can be computed explicitly

$$\begin{aligned}\widetilde{p}\,(s,t,x,y) &= (2\pi)^{-d/2}\,(\det\sigma\,(s,t,y))^{-1/2}\\ &\quad\times\exp\left(-\frac{1}{2}\,(y-x-m\,(s,t,y))^T\,\sigma^{-1}\,(s,t,y)\,(y-x-m\,(s,t,y))\right),\end{aligned} \tag{6}$$

where

$$\sigma\,(s,t,y) = \int_s^t \sigma\,(u,y)\,du, \qquad m\,(s,t,y) = \int_s^t m\,(u,y)\,du.$$

Note that the following differential operators L and $\tilde{L}$ correspond to the infinitesimal operators of Y or of the frozen process $\widetilde{Y}$, respectively, i.e.

$$Lf(s,t,x,y) = \lim_{h\to 0} h^{-1}\{E[f(s,t,Y(s+h),y) \mid Y(s) = x] - f(s,t,x,y)\},$$

$$\tilde{L}f(s,t,x,y) = \lim_{h\to 0} h^{-1}\{E[f(s,t,\tilde{Y}(s+h),y) \mid \widetilde{Y}(s) = x] - f(s,t,x,y)\}.$$

We put

$$H = (L - \tilde{L})\tilde{p}.$$

Then

$$\begin{aligned}H\,(s,t,x,y) &= \frac{1}{2}\sum_{i,j=1}^{d}(\sigma_{ij}\,(s,x) - \sigma_{ij}\,(s,y))\,\frac{\partial^2\widetilde{p}\,(s,t,x,y)}{\partial x_i \partial x_j}\\ &\quad+\sum_{i,j=1}^{d}(m_i\,(s,x) - m_i\,(s,y))\,\frac{\partial\widetilde{p}\,(s,t,x,y)}{\partial x_i}.\end{aligned}$$

In the following lemmas the k- fold convolution of H is denoted by $H^{(k)}$. The following results have been proved in Konakov and Mammen (2000).

Lemma 1. *Let $0 \le s < t \le T$. It holds*

$$p(s,t,x,y) = \sum_{r=0}^{\infty} \widetilde{p} \otimes H^{(r)}(s,t,x,y).$$

Lemma 2. *Let $0 \le s < t \le T$. There are constants C and C_1 such that*

$$|H(s,t,x,y)| \le C_1 \rho^{-1} \phi_{C,\rho}(y-x)$$

and

$$\left| \widetilde{p} \otimes H^{(r)}(s,t,x,y) \right| \le C_1^{r+1} \frac{\rho^r}{\Gamma(1+\frac{r}{2})} \phi_{C,\rho}(y-x),$$

where $\rho^2 = t-s$, $\phi_{C,\rho}(u) = \rho^{-d}\phi_C(u/\rho)$and

$$\phi_C(u) = \frac{\exp(-C\|u\|^2)}{\int \exp(-C\|v\|^2\, dv)}.$$

3.2 The parametrix method for Markov chains.

We now give a short overview on the parametrix method for Markov chains. This theory was developed in Konakov and Mammen (2000). For any $0 \le jh \le T$, $x, y \in \mathbb{R}^d$ we consider an additional family of "frozen" Markov chains defined for $jh \le ih \le T$ as

$$\widetilde{X}_{i+1,h} = \widetilde{X}_{i,h} + m\,(ih,y)\,h + \sqrt{h}\widetilde{\xi}_{i+1,h},\ \widetilde{X}_{j,h} = x \in \mathbb{R}^d,\ j \le i \le n, \tag{7}$$

where $\widetilde{\xi}_{j+1,h}, ..., \widetilde{\xi}_{n,h}$ is an innovation sequence such that the conditional density of $\widetilde{\xi}_{i+1,h}$ given the past $\widetilde{X}_{i,h} = x_i, ..., \widetilde{X}_{0,h} = x_0$ equals to $q\,(ih, y, \cdot)$. Let us introduce the infinitesimal operators corresponding to Markov chains (1) and (7) respectively,

$$L_h f\,(jh,kh,x,y) = h^{-1}\left(\int p_h\,(jh,(j+1)\,h,x,z)\, f\,((j+1)\,h,kh,z,y)\,dz - f\,((j+1)\,h,kh,x,y) \right)$$

and

$$\widetilde{L}_h f\,(jh,kh,x,y) = h^{-1}\left(\int \widetilde{p}_h^{y}\,(jh,(j+1)\,h,x,z)\, f\,((j+1)\,h,kh,z,y)\,dz - f\,((j+1)\,h,kh,x,y) \right),$$

where $\widetilde{p}_h^{y}\,(jh,j'h,x,\cdot)$ denotes the conditional density of $\widetilde{X}_{j',h}$ given $\widetilde{X}_{j,h} = x$. Similarly as above, for brevity for any z we write $\widetilde{p}_h\,(jh,j'h,x,z) = \widetilde{p}_h^{z}\,(jh,j'h,x,z)$, where the variable z acts here twice: as the argument of the density and as defining quantity of the process $\widetilde{X}_{i,h}$. For technical convenience the terms $f\,((j+1)\,h,kh,z,y)$ on the right hand side of $L_h f$ and $\widetilde{L}_h f$ appear instead of $f\,(jh,kh,z,y)$.

In analogy with the definition of H we put, for $k > j$,

$$H_h\,(jh,kh,x,y) = \left(L_h - \widetilde{L}_h\right) \widetilde{p}_h\,(jh,kh,x,y).$$

We also shall use the convolution type binary operation $\otimes_h$ which is a discrete version of $\otimes$:

$$g \otimes_h f\,(jh,kh,x,y) = \sum_{i=j}^{k-1} h \int_{\mathbb{R}^d} g\,(jh,ih,x,z)\, f\,(ih,kh,z,y)\, dz,$$

where $0 \le j < k \le n$. We write $g \otimes_h H_h^{(0)} = g$ and $g \otimes_h H_h^{(r)} = \left(g \otimes_h H_h^{(r-1)}\right) \otimes_h H_h$ for $r \ge 1$. For the higher order convolutions we use the convention $\sum_{i=j}^{l} = 0$ for $l < j$. One can show the following analog of the "parametrix " expansion for p_h [see Konakov and Mammen (2000)].

Lemma 3. *Let $0 \le jh < kh \le T$. It holds*

$$p_h(jh,kh,x,y) = \sum_{r=0}^{k-j} \widetilde{p}_h \otimes_h H_h^{(r)}(jh,kh,x,y),$$

where

$$\widetilde{p}_h(jh,jh,x,y) = p_h(kh,kh,x,y) = \delta(y-x)$$

and δ is the Dirac delta symbol.

4 Some technical tools.

4.1 Plugged in Edgeworth expansions for independent observations.

In this Section we will develop some tools that are helpful for the comparison of the expansion of p (see Lemma 1) and the expansion of p_h (see Lemma 3). These expansions are simple expressions in $\tilde{p}$ or $\tilde{p}_h$, respectively. Recall that $\tilde{p}$ is a Gaussian density, see (6), and that $\tilde{p}_h$ is the density of a sum of independent variables. The densities $\tilde{p}$ and $\tilde{p}_h$ can be compared by application of the classical Edgeworth expansions. This is done in Lemma 5 and this is the essential step for the comparison of the expansions of p and p_h. Lemmas 4 and 7 contain technical tools that will be used below. Lemma 7 contains bounds on derivatives of $\tilde{p}_h$ that will be used at several places in the proof of Theorem 1. Its proof makes use of Lemma 6 that is a generalisation of a result in Konakov and Molchanov (1984) (Lemma 4 on page 68). Lemma 5 is a higher order extension of the results from Section 3.3 in Konakov and Mammen (2000).

For the formulation of the lemmas we need some additional notations. Suppose that $X \in \mathbb{R}^d$ is a random vector having a density $q(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d$, $EX = 0, Cov(X,X) = \Sigma$, where Σ be a positively definite $d \times d$ matrix . Denote $A = \|a_{ij}\| = \Sigma^{-1/2}$ and let $\chi_\nu(Z)$ be a cumulant of the order $\nu = (\nu_1, ..., \nu_d)$ of a random vector $Z \in \mathbb{R}^d$, $\phi(x)$ denotes a function in $\mathbb{R}^d$ such that $D_x^\nu \phi(x)$ exist and are continuous for $|\nu| = 4$, and $A^{-1} = \left\|a^{ij}\right\| = \Sigma^{1/2}$.

Lemma 4. *The following relation holds for $s = 3$ and for $s = 4$*

$$\sum_{|\nu|=s} \frac{\chi_\nu(AX) D_z^\nu \phi(z)}{\nu!} = \sum_{|\nu|=s} \frac{\chi_\nu(X) D_x^\nu \phi(Ax)}{\nu!}$$

where $z = Ax$.

Denote

$$\mu_{j,k}(y) = h \sum_{i=j}^{k-1} m(ih,y), V_{j,k}(y) = h \sum_{i=j}^{k-1} \sigma(ih,y). \tag{8}$$

PROOF OF LEMMA 4. For $|\nu| = 3, \nu = (\nu_1, ..., \nu_d)$, each cumulant $\chi_\nu(AX)$ is a linear combination of $\chi_\mu(X)$ with $|\mu| = 3$ and with coefficients depending only on a_{ij}. It follows from the following relation

$$\chi_\nu(AX) = \mu_\nu(AX) = \int (a_{11}x_1 + ... + a_{1d}x_d)^{\nu_1} \times ... \times (a_{11}x_1 + ... + a_{1d}x_d)^{\nu_d} q(\mathbf{x})d\mathbf{x}.$$

Analogously, from the usual differentiation rule of a composite function and from the relation $\phi(z) = \phi(Ax)$, $x = A^{-1}z$, it follows that $D_z^\nu \phi(z) = D_z^\nu \phi(Ax)$ is a linear combination of $D_x^\nu \phi(Ax)$ with coefficients depending only on a^{ij}. As a result of such substitutions we obtain that

$$\sum_{|\nu|=3} \frac{\chi_\nu(AX) D_z^\nu \phi(z)}{\nu!} = \frac{1}{3!} \sum_{j=1}^d \left[\sum_{|\mu|=3} \frac{3!}{\mu_1!...\mu_d!} a_{1j}^{\mu_1} ... a_{dj}^{\mu_d} \chi_\mu(X) \right]$$

$$\times \left[\sum_{|\mu'|=3} \frac{3!}{\mu_1'!...\mu_d'!} (a^{j1})^{\mu_1'} ... (a^{jd})^{\mu_d'} D_x^{\mu'} \phi(AX) \right]$$

$$+ \frac{1}{2!1!} \sum_{\{i \neq, j\}} \left[\sum_{l=1}^d \sum_{|\mu|=2} \frac{2!}{\mu_1!...\mu_d!} a_{1j}^{\mu_1} ... a_{dj}^{\mu_d} a_{il} \chi_{\mu+e_l}(X) \right]$$

$$\times \left[\sum_{l'=1}^d \sum_{|\mu'|=2} \frac{2!}{\mu_1'!...\mu_d'!} (a^{j1})^{\mu_1'} ... (a^{jd})^{\mu_d'} a^{il'} D_x^{\mu'+e_{l'}} \phi(AX) \right]$$

$$+ \frac{1}{3!} \sum_{\{i \neq j \neq k\}} \left[\sum_{l,q=1}^d \sum_{|\mu|=1} \frac{1}{\mu_1!...\mu_d!} a_{1j}^{\mu_1} ... a_{dj}^{\mu_d} a_{il} a_{kq} \chi_{\mu+e_l+e_q}(X) \right]$$

$$\times \left[\sum_{l',q'=1}^d \sum_{|\mu'|=1} \frac{1}{\mu_1'!...\mu_d'!} (a^{j1})^{\mu_1'} ... (a^{jd})^{\mu_d'} a^{il'} a^{kq'} D_x^{\mu'+e_{l'}+e_{q'}} \phi(AX) \right]$$

where $\sum_{\{i \neq, j\}}$ ($\sum_{\{i \neq j \neq k\}}$) denotes the sum over all different pairs (triples) of $i, j \in \{1, 2, ..., d\}$ (of $i, j, k \in \{1, 2, ..., d\}$) and $e_i \in \mathbb{R}^d$ denotes the vector whose $i-$th coordinate is equal to 1 and other coordinates are zero. Collecting the similar terms in the last equation we obtain that for $\nu = 3e_k$, $\nu' = 3e_l$ the coefficient before $\chi_\nu(X) D_x^{\nu'} \phi(AX)$ is equal to $\frac{1}{3!}(a_{1k}a^{l1} + ... + a_{dk}a^{ld})^3 = \frac{1}{3!}\delta_{kl}$, for $\nu = e_q + 2e_r$, $\nu' = e_l + 2e_n$, $q \neq r$, the coefficient before $\chi_\nu(X) D_x^{\nu'} \phi(AX)$ is equal to $\frac{1}{2!}(a_{1q}a^{l1} + ... + a_{dq}a^{ld})(a_{1r}a^{n1} + ... + a_{dr}a^{nd})^2 = \frac{1}{2!}\delta_{ql}\delta_{rn}$, in particular , for $l = n$ the last expression is equal to zero. For $\nu = e_q + e_r + e_n, \nu' = e_{q'} + e_{r'} + e_{n'}$ $q \neq r, q \neq n, r \neq n$, the coefficient before $\chi_\nu(X) D_x^{\nu'} \phi(AX)$ is equal to $(a_{1q}a^{q'1} + ... + a_{dq}a^{q'd}) \times (a_{1r}a^{r'1} + ... + a_{dr}a^{r'd}) \times$ $(a_{1n}a^{n'1} + ... + a_{dn}a^{n'd}) = \delta_{qq'}\delta_{rr'}\delta_{nn'}$. This proves lemma for $|\nu| = 3$. The proof for $|\nu| = 4$ is quite similar. For this case we use the relation which enabes to express a cumulant $\chi_\nu(AX)$ as $\mu_\nu(AX)$ plus a second order polynomial of the moments $\mu_{\nu'}(AX)$, $|\nu'| = 2$. A necessary correction term for $\mu_\nu(X)$ to get a $\chi_\nu(X)$ comes from the derivation of $D_z^\nu \phi(z)$. This completes the proof of the lemma.

Lemma 5. *The following bound holds with a constant C for $\nu = (\nu_1, ...\nu_p)^T$ with $0 \leq |\nu| \leq 6$*

$$\left| D_z^\nu \widetilde{p}_h(jh, kh, x, y) - D_z^\nu \widetilde{p}(jh, kh, x, y) - \sqrt{h} D_z^\nu \widetilde{\pi}_1(jh, kh, x, y) - h D_z^\nu \widetilde{\pi}_2(jh, kh, x, y) \right|$$
$$\leq C h^{3/2} \rho^{-3} \zeta_\rho^{S-|\nu|}(y - x)$$

for all $j < k, x$ *and* y. *Here* D_z^ν *denotes the partial differential operator of order* ν *with respect to* $z = V_{j,k}^{-1/2}(y)(y - x - \mu_{j,k}(y))$. *The quantity* ρ *denotes again the term* $\rho = [h(k-j)]^{1/2}$ *and the functions* $\widetilde{\pi}_1$ *and* $\widetilde{\pi}_2$ *are defined in (4) and (5). We write* $\zeta_\rho^k(\cdot) = \rho^{-d}\zeta^k(\cdot/\rho)$ *where*

$$\zeta^k(z) = \frac{[1 + \|z\|^k]^{-1}}{\int [1 + \|z'\|^k]^{-1} dz'}.$$

PROOF OF LEMMA 5. We note first that $\tilde{p}_h(jh, kh, x, \cdot)$ is the density of the vector

$$x + \mu_{j,k}(y) + h^{1/2} \sum_{i=j}^{k-1} \widetilde{\xi}_{i+1,h},$$

where, as above in the definition of the "frozen" Markov chain $\widetilde{Y}_n$, $\widetilde{\xi}_{i+1,h}$ is a sequence of independent variables with densities $q(ih, y, \cdot)$, $\mu_{j,k}(y) = \sum_{i=j}^{k-1} hm(ih, y)$. Let $f_h(\cdot)$ be the density of the normalized sum

$$h^{1/2} \left[V_{j,k}(y)\right]^{-1/2} \sum_{i=j}^{k-1} \widetilde{\xi}_{i+1,h}.$$

Clearly, we have

$$\tilde{p}_h(jh, kh, x, \cdot) = \det\left[V_{j,k}(y)\right]^{-1/2} f_n\{\left[V_{j,k}(y)\right]^{-1/2}\left[\cdot - x - \mu_{j,k}(y)\right]\}.$$

We now argue that an Edgeworth expansion holds for f_h. This implies the following expansion for $\tilde{p}_h(jh, kh, x, \cdot)$

$$\tilde{p}_h(jh, kh, x, \cdot) \tag{9}$$

$$= \det\left[V_{j,k}(y)\right]^{-1/2} [\sum_{r=0}^{S-3} (k-j)^{-r/2} P_r(-\phi : \{\bar{\chi}_{\beta,r}\})\{\left[V_{j,k}(y)\right]^{-1/2}\left[\cdot - x - \mu_{j,k}(y)\right]\}$$

$$+[k-j]^{-(S-2)/2} O([1 + \left\|\{\left[V_{j,k}(y)\right]^{-1/2}[\cdot - x - \mu_{j,k}(y)]\}\right\|^S]^{-1})]$$

with standard notations, see Bhattacharya and Rao (1976), p. 53. In particular, P_r denotes a product of a standard normal density with a polynomial that has coefficients depending only on cumulants of order $\leq r+2$. Expansion (9) follows from Theorem 19.3 in Bhattacharya and Rao (1976). This can be seen as in the proof of Lemma 3.7 in Konakov and Mammen (2000).

It follows from (9) and Condition (A3) that

$$\left|\tilde{p}_h(jh, kh, x, y) - \tilde{p}(jh, kh, x, y) - h^{1/2}\widehat{\pi}_1(jh, kh, x, y) - h\widehat{\pi}_2(jh, kh, x, y)\right|$$

$$\leq Ch^{3/2}\rho^{-3}\zeta_\rho^{S-|\nu|}(y - x), \tag{10}$$

where

$$\tilde{p}(jh,kh,x,y) = \det\left[V_{j,k}(y)\right]^{-1/2}(2\pi)^{-p/2}$$
$$\exp\{-\frac{1}{2}(y-x-\mu_{j,k}(y))^T\left[V_{j,k}(y)\right]^{-1}(y-x-\mu_{j,k}(y))\},$$
$$\widehat{\pi}_1(jh,kh,x,y) = -\rho^{-1}\det\left[V_{j,k}(y)\right]^{-1/2}\sum_{|\nu|=3}\frac{\overline{\chi}_{\nu,j,k}(y)}{\nu!}D_z^\nu\phi\left\{\left[V_{j,k}(y)\right]^{-1/2}(y-x-\mu_{j,k}(y))\right\},$$
$$\widehat{\pi}_2(jh,kh,x,y) = \rho^{-2}\det\left[V_{j,k}(y)\right]^{-1/2}\left[\sum_{|\nu|=4}\frac{\overline{\chi}_{\nu,j,k}(y)}{\nu!}D_z^\nu\phi\left\{\left[V_{j,k}(y)\right]^{-1/2}(y-x-\mu_{j,k}(y))\right\}\right.$$
$$\left.+\frac{1}{2}\left\{\sum_{|\nu|=3}\frac{\overline{\chi}_{\nu,j,k}(y)}{\nu!}D_z^\nu\right\}^2\phi\left\{\left[V_{j,k}(y)\right]^{-1/2}(y-x-\mu_{j,k}(y))\right\}\right],$$

where $\overline{\chi}_{\nu,j,k}(y) = \frac{1}{k-j}\sum_{i=j}^{k-1}\chi_{\nu,j,k,i}(y)$, $\chi_{\nu,j,k,i}(y) = \nu-$th cumulant of $\rho\left[V_{j,k}(y)\right]^{-1/2}\widetilde{\xi}_{i+1,h} = \rho^{|\nu|}\times$ $\{\nu-$th cumulant of $\left[V_{j,k}(y)\right]^{-1/2}\widetilde{\xi}_{i+1,h}\}$, and $D_z^\nu\phi(z)$ denotes the $\nu-$th derivative of ϕ with respect to $z = \left[V_{j,k}(y)\right]^{-1/2}(y-x-\mu_{j,k}(y))$. It follows from the (conditional) independence of $\widetilde{\xi}_{i+1,h}, i = j,...,k-1$, that $\overline{\chi}_{\nu,j,k}(y) = \frac{\rho^{|\nu|}}{k-j}h^{-|\nu|/2}\times\chi_\nu(AX)$, where $A = h^{1/2}\left[V_{j,k}(y)\right]^{-1/2} = \Sigma^{-1/2}, \Sigma = Cov(X,X), X = \sum_{i=j}^{k-1}\widetilde{\xi}_{i+1,h}$. By Lemma 4 for $s = 3,4$

$$\begin{aligned}
\sum_{|\nu|=s}\frac{\overline{\chi}_{\nu,j,k}(y)}{\nu!}D_z^\nu\phi(z) &= \rho^s\frac{1}{k-j}\sum_{|\nu|=s}\frac{\chi_\nu(AX)}{\nu!}D_{h^{1/2}z}^\nu\phi_h(h^{1/2}z)\\
&= (-1)^s\rho^s\sum_{|\nu|=s}\frac{\overline{\chi}_\nu(X)}{\nu!}D_x^\nu\phi_h(A(y-x-\mu_{j,k}(y)))\\
&= (-1)^s\rho^s\sum_{|\nu|=s}\frac{\overline{\chi}_\nu(X)}{\nu!}D_x^\nu\phi(\left[V_{j,k}(y)\right]^{-1/2}(y-x-\mu_{j,k}(y))), \qquad (11)
\end{aligned}$$

where we put $\phi_h(z) = \phi(h^{-1/2}z), \overline{\chi}_\nu(X) = \frac{1}{k-j}\sum_{i=j}^{k-1}\chi_\nu(ih,y)$. It follows from (11) and the condition **B1** that up to the error term in the right hand side of (10) the functions $\widehat{\pi}_1$ and $\widehat{\pi}_2$ coincide with the functions $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ given at the beginning of Section 4. For $\nu = 0$ the statement of the lemma immediately follows from (10). For $\nu > 0$ one proceeds similarly. See the remark at the end of the proof of Lemma 3.7 in Konakov and Mammen (2000).

Lemma 6. *Let $L(d)$ be the set of symmetric matrices, and for $0 < \lambda^- < \lambda^+ < \infty$ let $D_{\lambda^+,\lambda^-} \subset L(d)$ be the open subset of $L(d)$ that contains all $\Lambda \in L(d)$ with $\lambda^- I < \Lambda < \lambda^+ I$. For $\Lambda \in L(d)$ define $A = A(\Lambda)$ as the symmetric solution of the equation $A^2 = \Lambda$. Then for any $k,l,i,j \leq d$ and $\Lambda \in D_{\lambda^+,\lambda^-}$ we have that with a constant C_m depending on m*

$$\left|\frac{\partial^m a_{ij}(\Lambda)}{(\partial\lambda_{kl})^m}\right| \leq C_m(\lambda^-)^{-(2m-1)/2}. \qquad (12)$$

Here $a_{ij}(\Lambda)$ are the elements of $A = A(\Lambda)$.

Proof of Lemma 6. For $m = 1$ the lemma was proved in Konakov and Molchanov (1984) (see Lemma 4). Suppose now that (12) holds for $m \leq l$. From the equality $AA = \Lambda$ we obtain for $m = l+1$

$$d^{l+1}(AA) = (d^{l+1}A)A + \binom{l+1}{1}(d^lA)dA + ... + \binom{l+1}{l}dA(d^lA) + A(d^{l+1}A) = 0,$$

where d denotes elementwise differentiation of a matrix with respect to a fixed element of Λ. This implies

$$(d^{l+1}A)A + A(d^{l+1}A) = -\begin{pmatrix} l+1 \\ 1 \end{pmatrix}(d^lA)dA - \ldots - \begin{pmatrix} l+1 \\ l \end{pmatrix} dA(d^lA). \tag{13}$$

Denote the symmetric matrix in the right hand side of (13) by $\widetilde{\Lambda}$. Then equality (13) determines a linear operator ℓ mapping $d^{l+1}A$ to $\widetilde{\Lambda}$. In the linear space of symmetric $d \times d$ matrices we introduce the scalar product $\langle X, Y \rangle = \mathrm{trace}(XY)$. The operator ℓ determines a quadratic form

$$\langle \ell X, X \rangle = \mathrm{trace}[(XA + AX)X] = 2\mathrm{trace}[XAX] \geq 2\sqrt{\lambda^-}\mathrm{trace}[XX] = 2\sqrt{\lambda^-}\langle X, X \rangle,$$

where in the inequality we have used that $A - \sqrt{\lambda^-}I$ positive definite implies that $X(A - \sqrt{\lambda^-}I)X = XAX - \sqrt{\lambda^-}XX$ is positive definite. Similarly, we get $\langle \ell X, X \rangle \leq 2\sqrt{\lambda^+}\langle X, X \rangle$. Hence,

$$2\sqrt{\lambda^-} \leq \|\ell\| = \sup_{X \neq 0} \frac{\|\ell X\|}{\|X\|} \leq 2\sqrt{\lambda^+}$$

and

$$\frac{1}{2\sqrt{\lambda^+}} \leq \|\ell^{-1}\| \leq \frac{1}{2\sqrt{\lambda^-}}.$$

We obtain

$$\|d^{l+1}A\| \leq \frac{1}{2\sqrt{\lambda^-}}\|\widetilde{\Lambda}\|.$$

Using the induction hypothesis we get from (13)

$$\|d^{l+1}A\| \leq C_{l+1}(\lambda^-)^{(2l+1)/2}.$$

This completes the proof.

From Lemmas 5 and 6 we get the following corollary. The statement of the next lemma is an extension of Lemma 3.7 in Mammen and Konakov (2000) where the result has been shown for $0 \leq |b| \leq 2, a = 0$.

Lemma 7. *The following bound holds:*

$$\left| D_y^a D_x^b \widetilde{p}_h(jh, kh, x, y) \right| \leq C\rho^{-|a|-|b|}\zeta_\rho^{S-|a|}(y - x)$$

for all $j < k$, for all x and y and for all a, b with $0 \leq |a| + |b| \leq 6$. Here, $\rho = [(k-j)h]^{1/2}$. The constant S has been defined in Assumption (A3).

Proof of Lemma 7. For two matrices A and B with elements a_{ij} or b_{kl}, respectively where $a_{ij}(B)$ are smooth functions of b_{kl} we write $\left|\frac{\partial A}{\partial B}\right| \leq C$ if $\left|\frac{\partial a_{ij}}{\partial b_{kl}}\right| \leq C$ for all $1 \leq i, j \leq d, 1 \leq k, l \leq d$. To obtain the assertion of the lemma we have to estimate the derivatives $D_y^a D_x^b z$, where $z = V_{j,k}^{-1/2}(y)(y - x - \mu_{j,k}(y))$. Note that $z = z(V_{j,k}^{-1/2}, \mu_{j,k}, x, y)$, where $V_{j,k}^{-1/2} = V_{j,k}^{-1/2}(y)$ and $\mu_{j,k} = \mu_{j,k}(y)$. For $l = 1, ..., 6$ it follows from condition (B1) and (8) that

$$\left|\frac{\partial^l \mu_{j,k}(y)}{(\partial y)^l}\right| \leq C\rho^2, \left|\frac{\partial^l V_{j,k}(y)}{(\partial y)^l}\right| \leq C\rho^2. \tag{14}$$

It follows from Lemma 6 that

$$\left|\frac{\partial^l V_{j,k}^{1/2}}{(\partial V_{j,k})^l}\right| \leq C\rho^{-(2l-1)/2}. \tag{15}$$

From inequalities (3.16) in Konakov and Mammen (2000) and from the representation of an inverse matrix in terms of cofactors divided by the determinant we obtain that

$$\left| \frac{\partial^l V_{j,k}^{-1/2}}{(\partial V_{j,k}^{1/2})^l} \right| \leq C\rho^{-(l+1)}. \tag{16}$$

From (14)-(16) and from the chain rule we get

$$\left| \frac{\partial^l V_{j,k}^{-1/2}(y)}{(\partial y)^l} \right| \leq C\rho^{-l}. \tag{17}$$

Now, Lemma 5 implies the assertion of Lemma 7.

4.2 Bounds on operator kernels used in the parametrix expansions.

In this Section we will present bounds for operator kernels appearing in the expansions based on the parametrix method. In Lemma 8 we compare the infinitesimal operators L_h and $\tilde{L}_h$ with the differential operators L and $\tilde{L}$. We give an approximation for the error if, in the definition of $H_h = (L_h - \tilde{L}_h)\tilde{p}_h$, the terms L_h and $\tilde{L}_h$ are replaced by L or $\tilde{L}$, respectively. We show that this term can be approximated by $K_h + M_h$, where $K_h = (L - \tilde{L})\tilde{p}_h$ and where M_h is defined in Remark 5 after Lemma 8 . The bounds obtained in Lemma 9 will be used in the proof of our theorem to show that in the expansion of p_h the terms $\widetilde{p}_h \otimes_h H_h^{(r)}$ can be replaced by $\widetilde{p}_h \otimes_h (K_h + M_h)^{(r)}$.

Lemma 8. *The following bound holds with a constant C*

$$\begin{aligned} &|H_h(jh,kh,x,y) - K_h'(jh,kh,x,y) - M_h'(jh,kh,x,y) - R_h(jh,kh,x,y)| \\ &\quad \leq Ch^{3/2}\rho^{-1}\zeta_\rho^S(y-x) \end{aligned}$$

with ζ_ρ^S as in Lemma 5 for all $j<k$, x and y. For $j<k-1$ we define

$$\begin{aligned}
&K_h'(jh,kh,x,y) = (L-\widetilde{L})\lambda(x), M_h'(jh,kh,x,y) \\
&\qquad = M_{h,1}(jh,kh,x,y) + M_{h,2}(jh,kh,x,y) + M_{h,3}'(jh,kh,x,y), \\
&M_{h,1}(jh,kh,x,y) = h^{1/2}\sum_{|\nu|=3}\frac{D_x^\nu\lambda(x)}{\nu!}(\chi_\nu(jh,x) - \chi_\nu(jh,y)), \\
&M_{h,2}(jh,kh,x,y) = h\sum_{|\nu|=4}\frac{D_x^\nu\lambda(x)}{\nu!}(\chi_\nu(jh,x) - \chi_\nu(jh,y)), \\
&M_{h,3}'(jh,kh,x,y) = \frac{h}{2}(L_\star^2 - \widetilde{L}^2)\lambda(x), \\
&R_h(jh,kh,x,y) = h^{3/2}\sum_{|\nu|=4}\frac{D_x^\nu\lambda(x)}{\nu!}\sum_{r=1}^d \nu_r[m_r(jh,x)\mu_{\nu-e_r}(jh,x) - m_r(jh,y)\mu_{\nu-e_r}(jh,y)] \\
&\quad +5\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{k=1}^d(m_k(jh,x) - m_k(jh,y))\left\{\nu_k\int q(jh,x,\theta)\widetilde{h}^{\nu-e_k}(\theta)\right. \\
&\quad \times\left[\int_0^1(1-u)^4 D^\nu\lambda(x+u\widetilde{h}(\theta))du\right]d\theta + \int q(jh,x,\theta)\widetilde{h}^\nu(\theta)\left.\left[\int_0^1(1-u)^4 uD^{\nu+e_k}\lambda(x+u\widetilde{h}(\theta))du\right]d\theta\right\} \\
&\quad +h^2\sum_{|\nu|=4}\frac{D_x^\nu\lambda(x)}{\nu!}\sum_{|\nu'|=2}\nu!N(\nu,\nu')[m^{\nu'}(jh,x)\mu_{\nu-\nu'}(jh,x) - m^{\nu'}(jh,y)\mu_{\nu-\nu'}(jh,y)].
\end{aligned}$$

Here $L_\star$ is defined as $\widetilde{L}$ but with the coefficients "frozen" at the point x, e_r denotes a d-dimensional vector with the r-th element equal to 1 and with all other elements equal to 0. Furthermore, for $|\nu| = 4, |\nu'| = 2$ we define

$$N(\nu,\nu') = 2^{\chi[\nu'!=1]+\chi[(\nu-\nu')!=1]-2},$$

where $\chi(\cdot)$ is the indicator function. We put $m(x)^\nu = m_1(x)^{\nu_1}\cdot\ldots\cdot m_d(x)^{\nu_d}$ and $m(x)^\nu = 0, \nu! = 0$. We define $\mu_\nu(t,x) = \int z^\nu q(t,x,z)dz$ and $\mu_\nu(t,x) = 0$ if at least one of the coordinates of $\nu = (\nu_1,...,\nu_d)$ is negative. We use also the following definitions

$$\begin{aligned} \lambda(x) &= \widetilde{p}_h((j+1)h, kh, x, y), \\ \widetilde{h}(\theta) &= m(jh,y)h + \theta h^{1/2}. \end{aligned}$$

Here again ρ denotes the term $\rho = [h(k-j)]^{1/2}$. For $j = k-1$ we define

$$K'_h(jh,kh,x,y) = R_h(jh,kh,x,y) = M_{h,2}(jh,kh,x,y) = M'_{h,3}(jh,kh,x,y) = 0$$

and

$$M_{h,1}(jh,kh,x,y) = h^{-(d+2)/2}\left[q\left\{jh,x,h^{-1/2}(y-x-m[jh,x]h)\right\} - q\left\{jh,y,h^{-1/2}(y-x-m[jh,y]h)\right\}\right].$$

Proof of Lemma 8. As in the proof of Lemma 3.9 in Konakov and Mammen (2000) we have

$$H_h(jh,kh,x,y) = H_h^1(jh,kh,x,y) - H_h^2(jh,kh,x,y),$$

where

$$H_h^1(jh,kh,x,y) = h^{-1}\int q(jh,x,\theta)[\lambda(x+h(\theta)) - \lambda(x)]d\theta, \tag{18}$$

$$H_h^2(jh,kh,x,y) = h^{-1}\int q(jh,y,\theta)[\lambda(x+\widetilde{h}(\theta)) - \lambda(x)]d\theta, \tag{19}$$

$$h(\theta) = m(jh,x)h + \theta h^{1/2}, \widetilde{h}(\theta) = m(jh,y)h + \theta h^{1/2}.$$

For $[\lambda(x+h(\theta)) - \lambda(x)]$ and $[\lambda(x+\widetilde{h}(\theta)) - \lambda(x)]$ in (18), (19) we use now the Taylor expansion up to order 5 with remaining term in integral form. To pass from moments to cumulants we use the well known relations (see e.g. relation (6.11) on page 46 in Bhattacharya and Rao (1986)). After long but simple calculations we come to the conclusion of the lemma.

Remark 5. We show now that the function $K'_h(jh,kh,x,y) + M'_{h,3}(jh,kh,x,y)$ in Lemma 8 is equal to $K_h(jh,kh,x,y) + \frac{h}{2}(L_\star^2 - 2L\widetilde{L} + \widetilde{L}^2)\lambda(x) + M''_{h,3}(jh,kh,x,y)$ where

$$\begin{aligned} M''_{h,3}(jh,kh,x,y) &= -h^2\sum_{|\mu|=2}\frac{m^\mu(jh,y)}{\mu!}(L-\widetilde{L})D^\mu\lambda(x) \\ &\quad -3\sum_{|\mu|=3}\int_0^1(1-\delta)^2 d\delta\int q(jh,y,\theta)\frac{\widetilde{h}(\theta)^\mu}{\mu!}(L-\widetilde{L})D^\mu\lambda(x+\delta\widetilde{h}(\theta))d\theta. \end{aligned} \tag{20}$$

Thus in Lemma 8 we can replace $K'_h(jh,kh,x,y) + M'_h(jh,kh,x,y)$ by $K_h(jh,kh,x,y) + M_h(jh,kh,x,y)$ where $K_h(jh,kh,x,y) = (L-\widetilde{L})\widetilde{p}_h(jh,kh,x,y)$, $M_h(jh,kh,x,y) = \frac{h}{2}(L_\star^2 - 2L\widetilde{L} + \widetilde{L}^2)\lambda(x) + M''_h$, $M''_h = M_{h,1}(jh,kh,x,y) + M_{h,2}(jh,kh,x,y) + M''_{h,3}(jh,kh,x,y)$ and

$$\max\{|M'_h(jh,kh,x,y)|, |M_h(jh,kh,x,y)|\} \le C\rho^{-1}\zeta_\rho(y-x),$$

$\rho^2 = kh - jh$. To show this we note that

$$\widetilde{p}_h(jh, kh, x, y) = \int q(jh, y, \theta)\lambda(x + \widetilde{h}(\theta))d\theta,$$

where $\widetilde{h}(\theta) = m(jh, y)h + h^{1/2}\theta$. From the Taylor expansion we get

$$\begin{aligned}\widetilde{p}_h(jh, kh, x, y) &= \lambda(x) + h\widetilde{L}\lambda(x) + h^2 \sum_{|\mu|=2} \frac{m^\mu(jh, y)}{\mu!} D^\mu \lambda(x) \\ &\quad +3 \sum_{|\mu|=3} \int_0^1 (1-\delta)^2 d\delta \int q(jh, y, \theta) \frac{\widetilde{h}(\theta)^\mu}{\mu!} D^\mu \lambda(x + \delta \widetilde{h}(\theta)) d\theta\end{aligned}$$

and, hence,

$$\begin{aligned}K_h'(jh, kh, x, y) &= K_h(jh, kh, x, y) + (L - \widetilde{L})[\lambda(x) - \widetilde{p}_h(jh, kh, x, y)] \\ &= K_h(jh, kh, x, y) + h(\widetilde{L}^2 - L\widetilde{L})\lambda(x) + M''_{h,3}(jh, kh, x, y).\end{aligned} \quad (21)$$

From

$$\begin{aligned}h(\widetilde{L}^2 - L\widetilde{L})\lambda(x) + M'_{h,3}(jh, kh, x, y) &= h(\widetilde{L}^2 - L\widetilde{L})\lambda(x) + \frac{h}{2}(L_\star^2 - \widetilde{L}^2)\lambda(x) \\ &= \frac{h}{2}(L_\star^2 - 2L\widetilde{L} + \widetilde{L}^2)\lambda(x)\end{aligned}$$

and from the definitions of the operators $L, \widetilde{L}$ and $L_\star$ and from the Lipschitz conditions on the coefficients $m(t, x)$ and $\sigma(t, x)$ we obtain that

$$\left| \frac{h}{2}(L_\star^2 - 2L\widetilde{L} + \widetilde{L}^2)\lambda(x) \right| \le Ch\rho^{-3}\zeta_\rho(y - x). \quad (22)$$

Analogously, we have

$$\left| h^2 \sum_{|\mu|=2} \frac{m^\mu(jh, y)}{\mu!}(L - \widetilde{L})D^\mu\lambda(x) \right| \le Ch^2\rho^{-3}\zeta_\rho(y - x), \quad (23)$$

$$\left| 3 \sum_{|\mu|=3} \int_0^1 (1-\delta)^2 d\delta \int q(jh, y, \theta) \frac{\widetilde{h}(\theta)^\mu}{\mu!}(L - \widetilde{L})D^\mu \lambda(x + \delta\widetilde{h}(\theta))d\theta \right| \le Ch^{3/2}\rho^{-4}\zeta_\rho(y - x). \quad (24)$$

Now (21)-(24) imply the assertion of this remark.

Lemma 9. *The following bound holds:*

$$\begin{aligned}&\left| \sum_{r=0}^n \widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0, T, x, y) - \sum_{r=0}^n \widetilde{p}_h \otimes_h (K_h + M_h)^{(r)}(0, T, x, y) \right| \\ &\quad \le C(\varepsilon)hn^{-1/2+\varepsilon}\zeta^S_{\sqrt{T}}(y - x),\end{aligned} \quad (25)$$

where $\lim_{\varepsilon \downarrow 0} C(\varepsilon) = +\infty$.

Proof of Lemma 9. For $r = 1$ we will show that for any $\varepsilon > 0$

$$\begin{aligned}&|\widetilde{p}_h \otimes_h (K_h + M_h + R_h)(0, kh, x, y) - \widetilde{p}_h \otimes_h (K_h + M_h)(0, kh, x, y)| \\ &= |\widetilde{p}_h \otimes_h R_h(0, kh, x, y)| \le Ch^{3/2-\varepsilon}(kh)^{-1/2+\varepsilon}B(\frac{1}{2}, \varepsilon)\zeta^S_\rho(y - x), \rho^2 = kh.\end{aligned} \quad (26)$$

Clearly, to estimate $\widetilde{p}_h \otimes_h R_h(0,T,x,y)$ it is enough to estimate

$$I_1 = h^{3/2}\sum_{j=0}^{k-2} h\int \widetilde{p}_h(0,kh,x,z)(f(jh,z)-f(jh,y))D_z^{\nu}\widetilde{p}_h((j+1)h,kh,z,y)dz$$

for ν, $|\nu|=4$, and

$$\begin{aligned} I_2 &= h^2\sum_{j=0}^{k-2} h\int \widetilde{p}_h(0,jh,x,z)(f(jh,z)-f(jh,y))\int q(jh,z,\theta)\widetilde{h}^{\nu-e_k}(\theta) \\ &\quad \times\int_0^1 (1-u)^4 D_z^{\nu}\lambda(z+u\widetilde{h}(\theta))dud\theta dz \end{aligned}$$

for $\nu, |\nu|=5$, $1\le k\le d$. Here $f(t,x)$ is a function whose first and second derivatives with respect to x are continuous and bounded uniformly in t and x. After integration by parts we obtain

$$\begin{aligned} I_1 &= -h^{3/2}\sum_{j=0}^{k-2} h\int D_z^{e_l}\widetilde{p}_h(0,jh,x,z)(f(jh,z)-f(jh,y))D_z^{\nu-e_l}\widetilde{p}_h((j+1)h,kh,z,y)dz \\ &\quad +h^{3/2}\sum_{j=0}^{k-2} h\int D_z^{e_s}\widetilde{p}_h(0,jh,x,z)D_z^{e_k}f(jh,z)D_z^{\nu-e_l-e_s}\widetilde{p}_h((j+1)h,kh,z,y)dz \\ &\quad +h^{3/2}\sum_{j=0}^{k-2} h\int \widetilde{p}_h(0,jh,x,z)D_z^{e_l+e_s}f(jh,z)D_z^{\nu-e_l-e_s}\widetilde{p}_h((j+1)h,kh,z,y)dz \end{aligned}$$

for $1\le l,s\le d$. Hence,

$$|I_1| \le Ch^{3/2}\sum_{j=0}^{k-2} h\frac{1}{\sqrt{jh}(kh-jh)}\zeta_{\rho}(y-x) \le Ch^{3/2-\varepsilon}(kh)^{-1/2+\varepsilon}B(\frac{1}{2},\varepsilon)\zeta^S_{\sqrt{T}}(y-x). \tag{27}$$

In the same way after integration by parts we get with $1\le l,s\le d$.

$$\begin{aligned} I_2 &= -h^2\sum_{j=0}^{k-2} h\int_0^1 (1-u)^4 du\int d\theta (m(jh,y)h^{1/2}+\theta)^{\nu-e_l}\int D_z^{e_l}\widetilde{p}_h(0,jh,x,z)(f(jh,z)-f(jh,y)) \\ &\quad \times q(jh,z,\theta)D_z^{\nu-e_l}\widetilde{p}_h((j+1)h,kh,z+u\widetilde{h}(\theta),y)dz + h^2\sum_{j=0}^{k-2} h\int_0^1 (1-u)^4 du\int d\theta (m(jh,y)h^{1/2}+\theta)^{\nu-e_l} \\ &\quad \times\int D_z^{e_s}[\widetilde{p}_h(0,jh,x,z)D^{e_l}f(jh,z)q(jh,z,\theta)]D_z^{\nu-e_l-e_s}\widetilde{p}_h((j+1)h,kh,z+u\widetilde{h}(\theta),y)dz \\ &\quad -h^2\sum_{j=0}^{k-2} h\int_0^1 (1-u)^4 du\int d\theta (m(jh,y)h^{1/2}+\theta)^{\nu-e_l}\widetilde{p}_h(0,jh,x,z)(f(jh,z)-f(jh,y)) \\ &\quad \times D_z^{e_l}q(jh,z,\theta)D_z^{\nu-e_l}\widetilde{p}_h((j+1)h,kh,z+u\widetilde{h}(\theta),y)dz. \end{aligned} \tag{28}$$

It follows from (28) that

$$|I_2| \le Ch^{3/2-\varepsilon}(kh)^{-1/2+\varepsilon}B(\frac{1}{2},\varepsilon)\zeta^S_{\rho}(y-x). \tag{29}$$

Claim (26) follows now from (27) and (29). For $r \geq 2$ we use the identity

$$
\begin{aligned}
&\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0,T,x,y) - \widetilde{p}_h \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) \\
&= \left[\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)} - \widetilde{p}_h \otimes_h (K_h + M_h)^{(r-1)}\right] \otimes_h (K_h + M_h)(0,T,x,y) \\
&\qquad + \widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)} \otimes_h R_h(0,T,x,y) \\
&= I + II. \qquad (30)
\end{aligned}
$$

For $r = 2$ we obtain from (26) and simple estimate $|(K_h + M_h)(jh, kh, z, y)| \leq C\rho_2^{-1}\zeta^S_{\rho_2}(y - z), \rho_2^2 = kh - jh$,

$$
\begin{aligned}
|I| &= |[\widetilde{p}_h \otimes_h (K_h + M_h + R_h) - \widetilde{p}_h \otimes_h (K_h + M_h)] \otimes_h (K_h + M_h)(0, kh, x, y)| \\
&\leq C^2 h^{3/2-\varepsilon} B(\frac{1}{2}, \varepsilon) \sum_{j=0}^{k-2} h(jh)^{-1/2+\varepsilon}(kh - jh)^{-1/2} \int \zeta^S_{\rho_1}(z - x)\zeta^S_{\rho_2}(y - z)dz \\
&\leq C^2 h^{3/2-\varepsilon} B(\frac{1}{2}, \varepsilon) B(\frac{1}{2}, \varepsilon + \frac{1}{2})(kh)^{\varepsilon}\zeta^S_{\rho}(y - x)
\end{aligned}
$$

with $\rho^2 = kh$. For $r \geq 3$ we obtain by induction

$$
\begin{aligned}
|I| &= \left|\left[\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)} - \widetilde{p}_h \otimes_h (K_h + M_h)^{(r-1)}\right] \otimes_h (K_h + M_h)(0, kh, x, y)\right| \\
&\leq C^r h^{3/2-\varepsilon} B(\frac{1}{2}, \varepsilon) B(\frac{1}{2}, \varepsilon + \frac{1}{2}) ... B(\frac{1}{2}, \varepsilon + \frac{r-1}{2})(kh)^{\varepsilon + (r-2)/2}\zeta^S_{\rho}(y - x) \\
&\leq \Gamma(\varepsilon) h^{3/2-\varepsilon} \frac{[C\Gamma(1/2)]^r}{\Gamma(\varepsilon + \frac{r}{2})}(kh)^{\varepsilon + (r-2)/2}\zeta^S_{\rho}(y - x) \qquad (31)
\end{aligned}
$$

with $\rho^2 = kh$. To estimate II we use the following estimates

$$
\left|D^a_y D^b_x \widetilde{p}_h(jh, kh, x, y)\right| \leq C\rho^{-|a|-|b|}\zeta^{S-|a|}_{\rho}(y - x), \left|D^b_x \widetilde{p}_h(jh, kh, x, x + v)\right| \leq C\zeta^S_{\rho}(v), \qquad (32)
$$

$$
\left|D^b_x (K_h + M_h + R_h)(jh, kh, x + v, x)\right| \leq C\rho^{-1}\zeta^S_{\rho}(v). \qquad (33)
$$

The inequalities (32) and (33) are obtained by using the same arguments as is the proof of Lemma 7. Using these inequalities and mimicking the proof of Theorem 2.3 in Konakov and Mammen (2002) we obtain the following bounds for $r = 0, 1, ...$

$$
\begin{aligned}
&\left|D^b_x D^a_y \widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0, kh, x, y)\right| \\
&\quad \leq C^r (kh)^{-|a|-|b|+r} B(\frac{1}{2}, \frac{1}{2}) B(1, \frac{1}{2}) ... B(\frac{r}{2}, \frac{1}{2})\zeta^{S-|a|}_{\rho}(y - x) \\
&\quad \leq \frac{[C\Gamma(1/2)]^r}{\Gamma(\frac{r+1}{2})}(kh)^{-|a|-|b|+r}\zeta^{S-|a|}_{\rho}(y - x). \qquad (34)
\end{aligned}
$$

Inequality (34) allows us to estimate $II = [\widetilde{p}_h \otimes_h (K_h + M''_h + R_h)^{(r-1)}] \otimes_h R_h(0, kh, x, y)$. For this it is enough to estimate

$$
\begin{aligned}
&h^{3/2} \sum h \int [\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)}](0, jh, x, z) \\
&\quad \times D^{\nu} \widetilde{p}_h((j + 1)h, kh, z, y)(f(jh, z) - f(jh, y))dz \qquad (35)
\end{aligned}
$$

for $r \geq 2$,$|\nu| = 4$, and

$$\sum_{j=0}^{n-2} h \int [\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)}](0, jh, x, z)(f(jh, z) - f(jh, y))$$
$$\times \int q(jh, z, \theta)\widetilde{h}^{\nu - e_l}(\theta) \int_0^1 (1-u)^4 D^\nu \widetilde{p}_h((j+1)h, kh, z + u\widetilde{h}(\theta), y) du d\theta dz \tag{36}$$

for $r \geq 2$, $|\nu| = 5$, $1 \leq l \leq d$. Here $f(t, x)$ is a function whose first and second derivatives with respect to x are continuous and bounded uniformly in t and x. The upper bound for (35) follows from (34) by integration by parts exactly in the same way as it was done to obtain the upper bound for I_1, see (27). This gives the estimate

$$h^{3/2} \Big| \sum h \int [\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)}](0, jh, x, z)$$
$$\times D^\nu \widetilde{p}_h((j+1)h, kh, z, y)(f(jh, z) - f(jh, y)) dz \Big|$$
$$\leq \Gamma(\varepsilon) h^{3/2 - \varepsilon} \frac{[C\Gamma(1/2)]^r}{\Gamma(\frac{r+1}{2})} (kh)^{\varepsilon + (r-2)/2} \zeta_\rho^S (y - x). \tag{37}$$

The upper bound for (36) follows from (34) by integration by parts in the same way as it was done to obtain an upper bound for I_2, see (29). This gives for (36) the same estimate as in (37) and, hence,

$$|II| \leq C\Gamma(\varepsilon) h^{3/2 - \varepsilon} \frac{[C\Gamma(1/2)]^r}{\Gamma(\frac{r+1}{2})} (kh)^{\varepsilon + (r-2)/2} \zeta_\rho^S (y - x). \tag{38}$$

The assertion of the lemma follows now from (26), (30), (31) and (38).

Lemma 10. *Let $A(s, t, x, y), B(s, t, x, y), C(s, t, x, y)$ be some functions with absolute value less than $C(t-s)^{-1/2} \zeta^\kappa_{\sqrt{t-s}}(y - x)$ for a constant C and an integer $\kappa \geq S'd$. Then*

$$\sum_{r=0}^{\infty} A \otimes_h (B + C)^{(r)}(ih, jh, x, y) - \sum_{r=0}^{\infty} A \otimes_h B^{(r)}(ih, jh, x, y)$$
$$= \sum_{r=1}^{\infty} [A \otimes_h \Phi] \otimes_h [C \otimes_h \Phi]^{(r)} (ih, jh, x, y),$$

where $\Phi = \sum_{r=0}^{\infty} B^{(r)}$.

Proof of Lemma 10. Under the conditions of the lemma all series are absolutely convergent. The assertion of this lemma is a consequence of the linearity of the operation $\otimes_h$ and of the possibility to permutate the terms in absolutely convergent series.

5 Proof of Theorem 1.

We now come to the proof of Theorem 1. Main tools for the proof have been given in Subsections 3.1, 3.2, 4.1 and 4.2. From Lemmas 1 and 2 we get that

$$p(0, T, x, y) = \sum_{r=0}^{n} \tilde{p} \otimes H^{(r)}(0, T, x, y) + o(h^2 T)\phi_{C, \sqrt{T}}(y - x).$$

With Lemma 3 this gives

$$p(0,T,x,y) - p_h(0,T,x,y) = T_1 + ... + T_7 + o(h^2 T)\phi_{C,\sqrt{T}}(y-x), \tag{39}$$

where

$$\begin{aligned}
T_1 &= \sum_{r=0}^{n} \tilde{p} \otimes H^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p} \otimes_h H^{(r)}(0,T,x,y),\\
T_2 &= \sum_{r=0}^{n} \tilde{p} \otimes_h H^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p} \otimes_h (H + M_h'' + \sqrt{h}N_1)^{(r)}(0,T,x,y),\\
T_3 &= \sum_{r=0}^{n} \tilde{p} \otimes_h (H + M_h'' + \sqrt{h}N_1)^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1)^{(r)}(0,T,x,y),\\
T_4 &= \sum_{r=0}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1)^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,T,x,y),\\
T_5 &= \sum_{r=0}^{n} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p}_h \otimes_h (K_h + M_h)^{(r)}(0,T,x,y),\\
T_6 &= \sum_{r=0}^{n} \tilde{p}_h \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0,T,x,y),\\
T_7 &= \sum_{r=0}^{n} \tilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p}_h \otimes_h H_h^{(r)}(0,T,x,y).
\end{aligned}$$

Here we put $N_1(s,t,x,y) = (L - \widetilde{L})\widetilde{\pi}_1(s,t,x,y)$.

We now discuss the asymptotic behaviour of the terms $T_1, ..., T_7$.

Asymptotic treatment of the term T_1.

We start from the recurrence relations for $r = 1, 2, 3, ...$

$$\begin{aligned}
&\left(\widetilde{p} \otimes H^{(r)}\right)(0,jh,x,y) - \left(\widetilde{p} \otimes_h H^{(r)}\right)(0,jh,x,y)\\
&= \left[\left(\widetilde{p} \otimes H^{(r-1)}\right) \otimes H - \left(\widetilde{p} \otimes H^{(r-1)}\right) \otimes_h H\right](0,jh,x,y)\\
&\quad + \left[\left(\widetilde{p} \otimes H^{(r-1)}\right) - \left(\widetilde{p} \otimes_h H^{(r-1)}\right)\right] \otimes_h H\,(0,jh,x,y)\,.
\end{aligned} \tag{40}$$

By summing up the identities in (40) from $r = 1$ to ∞ and by using the linearity of the operations $\otimes$ and $\otimes_h$ we get

$$\begin{aligned}
(p - p^d)\,(0,jh,x,y) &= (p \otimes H - p \otimes_h H)\,(0,jh,x,y)\\
&\quad + (p - p^d) \otimes_h H\,(0,jh,x,y)\,,
\end{aligned} \tag{41}$$

where we put

$$p^d(ih, i'h, x, y) = \sum_{r=0}^{\infty} (\widetilde{p} \otimes_h H^{(r)})(ih, i'h, x, y). \tag{42}$$

By iterative application of (41) we obtain

$$(p - p^d)\,(0,jh,x,y) = (p \otimes H - p \otimes_h H)\,(0,jh,x,y)$$

$$+\left(p\otimes H-p\otimes_h H\right)\otimes_h \Phi\left(0,jh,x,y\right), \tag{43}$$

where $\Phi(ih,i'h,z,z')=H(ih,i'h,z,z')+H\otimes_h H(ih,i'h,z,z')+...=\sum_{r=1}^{\infty}H^{(r)}(ih,i'h,z,z')$.

By application of a Taylor expansion we get

$$\begin{aligned}
&\left(p\otimes H-p\otimes_h H\right)(0,jh,x,z)\\
&=\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}du\int_{R^d}\left[\lambda\left(u\right)-\lambda\left(ih\right)\right]dv\\
&=\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)du\int_{R^d}\lambda'(ih)dv\\
&+\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}\lambda''(s)\mid_{s=s_i}dvd\delta du,
\end{aligned} \tag{44}$$

where $\lambda\left(u\right)=p(0,u,x,v)H(u,jh,v,z)$, $s_i=s_i(u,i,\delta,h)=ih+\delta(u-ih)$.

Note that

$$\begin{aligned}
&\int_{R^d}\lambda'(ih)dv=\int_{R^d}\frac{\partial}{\partial s}p(0,s,x,v)\mid_{s=ih}H(ih,jh,v,z)dv\\
&+\int_{R^d}p(0,ih,x,v)\frac{\partial}{\partial s}H(s,jh,v,z)\mid_{s=ih}dv=\int_{R^d}L^tp(0,ih,x,v)\\
&\times(L-\widetilde{L})\widetilde{p}(ih,jh,v,z)dv-\int_{R^d}p(0,ih,x,v)[(L-\widetilde{L})\widetilde{L}\widetilde{p}(ih,jh,v,z)\\
&-H_1(ih,jh,v,z)]dv=\int_{R^d}p(0,ih,x,v)H_1(ih,jh,v,z)dv\\
&+\int_{R^d}p(0,ih,x,v)(L^2-2L\widetilde{L}+\widetilde{L}^2)\widetilde{p}(ih,jh,v,z)dv,
\end{aligned} \tag{45}$$

where $H_1(s,t,v,z)$ is defined below in (53). We get from (45)

$$\begin{aligned}
&\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)du\int_{R^d}\lambda'(ih)dv=\frac{h}{2}(p\otimes_h H_1)(0,jh,x,z)\\
&+\frac{h}{2}(p\otimes_h A_0)(0,jh,x,z),
\end{aligned} \tag{46}$$

where $A_0(s,jh,v,z)=(L^2-2L\widetilde{L}+\widetilde{L}^2)\widetilde{p}(s,jh,v,z)$. The direct calculation shows that

$$\begin{aligned}
&A_0(s,jh,v,z)=\frac{1}{4}\sum_{p,q,r,l=1}^{d}(\sigma_{pq}(s,v)-\sigma_{pq}(s,z))(\sigma_{rl}(s,v)-\sigma_{rl}(s,z))\\
&\times\frac{\partial^4\widetilde{p}(s,jh,v,z)}{\partial v_p\partial v_q\partial v_r\partial v_l}+\sum_{p,q,r=1}^{d}(\sigma_{pq}(s,v)-\sigma_{pq}(s,z))(m_r(s,v)-m_r(s,z))\\
&\times\frac{\partial^3\widetilde{p}(s,jh,v,z)}{\partial v_p\partial v_q\partial v_r}+\frac{1}{2}\sum_{p,q,r,l=1}^{d}\sigma_{pq}(s,v)\frac{\partial\sigma_{rl}(s,v)}{\partial v_p}\frac{\partial^3\widetilde{p}(s,jh,v,z)}{\partial v_q\partial v_r\partial v_l}+(\leq 2),
\end{aligned} \tag{47}$$

where we denote by (≤ 2) the sum of terms containing the derivatives of $\widetilde{p}(s,jh,v,z)$ of the order less or equal than 2. Note that for a constant $C<\infty$ and any $0<\varepsilon<\frac{1}{2}$

$$\left|\frac{h}{2}(p\otimes_h H_1)(0,jh,x,z)\right| \leq Ch\phi_{C,\sqrt{jh}}\,(z-x)\,,$$

$$\left|\frac{h}{2}(p\otimes_h A_0)(0,jh,x,z)\right| \leq C(\varepsilon)h^{1/2}j^{-(1/2-\varepsilon)}\phi_{C,\sqrt{jh}}\,(z-x)\,. \tag{48}$$

First inequality (48) follows from (B1) and the well know estimates for the diffusion density p and for the kernel H_1 . The second inequality (48) follows from (B1), (47) and the following estimate

$$\begin{aligned}
&\frac{h}{2}\sum_{i=0}^{j-1}h\left|\int_{R^d}p(0,ih,x,v)\frac{\partial^3\widetilde{p}(ih,jh,v,z)}{\partial v_q\partial v_r\partial v_l}dv\right| \\
&\leq \frac{h^3}{2}\left|\frac{\partial^3\widetilde{p}(0,jh,x,z)}{\partial v_q\partial v_r\partial v_l}\right| + \frac{h}{2}\sum_{i=1}^{j-1}h\left|\int_{R^d}\frac{\partial p(0,ih,x,v)}{\partial v_q}\frac{\partial^2\widetilde{p}(ih,jh,v,z)}{\partial v_r\partial v_l}dv\right| \\
&\leq \frac{h^3}{2}\left|\frac{\partial^3\widetilde{p}(0,jh,x,z)}{\partial v_q\partial v_r\partial v_l}\right| + Ch^{1/2}j^{-(1/2-\varepsilon)}B(\frac{1}{2},\varepsilon)\phi_{C,\sqrt{jh}}\,(z-x)\,.
\end{aligned} \tag{49}$$

Now we shall estimate the second summand in the right hand side of (44). Clearly

$$\begin{aligned}
\lambda''(s) &= \frac{\partial^2}{\partial s^2}p(0,s,x,v)H(s,jh,v,z) + 2\frac{\partial}{\partial s}p(0,s,x,v) \\
&\times\frac{\partial}{\partial s}H(s,jh,v,z) + p(0,s,x,v)\frac{\partial^2}{\partial s^2}H(s,jh,v,z).
\end{aligned} \tag{50}$$

Using forward and backward Kolmogorov equations we get from (50) after long but simple calculations

$$\begin{aligned}
&\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}\lambda''(s)\mid_{s=s_i}dvd\delta du \\
&= \sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\sum_{k=1}^{4}\int_{R^d}p(0,s,x,v)A_k(s,jh,v,z)\mid_{s=s_i}dvd\delta du,
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
&A_1(s,jh,v,z) = (L^3 - 3L^2\widetilde{L} + 3L\widetilde{L}^2 - \widetilde{L}^3)\widetilde{p}(s,jh,v,z), \\
&A_2 = (L_1H + 2LH_1)(s,jh,v,z), \\
&A_3(s,jh,v,z) = [(L-\widetilde{L})\widetilde{L}_1 + 2(L_1-\widetilde{L}_1)\widetilde{L}]\widetilde{p}(s,jh,v,z), \\
&A_4(s,jh,v,z) = H_2(s,jh,v,z).
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&H_l(s,t,v,z) = (L_l - \widetilde{L}_l)\widetilde{p}(s,t,v,z) \\
&= \frac{1}{2}\sum_{i,j=1}^{d}\left(\frac{\partial^l\sigma_{ij}(s,v)}{\partial s^l} - \frac{\partial^l\sigma_{ij}(s,z)}{\partial s^l}\right)\frac{\partial^2\widetilde{p}(s,t,v,z)}{\partial v_i\partial v_j} \\
&+\sum_{i=1}^{d}\left(\frac{\partial^l m_i(s,v)}{\partial s^l} - \frac{\partial^l m_i(s,z)}{\partial s^l}\right)\frac{\partial\widetilde{p}(s,t,v,z)}{\partial v_i}, \quad l=1,2.
\end{aligned} \tag{53}$$

Using integration by parts and the definition (52) of A_2, A_3 and A_4 it is easy to get that for any $0<\varepsilon<1/2$ and for $k=2,3,4$

$$\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}p(0,s,x,v)A_k(s,jh,v,z)\mid_{s=s_i}dvd\delta du\right|$$

$$\leq C(\varepsilon)h^{3/2-\varepsilon}\phi_{C,\sqrt{jh}}(z-x)\,. \tag{54}$$

For $k=1$ we shall prove the following estimate for any $0<\varepsilon<\frac{1}{2}$

$$\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}p(0,s,x,v)A_1(s,jh,v,z)\mid_{s=s_i}dvd\delta du\right|$$

$$\leq C(\varepsilon)hj^{-(1/2-\varepsilon)}\phi_{C,\sqrt{jh}}(z-x)\,. \tag{55}$$

Note that the function $A_1(s,jh,v,z)$ can be written as the following sum

$$A_1(s,jh,v,z)=\frac{1}{8}\sum_{i,j,p,q,l,r=1}^{d}(\sigma_{ij}(s,v)-\sigma_{ij}(s,z))(\sigma_{pq}(s,v)-\sigma_{pq}(s,z))(\sigma_{lr}(s,v)$$

$$-\sigma_{lr}(s,z))\frac{\partial^6\widetilde{p}(s,jh,v,z)}{\partial v_i\partial v_j\partial v_p\partial v_q\partial v_l\partial v_r}+\frac{3}{4}\sum_{i,j,p,q,l=1}^{d}(\sigma_{ij}(s,v)-\sigma_{ij}(s,z))(\sigma_{pq}(s,v)$$

$$-\sigma_{pq}(s,z))(m_l(s,v)-m_l(s,z))\frac{\partial^5\widetilde{p}(s,jh,v,z)}{\partial v_i\partial v_j\partial v_p\partial v_q\partial v_l}+\frac{3}{4}\sum_{i,j,p,q,l,r=1}^{d}\sigma_{ij}(s,v)\frac{\partial\sigma_{pq}(s,v)}{\partial v_i}$$

$$(\sigma_{lr}(s,v)-\sigma_{lr}(s,z))\frac{\partial^5\widetilde{p}(s,jh,v,z)}{\partial v_j\partial v_p\partial v_q\partial v_l\partial v_r}+(\leq 4), \tag{56}$$

where we denote by (≤ 4) the sum of terms containing the derivatives of $\widetilde{p}(s,jh,v,z)$ of the order less or equal than 4. By (B1) and (56) it is clear that the estimate for the left hand side of (54) for $k=1$ will be the same up to a constant as for the following sum for fixed p,q,r,l

$$\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}p(0,s,x,v)\frac{\partial^4\widetilde{p}(s,jh,v,z)}{\partial v_p\partial v_q\partial v_l\partial v_r}\mid_{s=s_i}dvd\delta du\right|$$

After integration by parts w.r.t. v_p and with the substitution $hw=(u-ih)$ in each integral we obtain

$$\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}p(0,s,x,v)\frac{\partial^4\widetilde{p}(s,jh,v,z)}{\partial v_p\partial v_q\partial v_l\partial v_r}\mid_{s=s_i}dvd\delta du\right|$$

$$=\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}\frac{\partial p(0,s,x,v)}{\partial v_p}\frac{\partial^3\widetilde{p}(s,jh,v,z)}{\partial v_q\partial v_l\partial v_r}\mid_{s=s_i}dvd\delta du\right|$$

$$\leq Ch^2\phi_{C,\sqrt{jh}}(z-x)\int_0^1w^2\int_0^1(1-\delta)\sum_{i=0}^{j-1}h\frac{1}{\sqrt{ih+\delta hw}}\frac{1}{[(j-i)h-\delta hw)]^{3/2}}d\delta dw$$

$$\leq Ch^{3/2-\varepsilon}\phi_{C,\sqrt{jh}}(z-x)\int_0^1w^2\int_0^1(1-\delta)^{1/2-\varepsilon}\sum_{i=0}^{j-1}h\frac{1}{\sqrt{ih+\delta hw}}\frac{1}{[(j-\delta w)h-ih)]^{1-\varepsilon}}d\delta dw$$

$$\leq Ch^{3/2-\varepsilon}\phi_{C,\sqrt{jh}}(z-x)\int_0^1 w^2dw\int_0^1(1-\delta)^{1/2-\varepsilon}d\delta\int_0^{(j-1)h}\frac{dt}{\sqrt{t}[(j-1)h-t]^{1-\varepsilon}}$$
$$\leq Chj^{-(1/2-\varepsilon)}B(\frac{1}{2},\varepsilon)\phi_{C,\sqrt{jh}}(z-x)\,, \tag{57}$$

where $B(p,q)$ is a Beta function and $\phi_{C,\rho}(z-x)$ is defined in Lemma 2. As we mentioned above (55) follows now from (57). By (B2), (44), (46), (47), (49), (54) and (55) we obtain for any $0<\varepsilon<\frac{1}{2}$ and $j=1,2,...n$

$$|(p\otimes H-p\otimes_h H)(0,T,x,z)|\leq C(\varepsilon)h^{1/2}n^{-(1/2-\varepsilon)}\phi_{C,\sqrt{T}}(z-x). \tag{58}$$

We use now the following estimate for $\Phi(ih,i'h,z,z')$ that was proved in Konakov and Mammen (2002) (formula (5.7) on page 284)

$$|\Phi(ih,i'h,z,z')|\leq C\frac{1}{\sqrt{i'h-ih}}\phi_{C,\sqrt{i'h-ih}}(z'-z)\,. \tag{59}$$

From (B2), (44), (46), (57), (58) and (59) we obtain the following representation

$$(p-p^d)(0,T,x,y)=\frac{h}{2}(p\otimes_h H_1)(0,T,x,y)+\frac{h}{2}(p\otimes_h A_0)(0,T,x,y)$$
$$+\frac{h}{2}(p\otimes_h H_1\otimes_h\Phi)(0,T,x,y)+\frac{h}{2}(p\otimes_h A_0\otimes_h\Phi)(0,T,x,y)$$
$$+R(0,T,x,y), \tag{60}$$

where for any $0<\varepsilon<1/2$

$$|R(0,T,x,y)|\leq C(\varepsilon)(h^{3/2-\varepsilon}+hn^{-(1/2-\varepsilon)})\phi_{C,\sqrt{T}}(y-x)$$
$$=\phi_{C,\sqrt{T}}(y-x)\,o(h^{1+\delta}).$$

This representation implies that

$$\begin{aligned} T_1 \;&=\; \frac{h}{2}[p\otimes_h(L^2-2L\widetilde{L}+\widetilde{L}^2)\widetilde{p}\otimes_h\Phi)](0,T,x,y)\\ &\quad+\frac{h}{2}[p\otimes_h(L'-\widetilde{L}')\widetilde{p}\otimes_h\Phi](0,T,x,y)+R_T(0,T,x,y), \end{aligned} \tag{61}$$

where for any $0<\varepsilon<1/2$

$$|R_T(0,T,x,y)|\leq C(\varepsilon)hn^{-1/2+\varepsilon}\phi_{C,\sqrt{T}}(y-x)\leq C(\varepsilon)h^{1+\delta}\phi_{C,\sqrt{T}}(y-x) \tag{62}$$

for $\delta>0$ small enough and where $\Phi(s,t,x,y)=\sum_{r=0}^{\infty}H^{(r)}(s,t,x,y)$. Here the summand $H^{(0)}(s,t,x,y)$ is introduced to shorten the notation. By definition we suppose that $g\otimes_h H^{(0)}(s,t,x,y)=g(s,t,x,y)$ for a function g. Note, that in the homogenous case $\sigma_{ij}(s,x)=\sigma_{ij}(x), m_i(s,x)=m_i(x)$ and thus the second summand in (61) is equal to 0.

Asymptotic treatment of the term T_2. We will show that

$$\Bigg|T_2-3\sum_{r=0}^{\infty}\tilde{p}\otimes_h H^{(r)}(0,T,x,y)+\sum_{r=0}^{\infty}\tilde{p}\otimes_h(H+M_{h,1}+\sqrt{h}N_1)^{(r)}(0,T,x,y)$$
$$+\sum_{r=0}^{\infty}\tilde{p}\otimes_h(H+M_{h,2})^{(r)}(0,T,x,y)+\sum_{r=0}^{\infty}\tilde{p}\otimes_h(H+M''_{h,3})^{(r)}(0,T,x,y)\Bigg|$$
$$\leq Chn^{-\delta}\zeta_{\sqrt{T}}(y-x) \tag{63}$$

with some positive $\delta > 0$. Note that it is enough to consider the case $r \geq 2$ because for $r = 1, 2$ the left hand side of (63) is equal to zero. Note that (63) immediately follows from the following bounds for $r = 2, 3, ...$

$$\begin{aligned}
&\Big|\tilde{p} \otimes_h (H + M''_h + \sqrt{h}N_1)^{(r)}(0, T, x, y) \\
&\quad -\tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r)}(0, T, x, y) \\
&\quad -[\tilde{p} \otimes_h (H + M''_{h,3})^{(r)} - \tilde{p} \otimes_h H^{(r)}](0, T, x, y)\Big| \\
&\leq C(\varepsilon)h^{3/2-2\varepsilon} \frac{C^r}{\Gamma(\frac{r-1}{2})} T^{3\varepsilon+\frac{r-3}{2}} \zeta_{\sqrt{kh}}(v - x),
\end{aligned} \tag{64}$$

and

$$\begin{aligned}
&\Big|\tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r)}(0, T, x, y) \\
&\quad -\tilde{p} \otimes_h (H + M_{h,1} + \sqrt{h}N_1)^{(r)}(0, T, x, y) \\
&\quad -[\tilde{p} \otimes_h (H + M_{h,2})^{(r)} - \tilde{p} \otimes_h H^{(r)}](0, T, x, y)\Big| \\
&\leq C(\varepsilon)h^{3/2-2\varepsilon} \frac{C^r}{\Gamma(\frac{r-1}{2})} T^{3\varepsilon+\frac{r-4}{2}} \zeta_{\sqrt{T}}(v - x)
\end{aligned} \tag{65}$$

for all sufficiently small $\varepsilon > 0$ with a constant $C(\varepsilon)$ that fulfills $\lim_{\varepsilon \longrightarrow 0} C(\varepsilon) = +\infty$. First we prove the bound (64). Denote the expression under the sign of the absolute value in (64) by Γ_r. Note that $\Gamma_0 = \Gamma_1 = 0$. For $r \geq 2$ we make use of the following recurrence formula

$$\begin{aligned}
\Gamma_r &= \Gamma_{r-1} \otimes_h H + \Big[\tilde{p} \otimes_h (H + M''_h + \sqrt{h}N_1)^{(r-1)} \\
&\quad -\tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r-1)}\Big] \otimes_h (M''_h + \sqrt{h}N_1) \\
&\quad + \Big[\tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r-1)} - \tilde{p} \otimes_h (H + M''_{h,3})^{(r-1)}\Big] \otimes_h M''_{h,3} \\
&= I + II + III.
\end{aligned} \tag{66}$$

We start with bounding II . First we will give an estimate for

$$\Big|\tilde{p} \otimes_h (H + M''_h + \sqrt{h}N_1)^{(r-1)} - \tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r-1)}\Big|. \tag{67}$$

For $r = 2$ we have $\tilde{p} \otimes_h M''_{h,3}(0, kh, x, y)$. It follows from (20) that it is enough to estimate

$$J_1 = h^2 \sum_{i=0}^{k-2} h \int \widetilde{p}(0, ih, x, v)(f(ih, v) - f(ih, y))D_v^\nu \widetilde{p}_h((i + 1)h, kh, v, y)dv \tag{68}$$

for $|\nu| = 4$ and

$$\begin{aligned}
J_2 = h^{3/2} \sum_{i=0}^{k-2} h \int \widetilde{p}(0, ih, x, v)(f(ih, v) - f(ih, y)) \int q(ih, v, \theta)\theta^\nu \int_0^1 (1 - \delta)^2 \\
\times D_v^{\nu+e_l+e_q} \widetilde{p}_h((i + 1)h, kh, v + \delta\widetilde{h}(\theta), y)d\delta d\theta dv
\end{aligned} \tag{69}$$

for $|\nu| = 3$. Here $f(t, x)$ is a function with $D_x^\nu f(t, x), |\nu| = 0, 1, 2, 3$ bounded uniformly in (t, x). An estimate for J_1 follows from (27). This gives

$$|J_1| \leq Ch^{2-\varepsilon}(kh)^{\varepsilon-1/2} B(\frac{1}{2}, \varepsilon)\zeta^S_{\sqrt{kh}}(y - x). \tag{70}$$

The estimate for J_2 can be obtained analogously to the estimate of I_2 (see (29)). By integrating by parts we get

$$J_2 = h^{3/2}\sum_{i=0}^{k-2} h\int_0^1 (1-\delta)^2 d\delta \int d\theta\cdot\theta^\nu \int D_v^{e_l+e_q}[\widetilde{p}(0,ih,x,v) \times(f(ih,v)-f(ih,y))q(ih,v,\theta)]D_v^\nu\widetilde{p}_h((i+1)h,kh,v+\delta\widetilde{h}(\theta),y)dv.$$

The derivative

$$D_v^{e_l+e_q}[\widetilde{p}(0,ih,x,v)(f(ih,v)-f(ih,y))q(ih,v,\theta)]$$

is a sum of 9 summands. By using integration by parts once more for all summands which contain $D_\nu^\mu\widetilde{p}(0,ih,x,v)$ with $|\mu|<2$, we obtain

$$|J_2| \le Ch^{3/2-2\varepsilon}\zeta^{S-3}_{\sqrt{kh}}(y-x)\int \psi(\theta)\left\|\theta\right\|^3 (h^{(S-3)/2}\left\|\theta\right\|^{S-3}+1)d\theta \sum_{i=1}^{k-2} h\frac{1}{(ih)^{1-\varepsilon}}\times\frac{1}{(kh-ih)^{1-\varepsilon}} \le Ch^{3/2-2\varepsilon}B(\varepsilon,\varepsilon)(kh)^{2\varepsilon-1}\zeta^{S-3}_{\sqrt{kh}}(y-x) \tag{71}$$

for any $\varepsilon\in(0,1/4)$. It follows from (70) and (71) that for $r=2$ (67) does not exceed $Ch^{3/2-2\varepsilon}B(\varepsilon,\varepsilon)(kh)^{\varepsilon-1}\zeta^{S-3}_{\sqrt{kh}}(y-x)$. For $r\ge 3$ we use the recurrence relation

$$\begin{aligned}
&\tilde{p}\otimes_h(H+M''_h+\sqrt{h}N_1)^{(r-1)}-\tilde{p}\otimes_h(H+M_{h,1}+M_{h,2}+\sqrt{h}N_1)^{(r-1)}\\
&=\Big[\tilde{p}\otimes_h(H+M''_h+\sqrt{h}N_1)^{(r-2)}-\tilde{p}\otimes_h(H+M_{h,1}+M_{h,2}+\sqrt{h}N_1)^{(r-2)}\Big]\\
&\quad\otimes_h(H+M''_h+\sqrt{h}N_1)+[\tilde{p}\otimes_h(H+M_{h,1}+M_{h,2}+\sqrt{h}N_1)^{(r-2)}]\otimes_h M''_{h,3}\\
&=I'+II'.
\end{aligned}\tag{72}$$

From (72) we obtain for $r=3$

$$\begin{aligned}
|I'| &\le Ch^{3/2-2\varepsilon}B(\varepsilon,\varepsilon)\zeta^{S-3}_{\sqrt{kh}}(y-x)\sum_{i=1}^{k-2}h(ih)^{\varepsilon-1}(kh-ih)^{-1/2}\\
&\le Ch^{3/2-2\varepsilon}B(\varepsilon,\varepsilon)B(\frac{1}{2},\varepsilon)(kh)^{\varepsilon-1/2}\zeta^{S-3}_{\sqrt{kh}}(y-x).
\end{aligned}$$

To estimate II' we use the following estimates

$$\left|D_v^aD_x^b(H+M_{h,1}+M_{h,2}+\sqrt{h}N_1)(jh,kh,x,v)\right|\le C\rho^{-1-|a|-|b|}\zeta_\rho(v-x), \tag{73}$$

$$\left|D_x^b(H+M_{h,1}+M_{h,2}+\sqrt{h}N_1)(jh,kh,x,x+v)\right|\le C\rho^{-1}\zeta_\rho(v-x). \tag{74}$$

To prove (73) one can use the following estimates for the summands in $M_{h,1},M_{h,2}$ and $\sqrt{h}N_1$

$$\begin{aligned}
&\left|h^{1/2}D_v^aD_x^b\left[D_x^\nu\widetilde{p}_h((j+1)h,kh,x,v)(f(jh,x)-f(jh,v))\right]\right|\le C\rho^{-1-|a|-|b|}\zeta_\rho(v-x), |\nu|=3,\\
&\left|hD_v^aD_x^b\left[D_x^\nu\widetilde{p}_h((j+1)h,kh,x,v)(f(jh,x)-f(jh,v))\right]\right|\le C\rho^{-1-|a|-|b|}\zeta_\rho(v-x), |\nu|=4,\\
&\left|h^{1/2}D_v^aD_x^b\left[D_x^{\nu+e_p+e_q}\widetilde{p}_h((j+1)h,kh,x,v)\rho^2(f(jh,x)-f(jh,v))\right]\right|\le C\rho^{-1-|a|-|b|}\zeta_\rho(v-x), |\nu|=3,
\end{aligned}$$

for a function $f(t,x)$ with $|a|+|b|$ derivatives w.r.t. x that are uniformly bounded w.r.t. t. These estimates are direct consequences of Lemma 7. To prove (74) one can use the following estimates for the summands in $M_{h,1}, M_{h,2}$ and $\sqrt{h}N_1$

$$\left|h^{1/2}D_x^b\left[D_x^\nu\widetilde{p}_h((j+1)h,kh,x,y)\mid_{y=x+v}(f(jh,x)-f(jh,x+v))\right]\right| \le C\rho^{-1}\zeta_\rho(v-x), |\nu|=3,$$
$$\left|hD_x^b\left[D_x^\nu\widetilde{p}_h((j+1)h,kh,x,y)\mid_{y=x+v}(f(jh,x)-f(jh,x+v))\right]\right| \le C\rho^{-1}\zeta_\rho(v-x), |\nu|=4,$$
$$\left|h^{1/2}D_x^b\left[D_x^{\nu+e_p+e_q}\widetilde{p}_h((j+1)h,kh,x,v)\mid_{y=x+v}\rho^2(f(jh,x)-f(jh,x+v))\right]\right| \le C\rho^{-1}\zeta_\rho(v-x), |\nu|=3.$$

Again, these estimates follow from the estimates obtained in the proof of Lemma 7. Note that with $z(V_{j,k}^{-1/2}(y)\mu_{j,k}(y),x,y)=V_{j,k}^{-1/2}(y)(y-x-\mu_{j,k}(y))$ it holds

$$\left|\frac{\partial z}{\partial y}\right| \le \frac{C}{\rho}, \left|\frac{\partial z}{\partial x}\right| \le \frac{C}{\rho}$$

and that with $z(V_{j,k}^{-1/2}(x+v),\mu_{j,k}(x+v),x,x+v)=V_{j,k}^{-1/2}(x+v)(v-\mu_{j,k}(x+v))$ it holds

$$\left|\frac{\partial z}{\partial x}\right| \le \left|\frac{\partial z}{\partial V_{j,k}^{-1/2}}\right|\left|\frac{\partial V_{j,k}^{-1/2}}{\partial x}\right| + \left|\frac{\partial z}{\partial \mu_{j,k}}\right|\left|\frac{\partial \mu_{j,k}}{\partial x}\right| \le C(\|v\|+1).$$

Now with the inequalities (73),(74) we can proceed like in the proof of Theorem 2.3 in Konakov and Mammen (2002). This gives the following estimate for $r=3,4,...$

$$\begin{aligned}&\left|D_v^aD_x^b[\widetilde{p}\otimes_h(H+M_{h,1}+M_{h,2}+\sqrt{h}N_1)^{(r-2)}](jh,kh,x,v)\right|\\ &\le C^rB(1,\frac{1}{2})\times...\times B(\frac{r-1}{2},\frac{1}{2})\rho^{r-2-|a|-|b|}\zeta_\rho(v-x).\end{aligned} \tag{75}$$

Now we denote $\widetilde{p}_{1,r}=\widetilde{p}\otimes_h(H+M_{h,1}+M_{h,2}+\sqrt{h}N_1)^{(r)}, \widetilde{p}_0=\widetilde{p}$. To estimate $\widetilde{p}_{1,r-2}\otimes_h M''_{h,3}$ it is enough to make the same calculations with integration by parts as it was done above for J_1 and J_2. This gives

$$\begin{aligned}|II'| &\le \left|\widetilde{p}_{1,r-2}\otimes_h M''_{h,3}(0,kh,x,y)\right| &(76)\\ &\le C^rh^{3/2-2\varepsilon}B(1,\frac{1}{2})B(1+\frac{1}{2},\frac{1}{2})\times...\times B(1+\frac{r-3}{2},\frac{1}{2})B(\frac{r-2}{2},\varepsilon)(kh)^{\varepsilon+\frac{r-4}{2}}\zeta_{\sqrt{kh}}(v-x)\end{aligned}$$

and by induction

$$|I'| \le C^rh^{3/2-2\varepsilon}B(\varepsilon,\frac{1}{2})B(\varepsilon+\frac{1}{2},\frac{1}{2})\times....\times B(\varepsilon+\frac{r-3}{2},\frac{1}{2})B(\varepsilon,\varepsilon)(kh)^{\varepsilon+\frac{r-4}{2}}\zeta_{\sqrt{kh}}(v-x), \tag{77}$$

$r=3,4,....$ Comparing (76) and (77) we obtain that for $r\ge 3$

$$\begin{aligned}&\left|\tilde{p}\otimes_h(H+M''_h+\sqrt{h}N_1)^{(r-1)}-\tilde{p}\otimes_h(H+M_{h,1}+M_{h,2}+\sqrt{h}N_1)^{(r-1)}\right|\\ &\le C^rh^{3/2-2\varepsilon}B(\varepsilon,\frac{1}{2})B(\varepsilon+\frac{1}{2},\frac{1}{2})\times...\times B(\varepsilon+\frac{r-3}{2},\frac{1}{2})B(\varepsilon,\varepsilon)(kh)^{\varepsilon+\frac{r-4}{2}}\zeta_{\sqrt{kh}}(v-x).\end{aligned} \tag{78}$$

From (78) we get the following estimate for II

$$|II| \le C^rh^{3/2-2\varepsilon}B(\varepsilon,\varepsilon)B(\varepsilon,\frac{1}{2})B(\varepsilon+\frac{1}{2},\frac{1}{2})\times...\times B(\varepsilon+\frac{r-2}{2},\frac{1}{2})T^{\varepsilon+\frac{r-3}{2}}\zeta_{\sqrt{T}}(v-x). \tag{79}$$

To estimate III note that the following inequalities that are similar to (73), (74),(75) hold for $H+M''_{h,3}$,

$$\begin{aligned}&\left|D_v^aD_x^b(H+M''_{h,3})(jh,kh,x,v)\right| \le C\rho^{-1-|a|-|b|}\zeta_\rho(v-x),\\ &\left|D_x^b(H+M''_{h,3})(jh,kh,x,x+v)\right| \le C\rho^{-1}\zeta_\rho(v-x),\\ &\left|D_v^aD_x^b[\widetilde{p}\otimes_h(H+M''_{h,3})^{(r)}](jh,kh,x,v)\right|\\ &\quad\le C^rB(1,\frac{1}{2})\times...\times B(\frac{r+1}{2},\frac{1}{2})\rho^{r-|a|-|b|}\zeta_\rho(v-x).\end{aligned} \tag{80}$$

To prove the last three inequalities it is enough to get the corresponding estimates for summands in $M''_{h,3}$ (see (20)). These estimates can be proved by the same arguments as used in the proofs of (73), (74), and (75). To estimate III we have now to estimate $\widetilde{p}_{1,r} \otimes_h M''_{h,3}$ and $\widetilde{p}_{2,r} \otimes_h M''_{h,3}$ where

$$\widetilde{p}_{2,r} = \widetilde{p} \otimes_h (H + M''_{h,3})^{(r)}.$$

Using integration by parts and inequality (80), we obtain for $\widetilde{p}_{2,r-1} \otimes_h M''_{h,3}$ the same estimate as for $\widetilde{p}_{1,r} \otimes_h M''_{h,3}$

$$\begin{aligned}&\left|\widetilde{p}_{2,r} \otimes_h M''_{h,3}(0, kh, x, y)\right| \\ &\leq C^r h^{3/2-2\varepsilon} B(1, \frac{1}{2}) \times ... \times B(\frac{r+1}{2}, \frac{1}{2}) B(\frac{r}{2}, \varepsilon)(kh)^{\varepsilon + \frac{r-2}{2}} \zeta_{\sqrt{kh}}(y - x)\end{aligned}$$

for $i = 1, 2$. Hence for $r = 2, 3, ...$

$$\begin{aligned}|III| &\leq \left|\widetilde{p}_{1,r-1} \otimes_h M''_{h,3}(0, kh, x, y)\right| + \left|\widetilde{p}_{2,r-1} \otimes_h M''_{h,3}(0, kh, x, y)\right| \\ &\leq C^r h^{3/2-2\varepsilon} B(1, \frac{1}{2}) \times ... \times B(\frac{r}{2}, \frac{1}{2}) B(\frac{r-1}{2}, \varepsilon)(kh)^{\varepsilon + \frac{r-3}{2}} \zeta_{\sqrt{kh}}(y - x). \end{aligned} \tag{81}$$

From (66), (79) and (81) we get for $r = 2, 3, ...$

$$|\Gamma_r(0, kh, x, y)| \leq C^r h^{3/2-2\varepsilon} B(\varepsilon, \varepsilon) B(\varepsilon, \frac{1}{2}) \times .... \times B(\varepsilon + \frac{r-2}{2}, \frac{1}{2})(kh)^{\varepsilon + \frac{r-3}{2}} \zeta_{\sqrt{kh}}(v - x).$$

In particular,

$$|\Gamma_r(0, T, x, y)| \leq h^{3/2-2\varepsilon} \frac{\Gamma^3(\varepsilon)}{\Gamma(2\varepsilon)} \frac{C^r}{\Gamma(\varepsilon + \frac{r-1}{2})} T^{\varepsilon + \frac{r-3}{2}} \zeta_{\sqrt{kh}}(v - x), r = 2, 3... \tag{82}$$

for any $\varepsilon \in (0, 1/4)$. Now we estimate the left hand side of (65). Denote the expression under the sign of the absolute value in (65) by F_r. Note that $F_0 = F_1 = 0$. For $r \geq 2$ we make use of the following recurrence formula

$$\begin{aligned}F_r &= F_{r-1} \otimes_h H + \Big[\widetilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h} N_1)^{(r-1)} \\ &\quad - \widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-1)}\Big] \otimes_h (M_{h,1} + M_{h,2} + \sqrt{h} N_1) \\ &\quad + \Big[\widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-1)} - \widetilde{p} \otimes_h (H + M_{h,1})^{(r-1)}\Big] \otimes_h M_{h,2} \\ &= I + II + III.\end{aligned}$$

We start again from the estimation of

$$A_{r-1} = \widetilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h} N_1)^{(r-1)} - \widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-1)}.$$

For $r = 2$ we have $A_1 = (\widetilde{p} \otimes_h M_{h,2})(0, kh, x, y)$. It is enough to estimate

$$J_3 = h \sum_{i=0}^{k-2} h \int \widetilde{p}(0, ih, x, v)(f(ih, v) - f(ih, y)) D_v^\nu \widetilde{p}_h((i+1)h, kh, v, y) dv$$

for $|\nu| = 4$. Analogously to (27) we obtain that

$$|J_3| \leq C h^{1-\varepsilon}(kh)^{-1/2+\varepsilon} B(\frac{1}{2}, \varepsilon) \zeta^S_{\sqrt{kh}}(y - x).$$

and, hence,

$$|A_1| \leq C h^{1-\varepsilon} (kh)^{-1/2+\varepsilon} B(\frac{1}{2}, \varepsilon) \zeta^S_{\sqrt{kh}}(y-x). \tag{83}$$

For $r \geq 3$ we use the recurrence relation

$$\begin{aligned} A_{r-1} &= A_{r-2} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h} N_1) \\ &\quad + \left[\widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-2)} \right] \otimes_h M_{h,2} \\ &= I' + II'. \end{aligned} \tag{84}$$

From (83) and (84) we obtain for $r = 3$

$$\begin{aligned} |I'| &\leq C h^{1-\varepsilon} B(\frac{1}{2}, \varepsilon) \zeta^S_{\sqrt{kh}}(y-x) \sum_{i=0}^{k-2} h (ih)^{\varepsilon - 1/2} (kh - ih)^{-1/2} \\ &\leq C h^{1-\varepsilon} B(\frac{1}{2}, \varepsilon) B(\frac{1}{2}, \varepsilon + \frac{1}{2}) (kh)^{\varepsilon} \zeta^S_{\sqrt{kh}}(y-x). \end{aligned} \tag{85}$$

To estimate II' we use the following inequality for $r = 3, 4, ...$

$$\begin{aligned} &\left| D^a_v D^b_x [\widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-2)}](jh, kh, x, v) \right| \\ &\leq C^r B(1, \frac{1}{2}) \times ... \times B(\frac{r-1}{2}, \frac{1}{2}) \rho^{r-2-|a|-|b|} \zeta_\rho(v-x). \end{aligned} \tag{86}$$

This inequality follows from (75). We have

$$|II'| \leq C h^{1-\varepsilon} B(1, \varepsilon) (kh)^{\varepsilon} \zeta^S_{\sqrt{kh}}(y-x). \tag{87}$$

Comparing (85) and (87) we obtain that $|A_2| \leq C^2 h^{1-\varepsilon} B(\frac{1}{2}, \varepsilon) B(\frac{1}{2}, \varepsilon + \frac{1}{2}) (kh)^{\varepsilon} \zeta^S_{\sqrt{kh}}(y-x)$. By induction we easily get for $r = 2, 3...$

$$|A_{r-1}(0, kh, x, y)| \leq C^r h^{1-\varepsilon} B(\frac{1}{2}, \varepsilon) \times ... \times B(\frac{1}{2}, \varepsilon + \frac{r-2}{2}) (kh)^{\varepsilon + \frac{r-3}{2}} \zeta^S_{\sqrt{kh}}(y-x). \tag{88}$$

To estimate

$$A_{r-1} \otimes_h (M_{h,1} + M_{h,2} + \sqrt{h} N_1)$$

it is enough to estimate

$$J_4 = h^{1/2} \sum_{i=0}^{k-2} h \int A_{r-1}(0, ih, x, v)(f(ih, v) - f(ih, y)) D^\nu_v \widetilde{p}_h((i+1)h, kh, v, y) dv$$

for $|\nu| = 3$,

$$J_5 = h \sum_{i=0}^{k-2} h \int A_{r-1}(0, ih, x, v)(f(ih, v) - f(ih, y)) D^\nu_v \widetilde{p}_h((i+1)h, kh, v, y) dv$$

for $|\nu| = 4$, and

$$J_6 = h^{1/2} \sum_{i=0}^{k-2} h \int A_{r-1}(0, ih, x, v)(f(ih, v) - f(ih, y))(kh - ih) D^{\nu + e_p + e_q}_v \widetilde{p}(ih, kh, v, y) dv$$

for $|\nu| = 3$. It follows from (88) that

$$|J_4| \le C^r h^{3/2-2\varepsilon} B(\frac{1}{2},\varepsilon) \times ... \times B(\frac{1}{2},\varepsilon + \frac{r-2}{2}) B(\varepsilon, \varepsilon + \frac{r-1}{2})(kh)^{2\varepsilon+\frac{r-3}{2}} \zeta^S_{\sqrt{kh}}(y-x).$$

Clearly, the same estimate holds for J_5 and J_6 . Thus we obtain

$$|II| \le C^r h^{3/2-2\varepsilon} B(\frac{1}{2},\varepsilon) \times ... \times B(\frac{1}{2},\varepsilon + \frac{r-2}{2}) B(\varepsilon, \varepsilon + \frac{r-1}{2})(kh)^{2\varepsilon+\frac{r-3}{2}} \zeta^S_{\sqrt{kh}}(y-x).$$

Now we give an estimate for III. We write

$$B_{r-1} = \widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-1)} - \widetilde{p} \otimes_h (H + M_{h,1})^{(r-1)}.$$

Using the recurrence equation

$$B_{r-1} = B_{r-2} \otimes_h (H + M_{h,1} + \sqrt{h} N_1) + \widetilde{p} \otimes_h (H + M_{h,1})^{(r-2)} \otimes_h \sqrt{h} N_1, B_0 = 0$$

we obtain that

$$III = \sum_{l=0}^{r-2} \widetilde{p}_{3,l} \otimes_h \sqrt{h} N_1 \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-l-2)} \otimes_h M_{h,2}(0, T, x, y), \tag{89}$$

where $\widetilde{p}_{3,l} = \widetilde{p} \otimes_h (H + M_{h,1})^{(l)}$. To estimate III it is enough to estimate a typical term in the last sum. Thus we have to estimate

$$\begin{aligned} &h^{3/2} \sum_{k=0}^{n-2} h \int \Big\{ \sum_{j=0}^{k-1} h \Big[\int \sum_{i=0}^{j-1} h \int \widetilde{p}_{3,l}(0, ih, x, w)(jh - ih) D_w^{\mu+e_n+e_m} \widetilde{p}(ih, jh, w, z) \\ &\times (g(ih, w) - g(ih, z)) dw \Big] (H + M_{h,1} + \sqrt{h} N_1)^{(r-l-2)}(ih, kh, z, v) dz \Big\} (f(kh, v) - f(kh, y)) \\ &\times D_v^{\nu} \widetilde{p}_h((k+1)h, T, v, y) dv. \end{aligned}$$

To estimate this term we apply two times an integration by parts in the internal integral $\int ...dw$ and then we make two times an integration by parts in $\int ...dv$. We also use the following estimates

$$\begin{aligned} &\left| D_w^a D_x^b \widetilde{p}_{3,l}(0, ih, x, w) \right| \\ &\quad \le C^l B(1, \frac{1}{2}) \times ... \times B(\frac{l+1}{2}, \frac{1}{2})(ih)^{\frac{l-|a|-|b|}{2}} \zeta_{\sqrt{ih}}(w - x), \\ &\left| D_v^a D_z^b (H + M_{h,1} + \sqrt{h} N_1)^{(r-l-2)}(ih, kh, z, v) \right| \\ &\quad \le C^{r-l-2} B(\frac{1}{2}, \frac{1}{2}) \times ... \times B(\frac{1}{2}, \frac{r-l-3}{2})(kh - ih)^{\frac{r-l-4-|a|-|b|}{2}} \zeta_{\sqrt{kh-ih}}(v - z) \end{aligned}$$

for $0 \le l \le r-3$ with $B(\frac{1}{2}, 0) = 1$. This gives the following estimate for any $0 \le l \le r-3, r \ge 2$

$$\begin{aligned} &\left| \widetilde{p}_{3,l} \otimes_h \sqrt{h} N_1 \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-l-2)} \otimes_h M_{h,2}(0, T, x, y) \right| \\ &\quad \le C^r h^{3/2-3\varepsilon} \frac{\Gamma(\varepsilon)}{\Gamma(3\varepsilon + \frac{r-1}{2})} T^{3\varepsilon+\frac{r-3}{2}} \zeta_{\sqrt{T}}(y - x). \end{aligned} \tag{90}$$

For $l = r-2$ we have to estimate

$$\widetilde{p}_{3,r-2} \otimes_h \sqrt{h} N_{1h} \otimes_h M_{h,2}(0, T, x, y).$$

This is a finite sum of terms corresponding to the different summands in $N_{1,h}$ and $M_{h,2}$. To estimate a typical term

$$h^{3/2}\sum_{k=0}^{n-2} h\int\Big\{\sum_{j=0}^{k-1} h\int \widetilde{p}_{3,r-2}(0,jh,x,w)(kh-jh)D_w^{\mu+e_n+e_m}\widetilde{p}(jh,kh,w,v)$$
$$\times(g(jh,w)-g(jh,v))dw\Big\}(f(kh,v)-f(kh,y))D_v^{\nu}\widetilde{p}_h((k+1)h,T,v,y)dv$$

again we apply integration by parts and after direct calculations we obtain the following estimate for $r=2,3,...$

$$\Big|\widetilde{p}_{3,r-2}\otimes_h\sqrt{h}N_{1h}\otimes_h M_{h,2}(0,T,x,y)\Big|$$
$$\leq C^r h^{3/2-3\varepsilon}\frac{\Gamma(\varepsilon+\frac{r-2}{2})}{\Gamma(3\varepsilon+\frac{r-2}{2})}\Gamma^2(\varepsilon)\frac{1}{\Gamma(\frac{r}{2})}T^{3\varepsilon+\frac{r-4}{2}}\zeta_{\sqrt{T}}(y-x). \qquad (91)$$

The inequalities (64) and (65) follow now from (82), (90) and (91).

Asymptotic treatment of the term T_3. We will show that

$$\left|T_3-\left[\sum_{r=0}^{\infty}\tilde{p}\otimes_h(H+A)^{(r)}(0,T,x,y)-\sum_{r=0}^{\infty}\tilde{p}\otimes_h H^{(r)}(0,T,x,y)\right]\right|$$
$$\leq Chn^{-\delta}\zeta_{\sqrt{T}}(y-x), \qquad (92)$$

where $A=M_h''-M_h=-\frac{h}{2}(L_\star^2-2L\widetilde{L}+\widetilde{L}^2)\lambda(x)$. Write

$$\begin{aligned} C_r &= \tilde{p}\otimes_h(H+M_h''+\sqrt{h}N_1)^{(r)}(0,T,x,y)\\ &\quad-\tilde{p}\otimes_h(H+M_h+\sqrt{h}N_1)^{(r)}(0,T,x,y)\\ &\quad-[\tilde{p}\otimes_h(H+A)^{(r)}-\tilde{p}\otimes_h H^{(r)}](0,T,x,y).\end{aligned}$$

Similarly as in (66) we have the following recurrence relation

$$\begin{aligned} C_r &= C_{r-1}\otimes_h H+\Big[\tilde{p}\otimes_h(H+M_h''+\sqrt{h}N_1)^{(r-1)}\\ &\quad-\tilde{p}\otimes_h(H+M_h+\sqrt{h}N_1)^{(r-1)}\Big]\otimes_h(M_h''+\sqrt{h}N_1)\\ &\quad+\Big[\tilde{p}\otimes_h(H+M_h+\sqrt{h}N_1)^{(r-1)}-\tilde{p}\otimes_h(H+A)^{(r-1)}\Big]\otimes_h A\\ &= I+II+III. \end{aligned} \qquad (93)$$

With the notation

$$D_{r-1}=\tilde{p}\otimes_h(H+M_h+\sqrt{h}N_1)^{(r-1)}-\tilde{p}\otimes_h(H+A)^{(r-1)}$$

we get

$$D_{r-1}=D_{r-2}\otimes_h(H+M_h+\sqrt{h}N_1)+\tilde{p}_h\otimes_h(H+A)^{(r-2)}\otimes_h(M_h-A+\sqrt{h}N_1).$$

Iterative application gives

$$\begin{aligned} III &= D_{r-1}\otimes_h A\\ &= \sum_{l=0}^{r-2}\widetilde{p}_{4,l}\otimes_h(M_h-A+\sqrt{h}N_1)\otimes_h(H+M_h+\sqrt{h}N_1)^{(r-l-2)}\otimes_h A(0,T,x,y),\end{aligned}$$

where $\widetilde{p}_{4,l} = \widetilde{p} \otimes_h (H + A)^{(l)}$. This sum can be estimated in exactly the same way as the sum in (89). This gives for $r = 2, 3, ...$

$$|III| \leq C(\varepsilon)h^{3/2-2\varepsilon}\frac{C^r}{\Gamma(\frac{r-1}{2})}T^{3\varepsilon+\frac{r-4}{2}}\zeta_{\sqrt{T}}(v-x). \tag{94}$$

To estimate II we write

$$E_{r-1} = \tilde{p} \otimes_h (H + M_h'' + \sqrt{h}N_1)^{(r-1)} - \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1)^{(r-1)}.$$

For $r = 2$ we have $E_1 = \tilde{p} \otimes_h A$ and analogously to (70)

$$|E_1| \leq Ch^{1-\varepsilon}(kh)^{\varepsilon-1/2}B(\frac{1}{2},\varepsilon)\zeta^S_{\sqrt{kh}}(y-x).$$

For $r \geq 3$ similarly as in (72) we use the recurrence relation

$$\begin{aligned} E_{r-1} &= E_{r-2} \otimes_h (H + M_h'' + \sqrt{h}N_1) + [\tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1)^{(r-2)}] \otimes_h A \\ &= I' + II'. \end{aligned}$$

The terms I' and II' have a similar structure as the corresponding terms in (84) and they can be estimated similarly. This gives the following estimates for $r = 2, 3, ...$

$$\begin{aligned} |E_{r-1}| &\leq C^r h^{1-\varepsilon}B(\frac{1}{2},\varepsilon) \times ... \times B(\frac{1}{2},\varepsilon + \frac{r-2}{2})(kh)^{\varepsilon+\frac{r-3}{2}}\zeta^S_{\sqrt{kh}}(y-x), \\ |II| &= \left|E_{r-1} \otimes_h (M_h'' + \sqrt{h}N_1)(0,T,x,y)\right| \\ &\leq C(\varepsilon)h^{3/2-2\varepsilon}\frac{C^r}{\Gamma(\frac{r-1}{2})}T^{3\varepsilon+\frac{r-4}{2}}\zeta_{\sqrt{T}}(v-x). \end{aligned}$$

The claim (92) follows from (93), (94) and the last two inequalities.

Asymptotic treatment of the term T_4. We will show that

$$\begin{aligned} T_4 &= \sum_{r=1}^{\infty}\tilde{p} \otimes_h H^{(r)}(0,T,x,y) \\ &\quad - \sum_{r=1}^{\infty}\tilde{p} \otimes_h [H + hN_2]^{(r)}(0,T,x,y) + R_h^*(x,y), \end{aligned} \tag{95}$$

with $N_2(s,t,x,y) = (L - \widetilde{L})\widetilde{\pi}_2(s,t,x,y)$, $|R_h^*(x,y)| \leq Chn^{-\delta}\zeta^S_{\sqrt{T}}(y-x)$ for $\delta > 0$ small enough and with a constant C depending on δ. For the proof of (95) it suffices to show that for δ small enough

$$\left|\sum_{r=1}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(r)}(0,T,x,y) - \sum_{r=1}^{n} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,T,x,y)\right| \le \left[\sum_{k=1}^{n} \frac{C^k}{\Gamma(\frac{k}{2})}\right] hn^{-\delta}\zeta^S_{\sqrt{T}}(y-x), \tag{96}$$

$$\left|\sum_{r=1}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1)^{(r)}(0,T,x,y) - \sum_{r=1}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(r)}(0,T,x,y) - \left[\sum_{r=1}^{n} \tilde{p} \otimes_h H^{(r)}(0,T,x,y) - \sum_{r=1}^{n} \tilde{p} \otimes_h [H + hN_2]^{(r)}(0,T,x,y)\right]\right| \le \left[\sum_{k=1}^{n} \frac{C^k}{\Gamma(\frac{k}{2})}\right] Chn^{-\delta}\zeta^S_{\sqrt{T}}(y-x). \tag{97}$$

Denote $D_{3,0} \equiv 0$ and

$$\begin{aligned} D_{3,m}(0,jh,x,y) &= \sum_{r=1}^{m} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,jh,x,y) \\ &\quad - \sum_{r=1}^{m} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(r)}(0,jh,x,y). \end{aligned}$$

Then (96) can be rewritten as

$$|D_{3,n}(0,T,x,y)| \le Chn^{-\delta}\zeta^S_{\sqrt{T}}(y-x).$$

We now make iterative use of

$$D_{3,m} = D_{3,m-1} \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2) + g_{m-1}, \tag{98}$$

for $m = 1, 2, ...,$ where

$$\begin{aligned} g_m(0,jh,x,y) &= -\left[\sum_{r=0}^{m} \tilde{p} \otimes_h (K_h + M_h)^{(r)}\right] \otimes_h (H - K_h + \sqrt{h}N_1 + hN_2)(0,jh,x,y) \\ &= S_{h,m} \otimes_h (L - \widetilde{L})d_h(0,jh,x,y) \end{aligned}$$

with

$$\begin{aligned} g_0(0,jh,x,y) &= -\tilde{p} \otimes_h (H - K_h + \sqrt{h}N_1 + hN_2)(0,jh,x,y), \\ d_h &= \widetilde{p}_h - \widetilde{p} - \sqrt{h}\widetilde{\pi}_1 - h\widetilde{\pi}_2, \\ S_{h,m}(0,ih,x,y) &= \sum_{r=0}^{m} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,ih,x,y). \end{aligned}$$

Iterative application of (98) gives

$$D_{3,n}(0,T,x,y) = \sum_{r=0}^{n-1} g_r \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(0,T,x,y).$$

To prove (96) we will show that

$$\begin{aligned} &\left| g_r \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(0,T,x,y) \right| \\ &\quad \le \frac{C^{n-r}}{\Gamma(\frac{n-r}{2})} h n^{-\delta} \zeta^S_{\sqrt{T}}(y-x). \end{aligned} \tag{99}$$

For this purpose we decompose the left handside of (99) into four terms

$$\begin{aligned} a_{r,1} &= \sum_{0\le i\le n/2} h \int g_r(0,ih,x,u)(H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(ih,T,u,y)du, \\ a_{r,2} &= \sum_{n/2<i\le n} h^2 \sum_{0\le k\le i/2} \int\int S_{h,r}(0,kh,x,v)(L-\widetilde{L})d_h(kh,ih,v,u) \\ &\quad \times (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(ih,T,u,y)dvdu, \\ a_{r,3} &= \sum_{n/2<i\le n} h^2 \sum_{i/2<k\le i-n^{\delta'}} \int\int (L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)d_h(kh,ih,v,u) \\ &\quad \times (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(ih,T,u,y)dvdu, \\ a_{r,4} &= \sum_{n/2<i\le n} h^2 \sum_{i-n^{\delta'}<k\le i-1} \int\int (L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)d_h(kh,ih,v,u) \\ &\quad \times (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(ih,T,u,y)dvdu. \end{aligned}$$

Here L^T and $\widetilde{L}^T$ denote the adjoint operators of L and $\widetilde{L}$, and δ' satisfies inequalities $2\varkappa < \delta' < \frac{3}{5}(1-\varkappa)$, where $\varkappa$ is defined in (B2). For the proof of (99) it suffices to show for $l = 1,2,3,4$

$$\begin{aligned} |a_{r,l}| &\le h n^{-\delta} C^{n-r} B(1,\frac{1}{2}) \times ... \times B(\frac{n-r-1}{2},\frac{1}{2}) \zeta^S_{\sqrt{T}}(y-x) \\ &\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r}{2})}\right] h n^{-\delta} \zeta^S_{\sqrt{T}}(y-x) \end{aligned} \tag{100}$$

for some $\delta > 0$.

Proof of (100) for $l=2$. Note that $k \le i/2, i > n/2$ imply $ih - kh \ge \frac{T}{4}$. The claim follows from the inequalities

$$\begin{aligned} &\max\{|K_h(ih,jh,x,y)|, |M_h(ih,jh,x,y)|, \left|\sqrt{h}N_1(ih,jh,x,y)\right| \\ &\quad |hN_2(ih,jh,x,y)|, |H(ih,jh,x,y)|\} \\ &\le C\rho^{-1}\zeta_\rho(y-x) \text{ with } \rho^2 = jh - ih \text{ for } 0 \le i < j \le n, \end{aligned} \tag{101}$$

$$|S_{h,m}(0,kh,x,v)| \leq C\zeta^{S-2}_{\sqrt{kh}}(v-x), \tag{102}$$

$$\left|(L-\widetilde{L})d_h(kh,ih,v,u)\right| \leq Ch^{3/2}(ih-kh)^{-2}\zeta^{S-8}_{\sqrt{ih-kh}}(u-v) \tag{103}$$

$$= O(hn^{-1/2+3/2\varkappa})\zeta^{S-8}_{\sqrt{ih-kh}}(u-v),$$

$$\left|(H+M_h+\sqrt{h}N_1+hN_2)^{(n-r-1)}(ih,T,u,y)\right| \tag{104}$$

$$\leq C^{n-r}\rho^{n-r-3}B(\frac{1}{2},\frac{1}{2})\times ... \times B(\frac{n-r-2}{2},\frac{1}{2})\zeta^{S-2}_{\sqrt{T-ih}}(y-u)$$

$$\leq \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right](T-ih)^{-1/2}\zeta^{S-2}_{\sqrt{T-ih}}(y-u)$$

for $n-r-3=-1,0,1,...,n-3$ with $\rho^2=T-ih$. We put $B(\frac{1}{2},0)=1$). Inequality (101) follows from the definitions of the functions $K_h,...,H$. Inequalities (102) and (104) can be proved by the same method as used in the proof of Theorem 2.3 in Konakov and Mammen (2002) (pp. 282 - 284). Inequality (103) follows from the inequality $ih-kh \geq \frac{T}{4}$, Lemma 5 and the arguments used in the proof of Lemma 7.

Proof of (100) for $l=3$. Note that $n/2<i, k>i/2$ imply $kh>\frac{T}{4}$. We use the following inequalities

$$|d_h(kh,ih,v,u)| \leq Ch^{3/2}(ih-kh)^{-3/2}\zeta^{S-6}_{\sqrt{ih-kh}}(u-v), \tag{105}$$

$$\left|(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)\right| \leq CT^{-1}\zeta^{S-2}_{\sqrt{kh}}(v-x),$$

$$\left|h\sum_{i/2<k\leq i-n^{\delta'}}\int(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)d_h(kh,ih,v,u)dv\right| \tag{106}$$

$$\leq Ch^{3/2}T^{-1}\sum_{i/2<k\leq i-n^{\delta'}}h\frac{1}{(ih-kh)^{3/2}}\zeta_{\sqrt{ih}}(u-x)$$

$$\leq Ch^{3/2}T^{-1}\int_{ih/2}^{ih-n^{\delta'}h}\frac{du}{(ih-u)^{3/2}}\zeta_{\sqrt{ih}}(u-x) \leq Ch^{3/2}T^{-3/2}n^{(1-\delta')/2}\zeta_{\sqrt{ih}}(u-x)$$

$$\leq Chn^{-\delta''}\zeta_{\sqrt{ih}}(u-x),$$

where $\delta''=\delta'/2-\varkappa>0$. Claim (100) for $l=3$ now follows from (106) and (104).

Proof of (100) for $l=4$. For $i-n^{\delta'}<k\leq i-1, n/2<i$ we have $ih>T/2$, $kh>T/3$, $(i-k)<n^{\delta'}$ for sufficiently large n. The integral

$$\int(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)\widetilde{p}_h(kh,ih,v,u)dv$$

is a finite sum of integrals. We show how to estimate a typical term of this sum. The other terms can be estimated analogously. We consider for fixed j,l

$$\int\frac{\partial^2 S_{h,r}(0,kh,x,v)}{\partial v_j\partial v_l}(\sigma_{jl}(kh,v)-\sigma_{jl}(kh,u))h^{-d/2} \tag{107}$$

$$\times q^{(i-k)}[kh,u,h^{-1/2}(u-v-h\sum_{l=k}^{i-1}m(lh,u))]dv$$

$$=\int\frac{\partial^2 S_{h,r}(0,kh,x,v)}{\partial v_j\partial v_l}\mid_{v=u^*-\sqrt{h}w}$$

$$\times[\sigma_{jl}(kh,u^*-\sqrt{h}w)-\sigma_{jl}(kh,u)]q^{(i-k)}(kh,u,w)dw,$$

where $u^* = u - h\sum_{l=k}^{i-1} m(lh,u)$. Now using a Tailor expansion we obtain that the right hand side of (107) is equal to

$$\int \left[\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j \partial v_l} - \sqrt{h}\sum_{|\nu|=1}\frac{w^\nu}{\nu!}\int_0^1 D_v^\nu \frac{\partial^2 S_{h,r}(0,kh,x,u^*-\delta\sqrt{h}w)}{\partial v_j\partial v_l}d\delta\right]$$
$$\times\left[-\sqrt{h}\sum_{|\nu|=1}\frac{[w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u)]^\nu}{\nu!}D_u^\nu\sigma_{jl}(kh,u)\right.$$
$$+2h\sum_{|\nu|=2}\frac{[w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u)]^\nu}{\nu!}$$
$$\left.\times\int_0^1 D_u^\nu\sigma_{jl}(kh,u-\delta\sqrt{h}w-\delta h\sum_{l=k}^{i-1}m(lh,u))d\delta\right]q^{(i-k)}(kh,u,w)dw.$$

Note that

$$-\sqrt{h}\int\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j\partial v_l}(w_p+\sqrt{h}\sum_{l=k}^{i-1}m_p(lh,u))q^{(i-k)}(kh,u,w)dw$$
$$= -h\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j\partial v_l}\sum_{l=k}^{i-1}m_p(lh,u),$$
$$h\int\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j\partial v_l}(w_p+\sqrt{h}\sum_{l=k}^{i-1}m_p(lh,u))(w_q+\sqrt{h}\sum_{l=k}^{i-1}m_q(lh,u)) \tag{108}$$
$$\times\left\{\int_0^1 D_u^\nu\sigma_{jl}(kh,u)d\delta+\int_0^1\left[D_u^\nu\sigma_{jl}(kh,u-\delta\sqrt{h}w\right.\right.$$
$$\left.\left.-\delta h\sum_{l=k}^{i-1}m(lh,u))-D_u^\nu\sigma_{jl}(kh,u)\right]d\delta\right\}q^{(i-k)}(kh,u,w)dw$$
$$= h\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j\partial v_l}D_u^\nu\sigma_{jl}(kh,u)\int w_p w_q q^{(i-k)}(kh,u,w)dw$$
$$+h^2\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j\partial v_l}\sum_{l=k}^{i-1}m_p(lh,u)\sum_{l=k}^{i-1}m_q(lh,u)+R,$$

where by (A3′) we have for $j_0 < (i-k) < n^{\delta'}$, $w' = (i-k)^{-1/2}w$

$$\begin{aligned}
|R| &\le Ch^{3/2}\left|\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j\partial v_l}\right|\int\left(n^{\delta'/2}\|w'\|+O(T^{1/2}n^{-1/2+\delta'})\right)^3\psi(w')dw' \\
&\le Ch\zeta_{\sqrt{ih}}(u-x)(h^S n^{S\delta'}+1)T^{-3/2}n^{-1/2+3\delta'/2}\int\|w'\|^3\psi(w')dw' \qquad (109)\\
&\le Chn^{-1/2+\varkappa/2+3\delta'/2}\zeta_{\sqrt{ih}}(u-x)\le Chn^{-1/2(1-3\delta'-\varkappa)}\zeta_{\sqrt{ih}}(u-x),
\end{aligned}$$

We obtain analogously

$$h\int w_p(w_q+\sqrt{h}\sum_{l=k}^{i-1}m_q(lh,u))D_u^{e_q}\sigma_{jl}(kh,u)\int_0^1\frac{\partial^3 S_{h,r}(0,kh,x,u^*-\delta\sqrt{h}w)}{\partial v_p\partial v_j\partial v_l}d\delta$$
$$\times q^{(i-k)}(kh,u,w)dw$$
$$=h\frac{\partial\sigma_{jl}(kh,u)}{\partial u_q}\frac{\partial^3 S_{h,r}(0,kh,x,u^*)}{\partial v_p\partial v_j\partial v_l}\int w_p w_q q^{(i-k)}(kh,u,w)dw+R,$$

where

$$|R|\le Chn^{-1/2(1-3\delta'-3\varkappa)}\zeta_{\sqrt{ih}}(u-x), 1-3\delta'-3\varkappa>0$$

and, for $1-3\delta'-2\varkappa>0$

$$\Big|h^{3/2}\int w_p\int_0^1\frac{\partial^3 S_{h,r}(0,kh,x,u^*-\delta\sqrt{h}w)}{\partial v_p\partial v_j\partial v_l}d\delta \tag{110}$$
$$\int_0^1\frac{\partial^2\sigma_{jl}(kh,u-\delta\sqrt{h}w-\delta h\sum_{l=k}^{i-1}m_q(lh,u))}{\partial u_r\partial u_s}d\delta$$
$$\times(w_r+\sqrt{h}\sum_{l=k}^{i-1}m_r(lh,u))(w_s+\sqrt{h}\sum_{l=k}^{i-1}m_s(lh,u))q^{(i-k)}(kh,u,w)dw\Big|$$
$$\le Chn^{-1/2(1-3\delta'-2\varkappa)}\zeta_{\sqrt{ih}}(u-x).$$

For $1\le i-k\le j_0$ the same estimates remain true because the following bound holds

$$\int\|w\|^S q^{(j)}(t,x,w)dw\le C(j_0). \tag{111}$$

The same estimates hold for $\widetilde{p}(kh,ih,v,u)$ with $\phi^{(i-k)}(kh,u,w)$ instead of $q^{(i-k)}(kh,u,w)$, where $\phi(kh,u,w)$ is a gaussian density with the mean 0 and with the covariance matrix equal to $\sigma(kh,u)$. The first two moments of $q^{(i-k)}$ and $\phi^{(i-k)}$ coinside so after substraction we obtain uniformly for $i-n^{\delta'}<k\le i-1$

$$\Big|\sum_{n/2<i\le n}h^2\sum_{i-n^{\delta'}<k\le i-1}\int\int(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v) \tag{112}$$
$$\times(\widetilde{p}_h(kh,ih,v,u)-\widetilde{p}(kh,ih,v,u))dv$$
$$\times(H+M_h+\sqrt{h}N_1+hN_2)^{(n-r-1)}(ih,T,u,y)du\Big|$$
$$\le\left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right]hT^{3/2}n^{-3/2(1-\varkappa-5\delta'/3)}\zeta_{\sqrt{ih}}(u-x).$$

To estimate the other terms in $d_h(kh,ih,v,u)$ we need bounds for the following expressions

$$h\sum_{i-n^{\delta'}<k\le i-1}\int(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)\sqrt{h}(ih-kh)$$
$$\times D_v^{\nu}\widetilde{p}(kh,ih,v,u)dv \text{ for } |\nu|=3,$$
$$h\sum_{i-n^{\delta'}<k\le i-1}\int(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)h(ih-kh)$$
$$\times D_v^{\nu}\widetilde{p}(kh,ih,v,u)dv \text{ for } |\nu|=4,$$
$$h\sum_{i-n^{\delta'}<k\le i-1}\int(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)h(ih-kh)^2$$
$$\times D_v^{\nu}\widetilde{p}(kh,ih,v,u)dv \text{ for } |\nu|=6.$$

We have

$$\Bigg|h\sum_{i-n^{\delta'}<k\le i-1}\int(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)\sqrt{h}(ih-kh) \tag{113}$$
$$D_v^{\nu}\widetilde{p}(kh,ih,v,u)dv|$$
$$=\Bigg|h\sum_{i-n^{\delta'}<k\le i-1}\int D^{e_p+e_q}(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)\sqrt{h}(ih-kh)$$
$$D_v^{\nu-e_p-e_q}\widetilde{p}(kh,ih,v,u)dv\Big|$$
$$\le CT^{-2}n^{\delta'}h^{3/2}\sum_{i-n^{\delta'}<k\le i-1}\frac{h}{\sqrt{ih-kh}}\zeta_{\sqrt{ih}}(u-x)$$
$$\le Chn^{-(1-\varkappa-3\delta'/2)}\zeta_{\sqrt{ih}}(u-x).$$

Clearly, the same estimate (113) holds for $|\nu|=4$ and $|\nu|=6$. Now (100) for $l=4$ follows from this remark and (112) and (113).

Proof of (100) for $l=1$. Note that for this case $T-ih\ge T/2$.

$$a_{r,1}=\sum_{0\le i\le n/2}h^2\sum_{0\le k\le i-1}\int\int(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v) \tag{114}$$
$$\times d_h(kh,ih,v,u)\Psi_{h,r}(ih,T,u,y)dvdu$$
$$=\sum_{0\le k\le n/2-1}h\int(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)$$
$$\times\Bigg\{\sum_{k+1\le i\le k+n^{\delta'}}h\int d_h(kh,ih,v,u)\Psi_{h,r}(ih,T,u,y)du$$
$$+\sum_{k+n^{\delta'}<i\le n/2}h\int d_h(kh,ih,v,u)\Psi_{h,r}(ih,T,u,y)du\Bigg\}dv,$$

where we denote

$$\Psi_{h,r}(ih,T,u,y)=(H+M_h+\sqrt{h}N_1+hN_2)^{(n-r-1)}(ih,T,u,y).$$

We consider

$$
\begin{aligned}
&\sum_{k+1\le i\le k+n^{\delta'}} h\int h^{-d/2}q^{(i-k)}(kh,u,h^{-1/2}[u-v-h\sum_{l=k}^{i-1}m(lh,u)])\Psi_{h,r}(ih,T,u,y)du\\
&=\sum_{k+1\le i\le k+n^{\delta'}} h\int\Bigg\{q^{(i-k)}(kh,v,w)+\sqrt{h}\sum_{|\nu|=1}(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}D_v^{\nu}q^{(i-k)}(kh,v,w)\\
&\quad+h\sum_{|\nu|=2}\frac{(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}}{\nu!}D_v^{\nu}q^{(i-k)}(kh,v,w)\\
&\quad+3h^{3/2}\sum_{|\nu|=3}\frac{(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}}{\nu!}\\
&\quad\times\int_0^1(1-\delta)^2D_v^{\nu}q^{(i-k)}(kh,v+\delta h^{1/2}w+\delta h\sum_{l=k}^{i-1}m(lh,u),w)d\delta\Bigg\}\\
&\quad\times\Bigg\{\Psi_{h,r}(ih,T,v,y)+\sqrt{h}\sum_{|\nu|=1}(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}D_v^{\nu}\Psi_{h,r}(ih,T,v,y)\\
&\quad+h\sum_{|\nu|=2}\frac{(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}}{\nu!}D_v^{\nu}\Psi_{h,r}(ih,T,v,y)\\
&\quad+3h^{3/2}\sum_{|\nu|=3}\frac{(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}}{\nu!}\\
&\quad\times\int_0^1(1-\delta)^2D_v^{\nu}\Psi_{h,r}(ih,T,v+\delta h^{1/2}w+\delta h\sum_{l=k}^{i-1}m(lh,u),y)d\delta\Bigg\}dw
\end{aligned}
$$

This integral is a sum of $4\times 4=16$ integrals. We estimate only two of them. Other integrals can be estimated by similar methods. First, we estimate

$$
\sum_{k+1\le i\le k+n^{\delta'}} h\int q^{(i-k)}(kh,v,w)\Psi_{h,r}(ih,T,v,y)dw=\sum_{k+1\le i\le k+n^{\delta'}} h\Psi_{h,r}(ih,T,v,y)dw.
$$

Note that we get the same term when we replace $q^{(i-k)}(kh,v,w)$ by $\phi^{(i-k)}(kh,v,w)$. After the replacement this term disappears. Second, we estimate

$$\sum_{k+1\le i\le k+n^{\delta'}} h\int q^{(i-k)}(kh,v,w)\sqrt{h}\sum_{|\nu|=1}(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}D_v^{\nu}\Psi_{h,r}(ih,T,v,y)dw$$

$$= h^{3/2}\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\int q^{(i-k)}(kh,v,w)[w_j+\sqrt{h}\sum_{l=k}^{i-1}m_j(lh,v)$$

$$+O(hn^{\delta'}\|w\|+h^{3/2}n^{2\delta'})]dw$$

$$= h^2\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\sum_{l=k}^{i-1}m_j(lh,v)$$

$$+O\left(h^2n^{2\delta'}\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} h\left|D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\right|\right)$$

$$+O\left(h^{3/2}n^{\delta'}\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} h\left|D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\right|\int q^{(i-k)}(kh,v,w)\|w\|\,dw\right)$$

$$= h^2\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\sum_{l=k}^{i-1}m_j(lh,v)+R,$$

where

$$|R|\le\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}T^{1/2}hn^{-3/2+2\delta'}\zeta_{\sqrt{T-kh}}(y-v).$$

The first term in the right hand side of this equation will be the same if we replace $q^{(i-k)}(kh,v,w)$ by $\phi^{(i-k)}(kh,v,w)$. After the replacement this term disappears. For a proof of this equation we consider the function $u(w)$ that is defined as an implicit function and we used the following change of variables

$$h^{1/2}w=u-v-h\sum_{l=k}^{i-1}m(lh,u)$$

to obtain

$$\sqrt{h}\sum_{l=k}^{i-1}m(lh,u(w))=\sqrt{h}\sum_{l=k}^{i-1}m(lh,v)+O\left(h(i-k)\|w\|+h^{3/2}(i-k)^2\right)$$

because of $(i-k)\le n^{\delta'}$. By similar methods we get

$$\left|\sum_{k+1\le i\le k+n^{\delta'}} h\int[\sqrt{h}\widetilde{\pi}_1(kh,ih,v,u)+h\widetilde{\pi}_2(kh,ih,v,u)]\Psi_{h,r}(ih,T,u,y)du\right| \tag{115}$$

$$\le\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}hn^{-3/2+2\delta'+\varkappa/2}\zeta_{\sqrt{T-kh}}(y-v).$$

It remains to estimate

$$\sum_{k+n^{\delta'}<i\le n/2} h\int d_h(kh,ih,v,u)\Psi_{h,r}(ih,T,u,y)du.$$

From (104) and (105) we obtain

$$\left|\sum_{k+n^{\delta'}<i\le n/2} h\int d_h(kh,ih,v,u)\Psi_{h,r}(ih,T,u,y)du\right| \tag{116}$$
$$\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] T^{-1/2}h^{3/2}\int_{kh+n^{\delta'}h}^{T/2}\frac{du}{(u-kh)^{3/2}}\zeta_{\sqrt{T-kh}}(y-v)$$
$$\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] T^{-1/2}hn^{-\delta'/2}\zeta_{\sqrt{T-kh}}(y-v)$$
$$\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] hn^{-1/2(\delta'-\varkappa)}\zeta_{\sqrt{T-kh}}(y-v).$$

Now we substitute the estimate (116) into (114). This gives the following estimate for any $0<\varepsilon<\varkappa$

$$\left|\sum_{k=1}^{n/2-1} h\int (L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)\right. \tag{117}$$
$$\left.\sum_{k+n^{\delta'}<i\le n/2} h\int d_h(kh,ih,v,u)\Psi_{h,r}(ih,T,u,y)du\right|$$
$$\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] hn^{-1/2(\delta'-\varkappa)}h^{-\varepsilon}\sum_{k=1}^{n/2}h(kh)^{\varepsilon-1}\zeta_{\sqrt{T}}(y-x)$$
$$\le C(\varepsilon)\left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] hn^{-1/2(\delta'-\varkappa-\varepsilon)}\zeta_{\sqrt{T}}(y-x).$$

For $k=0$ we get with $S_{h,r}(0,0,x,v)=\delta(x-v)$ where $\delta(\cdot)$ is the Dirac function that

$$\left|\sum_{1\le i\le i/2} h^2\int\int S_{h,r}(0,0,x,v)(L-\widetilde{L})d_h(0,ih,v,u)\Psi_{h,r}(ih,T,u,y)du\right|$$
$$\le C(\varepsilon)\left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] hn^{-(1/2-\varepsilon)}\zeta_{\sqrt{T}}(y-x).$$

This completes the proof (100) for $l=1$. The estimate (97) may be proved by the same arguments as were used in the treatment of T_3.

Asymptotic treatment of the term T_5. We will show that,

$$T_5 \;=\; -\sqrt{h}\sum_{r=0}^{\infty}\widetilde{\pi}_1\otimes_h (H+M_{h,1}+\sqrt{h}N_1)^{(r)}(0,T,x,y)$$
$$-h\sum_{r=0}^{\infty}\widetilde{\pi}_2\otimes_h H^{(r)}(0,T,x,y)+R_h(x,y), \tag{118}$$

where $|R_h(x,y)|\le Chn^{-\gamma}\zeta_{\sqrt{T}}^{S-2}(y-x)$ for some $\gamma>0$. Note that with $S_h(s,t,x,y)=\sum_{r=1}^{n}(K_h+M_h)^{(r)}(s,t,x,y)$ the term T_5 can be rewritten as

$$T_5=(\widetilde{p}-\widetilde{p}_h)(0,T,x,y)+(\widetilde{p}-\widetilde{p}_h)\otimes_h S_h(0,T,x,y).$$

We start by showing that for $\varkappa < \delta < \frac{1-\varkappa}{4}$ uniformly for $x, y \in R$

$$\left| h \sum_{1 \le j \le n^\delta} \int (\widetilde{p}_h - \widetilde{p})(0, jh, x, u) S_h(jh, T, u, y) du \right| \le O(hn^{-1/2(1-\varkappa-4\delta)}) \zeta_{\sqrt{T}}^{S-2}(y - x) \tag{119}$$

for δ small enough. For the proof of (119) we will show that uniformly for $1 \le j \le n^\delta$ and for $x, y \in R^d$

$$\begin{aligned} &\int \widetilde{p}_h(0, jh, x, u) S_h(jh, T, u, y) du = S_h(jh, T, x, y) \\ &\quad + O[h^{1/2} T^{-1/2} n^{-1/2+\delta} + h^{1/2} T^{-1} + n^{\delta/2} h^{1/2}] \zeta_{\sqrt{T}}^{S-2}(y - x), \end{aligned} \tag{120}$$

$$\begin{aligned} &\int \widetilde{p}(0, jh, x, u) S_h(jh, T, u, y) du = S_h(jh, T, x, y) \\ &\quad + O[h^{1/2} T^{-1/2} n^{-1/2+\delta} + h^{1/2} T^{-1} + n^{\delta/2} h^{1/2}] \zeta_{\sqrt{T}}^{S-2}(y - x). \end{aligned} \tag{121}$$

Claim (119) immediately follows from (120)-(121). For the proof we will make use of the fact that for all $1 \le j \le n^\delta$ and for all $x, y \in R^d$ and $|\nu| = 1$

$$|D_x^\nu S_h(jh, T, x, y)| \le C(T - jh)^{-1} \zeta_{\sqrt{T-jh}}^{S-2}(y - x). \tag{122}$$

Claim (122) can be shown with the same arguments as in the proof of (5.7) in Konakov and Mammen (2002). Note that the function Φ in that paper has a similar structure as S_h. For $1 \le j \le n^\delta$ the bound (122) immediately implies for a constant C'

$$|D_x^\nu S_h(jh, T, x, y)| \le C' T^{-1} \zeta_{\sqrt{T}}^{S-2}(y - x). \tag{123}$$

We have $\widetilde{p}_h(0, jh, x, u) = h^{-d/2} q^{(j)}[0, u, h^{-1/2}(u - x - h \sum_{i=0}^{j-1} m(ih, u))]$. Denote the determinant of the Jacobian matrix of $u - h \sum_{i=0}^{j-1} m(ih, u)$ by Δ_h. From the condition (A3) and (123) we get that for $1 \le j \le n^\delta$

$$\begin{aligned} &\int \widetilde{p}_h(0, jh, x, u) S_h(jh, T, u, y) du \\ &= \int h^{-d/2} q^{(j)}[0, u, h^{-1/2}(u - x - h \sum_{i=0}^{j-1} m(ih, u))] S_h(jh, T, u, y) du \\ &= \int q^{(j)}(0, x + h^{1/2} w + h \sum_{i=0}^{j-1} m(ih, u(w)), w) \left| \Delta_h^{-1} \right| S_h(jh, T, x + h^{1/2} w + h \sum_{i=0}^{j-1} m(ih, u(w)), y) dw \\ &= \int [q^{(j)}(0, x, w) + O(j^{-d/2} h^{1/2})(\|w\| + 1) \psi(j^{-1/2} w)][1 + O(jh)][S_h(jh, T, x, y) \\ &\quad + O(h^{1/2} T^{-1}) \zeta_{\sqrt{T}}^{S-2}(y - x)(1 + h^{(S-2)/2} \|w\|^{S-2})(\|w\| + 1)] dw \\ &= S_h(jh, T, x, y) + O[h^{1/2} T^{-1/2} n^{-1/2+\delta} + h^{1/2} T^{-1} + h^{1/2} n^{\delta/2}] \zeta_{\sqrt{T}}^{S-2}(y - x) \end{aligned}$$

with $u = u(w)$ in $\sum_{i=0}^{j-1} m(ih, u)$ defined by the Inverse Function Theorem from the substitution $w = h^{-1/2}(u - x - h \sum_{i=0}^{j-1} m(ih, u))$. This proves (120). Claim (121) follows by similar arguments. From (119) we get that for $\delta < \frac{1-\varkappa}{4}$ (with $\varkappa$ defined as in (B2))

$$T_5 = (\widetilde{p} - \widetilde{p}_h)(0, T, x, y) + h \sum_{n^\delta < j < n} \int (\widetilde{p} - \widetilde{p}_h)(0, jh, x, u) S_h(jT, u, y) du + R_h(x, y)$$

with $|R_h(x,y)| \le O(hn^{-1/2(1-\varkappa-4\delta)})\zeta^{S-2}_{\sqrt{T}}(y-x)$. We now make use of the expansion of $\widetilde{p}_h - \widetilde{p}$ given in Lemma 5. We have with $\rho = (jh)^{1/2} \ge h^{1/2}n^{\delta/2}$

$$\left| h\sum_{j=n^\delta}^{n} h^{3/2}\rho^{-3}\int \zeta^S_\rho(u-x)S_h(jh,T,u,y)du \right| \le Ch^2T^{-\delta'}n^{-\delta''}\sum_{j=n^\delta}^{n}\rho^{-2+2\delta'}\int \left|\zeta^S_\rho(u-x)S_h(jh,T,u,y)\right|du, \tag{124}$$

where $\delta' < \frac{1}{2}\delta(1-\delta)^{-1}, 2\ \delta'' = \delta + 2\delta\delta' - 2\delta'$. Now we get that

$$h\sum_{j=n^\delta}^{n}\rho^{-2+2\delta'}\int \left|\zeta^S_\rho(u-x)S_h(jh,T,u,y)\right|du \le CB(\delta',1/2)T^{\delta'-1/2}\zeta^{S-2}_{\sqrt{T}}(y-x) \tag{125}$$

for a constant C. This shows that for $\delta' > 0$ small enough

$$\begin{aligned} T_5 &= -[\sqrt{h}\widetilde{\pi}_1 + h\widetilde{\pi}_2](0,T,x,y) \\ &\quad -h\sum_{n^\delta<j<n}\int[\sqrt{h}\widetilde{\pi}_1 + h\widetilde{\pi}_2](0,jh,x,u)S_h(jh,T,u,y)du + R'_h(x,y) \end{aligned}$$

with $|R'_h(x,y)| \le O(hn^{-(\delta''-\varkappa/2)})\zeta^{S-2}_{\sqrt{T}}(y-x)$ with a constant in $O(\cdot)$ depending on δ'. It follows from (119), (124) and (125) that

$$T_5 = -\sum_{r=0}^{\infty}[\sqrt{h}\widetilde{\pi}_1 + h\widetilde{\pi}_2] \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) + R''_h(x,y), \tag{126}$$

where $|R''_h(x,y)| \le O(hn^{-(\delta''-\varkappa/2)})\zeta^{S-2}_{\sqrt{T}}(y-x)$. Now we apply Lemma 10 with $A = \sqrt{h}\widetilde{\pi}_1$, $B = H + M_{h,1} + \sqrt{h}N_1, C = (K_h - H - \sqrt{h}N_1) + (M_h - M_{h,1})$ to

$$-\sum_{r=0}^{\infty}\sqrt{h}\widetilde{\pi}_1 \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) + \sum_{r=0}^{\infty}\sqrt{h}\widetilde{\pi}_1 \otimes_h (H + M_{h,1} + \sqrt{h}N_1)^{(r)}(0,T,x,y) \tag{127}$$

and with $A = h\widetilde{\pi}_2, B = H, C = (K_h - H) + M_h$ to

$$-\sum_{r=0}^{\infty}h\widetilde{\pi}_2 \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) + \sum_{r=0}^{\infty}h\widetilde{\pi}_2 \otimes_h H^{(r)}(0,T,x,y). \tag{128}$$

The estimate (118) follows from (125), (127), (128), Lemma 10 and Lemma 5 .

Asymptotic treatment of the term T_6. By application of Lemma 9 we get that

$$|T_6| \le C(\varepsilon)hn^{-1/2+\varepsilon}\zeta^S_{\sqrt{T}}(y-x).$$

Asymptotic treatment of the term T_7. From the recurrence relation for $r = 2, 3, \ldots$

$$\begin{aligned} &\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0,T,x,y) - \widetilde{p}_h \otimes_h {H_h}^{(r)}(0,T,x,y) \\ &= \left[\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)} - \widetilde{p}_h \otimes_h {H_h}^{(r-1)}\right] \otimes_h H_h(0,T,x,y) \\ &\quad +[\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)} \otimes_h (K_h + M_h + R_h - H_h)](0,T,x,y) \end{aligned}$$

and from Lemma 8 with $r=1$ we get by similar arguments as in the proof of Lemma 9 that

$$|T_7| \leq Ch^{3/2}T^{-1/2}\zeta^S_{\sqrt{T}}(y-x) = Chn^{-1/2}\zeta^S_{\sqrt{T}}(y-x).$$

Plugging in the asymptotic expansions of $T_1, ..., T_7$. We now plug the asymptotic expansions of $T_1, ..., T_7$ into (39). Using Lemma 10, Theorem 2.1 in Konakov and Mammen (2002) we get

$$\begin{aligned}
& p_h(0,T,x,y) - p(0,T,x,y) \\
& = \sqrt{h}\left[\widetilde{\pi}_1 + p^d \otimes_h \Re_1\right] \otimes_h \Phi(0,T,x,y) \\
& \quad + h\Big\{ \left[\widetilde{\pi}_2 + \widetilde{\pi}_1 \otimes_h \Phi \otimes_h \Re_1 + p^d \otimes_h \Re_2 + p^d \otimes_h \Re_3\right] \otimes_h \Phi(0,T,x,y) \\
& \quad + p^d \otimes_h (\Re_1 \otimes_h \Phi)^{(2)}(0,T,x,y) \\
& \quad + \frac{1}{2} p \otimes_h (L_\star^2 - L^2) p^d(0,T,x,y) - \frac{1}{2} p \otimes_h (L' - \widetilde{L}') p^d(0,T,x,y)\Big\} \\
& \quad + O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)),
\end{aligned} \tag{129}$$

where

$$\begin{aligned}
p^d(ih, i'h, x, y) &= \sum_{r=0}^{\infty} \widetilde{p} \otimes_h H^{(r)}(ih, i'h, x, y), \\
\Re_1(s,t,x,y) &= N_1(s,t,x,y) + M_1(s,t,x,y) - \widetilde{M}_1(s,t,x,y), \\
\Re_2(s,t,x,y) &= N_2(s,t,x,y) + \Pi_1(s,t,x,y) - \widetilde{\Pi}_1(s,t,x,y), \\
\Re_3(s,t,x,y) &= \sum_{|\nu|=4} \frac{\chi_\nu(s,x) - \chi_\nu(s,y)}{\nu!} D_x^\nu \widetilde{p}(s,t,x,y), \\
M_1(s,t,x,y) &= \sum_{|\nu|=3} \frac{\chi_\nu(s,x)}{\nu!} D_x^\nu \widetilde{p}(s,t,x,y), \\
\widetilde{M}_1(s,t,x,y) &= \sum_{|\nu|=3} \frac{\chi_\nu(s,y)}{\nu!} D_x^\nu \widetilde{p}(s,t,x,y), \\
\Pi_1(s,t,x,y) &= \sum_{|\nu|=3} \frac{\chi_\nu(s,x)}{\nu!} D_x^\nu \widetilde{\pi}_1(s,t,x,y), \\
\widetilde{\Pi}_1(s,t,x,y) &= \sum_{|\nu|=3} \frac{\chi_\nu(s,y)}{\nu!} D_x^\nu \widetilde{\pi}_1(s,t,x,y).
\end{aligned}$$

Note that for the homogenous case and $T = [0,1]$ (129) coincides with formula (53) on page 623 in Konakov and Mammen (2005).

Asymptotic replacement of p^d by p. It follows from (42), (57) and (58) that

$$\left|(p^d - p)(ih, jh, x, z)\right| \leq C(\varepsilon) h^{1-\varepsilon}(jh - ih)^{\varepsilon - 1/2} \phi_{\sqrt{(j-i)h}}(z - x) \tag{130}$$

for any $0 < \varepsilon < 1/2$. Using (130) and making an integration by parts we can replace p^d by p in (129). For example the operator $L_\star^2 - L^2$ is an operator of order three. Applying integration by parts we get for

$|\nu| = 3$

$$\left|\sum_{i=1}^{n-1} h \int D_z^\nu p(0,ih,x,z)(p^d - p)(ih,T,z,y)dz\right| \leq C(\varepsilon)h^{1-\varepsilon}\sum_{i=1}^{n-1} h \frac{1}{(ih)^{3/2}}\frac{1}{(T-ih)^{1/2-\varepsilon}}\phi_{\sqrt{T}}(y-x)$$
$$\leq C(\varepsilon)h^{1/2-2\varepsilon}T^{2\varepsilon-1/2}B(\varepsilon,\varepsilon+\frac{1}{2})\phi_{\sqrt{T}}(y-x).$$

By (B2) we have $0 < \varkappa < 1-4\varepsilon$. This implies

$$\left|\frac{h}{2}p \otimes_h (L_\star^2 - L^2)(p^d - p)(0,T,x,y)\right| \leq C(\varepsilon)hT^{1/2}n^{-(1/2-2\varepsilon-\varkappa/2)}\phi_{\sqrt{T}}(y-x)$$
$$\leq C(\varepsilon)h^{1+\delta}\phi_{\sqrt{T}}(y-x)$$

for some $0 < \delta < 1/2$. The other terms in (129) containing p^d can be estimated analogously. Thus we get the following representation

$$\begin{aligned}
&p_h(0,T,x,y) - p(0,T,x,y)\\
&= \sqrt{h}\,[\widetilde{\pi}_1 + p \otimes_h \Re_1] \otimes_h \Phi(0,T,x,y)\\
&\quad + h\Big\{ [\widetilde{\pi}_2 + \widetilde{\pi}_1 \otimes_h \Phi \otimes_h \Re_1 + p \otimes_h \Re_2 + p \otimes_h \Re_3] \otimes_h \Phi(0,T,x,y)\\
&\quad + p \otimes_h (\Re_1 \otimes_h \Phi)^{(2)}\,(0,T,x,y)\\
&\quad + \frac{1}{2}p \otimes_h (L_\star^2 - L^2)p(0,T,x,y) - \frac{1}{2}p \otimes_h (L' - \widetilde{L}')p(0,T,x,y)\Big\} + O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)).
\end{aligned}$$

In the further analysis we make use of the following binary operation $\otimes_h'$. This operator generalizes the binary operation $\otimes$ introduced in Konakov and Mammen (2005). For $s \in [0, t-h]$ and $t \in \{h, 2h, ..., T\}$ the operation $\otimes_h'$ is defined as follows

$$f \otimes_h' g(s,t,x,y) = \sum_{s \leq jh \leq t-h} h \int f(s,jh,x,z)g(jh,t,z,y)dz.$$

Note that for $s \in \{0, h, 2h, ..., T\}$ the two operations $\otimes_h'$ and $\otimes_h$ coincide.

Asymptotic replacement of $(p\otimes_h\Re_i)\otimes_h\Phi(0,T,x,y)$ *by* $p\otimes(\Re_i\otimes_h'\Phi)(0,T,x,y) = (p\otimes\Re_i)\otimes_h\Phi(0,T,x,y), i = 1,2,3,$ $[p \otimes_h (\Re_1 \otimes_h \Phi)] \otimes_h (\Re_1 \otimes_h \Phi)\,(0,T,x,y)$ *by* $p \otimes [(\Re_i \otimes_h' \Phi) \otimes_h' (\Re_i \otimes_h' \Phi)](0,T,x,y)$, $p \otimes_h (L_\star^2 - L^2)p(0,T,x,y)$ *by* $p \otimes (L_\star^2 - L^2)p(0,T,x,y)$ *and* $p \otimes_h (L' - \widetilde{L}')p(0,T,x,y)$ *by* $p \otimes (L' - \widetilde{L}')p(0,T,x,y)$.

These replacements follow from the definitions of $\Re_i, i = 1,2,3,$ and can be proved by the same method as in the treatment of T_1. There an estimate for the replacement error of $p \otimes_h H$ by $p \otimes H$ is given. Linearity of the operation $\otimes_h$ implies that it is enough to consider the functions $p \otimes_h \Im$ where $\Im(u,t,z,v)$ is a function that has one of the following forms:

$$\frac{\chi_\nu(u,z) - \chi_\nu(u,v)}{\nu!}D_x^\nu \widetilde{p}(u,t,z,v) \text{ with } |\nu| = 3,4,$$
$$\frac{\chi_\nu(u,z) - \chi_\nu(u,v)}{\nu!}D_x^\nu \widetilde{\pi}_1(u,t,z,v) \text{ with } |\nu| = 3,$$
$$(L - \widetilde{L})\widetilde{\pi}_1(u,t,z,v) \text{ or } (L - \widetilde{L})\widetilde{\pi}_2(u,t,z,v).$$

We consider the case $\Im(u,t,z,v) = (L - \widetilde{L})\widetilde{\pi}_1(u,t,z,v)$. The other cases can be treated similarly. It is enough to consider a typical term of $(L-\widetilde{L})\widetilde{\pi}_1(u,t,z,v)$. We will give bounds for

$$\int_0^{jh} du \int p(0,u,x,z)\left(\int_u^{jh}\chi_\nu(w,v)dw\right) D_z^\nu (L-\widetilde{L})\widetilde{p}(u,jh,z,v)dz$$
$$-\sum_{i=0}^{j-1} h \int p(0,ih,x,z)\left(\int_{ih}^{jh}\chi_\nu(w,v)dw\right) D_z^\nu (L-\widetilde{L})\widetilde{p}(ih,jh,z,v)dz$$
$$=\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h} du \int [\lambda(u)-\lambda(ih)]dz$$
$$=\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)du\int \lambda'(ih)dz$$
$$+\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2du\int_0^1(1-\delta)\int \lambda''(s)\mid_{s=s_i} dzd\delta du \tag{131}$$

where $\lambda(u)=p(0,u,x,z)\left(\int_u^{jh}\chi_\nu(w,v)dw\right) D_z^\nu H(u,jh,z,v)$, $s_i=ih+\delta(u-ih)$. As in the treatment of T_1, we obtain that

$$\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)du\int \lambda'(ih)dz$$
$$=\frac{h}{2}\sum_{i=0}^{j-1} h\int_{ih}^{jh}\chi_\nu(s,v)ds\int p(0,ih,x,z)D_z^\nu A_0(ih,jh,z,v)dz$$
$$+\frac{h}{2}\sum_{i=0}^{j-1} h\int_{ih}^{jh}\chi_\nu(s,v)ds\int p(0,ih,x,z)D_z^\nu H_1(ih,jh,z,v)dz$$
$$-\frac{h}{2}\sum_{i=0}^{j-1} h\chi_\nu(ih,v)\int p(0,ih,x,z)D_z^\nu H(ih,jh,z,v)dz, \tag{132}$$

where

$$\begin{aligned}
A_0(s,jh,z,v) &= (L^2-2L\widetilde{L}+\widetilde{L}^2)\widetilde{p}(s,jh,z,v),\\
H_l(s,t,z,v) &= \frac{1}{2}\sum_{i,j=1}^{d}\left(\frac{\partial^l\sigma_{ij}(s,z)}{\partial s^l}-\frac{\partial^l\sigma_{ij}(s,v)}{\partial s^l}\right)\frac{\partial^2\widetilde{p}(s,t,z,v)}{\partial z_i\partial z_j}\\
&\quad \sum_{i=1}^{d}\left(\frac{\partial^l m_i(s,z)}{\partial s^l}-\frac{\partial^l m_i(s,v)}{\partial s^l}\right)\frac{\partial\widetilde{p}(s,t,z,v)}{\partial z_i}
\end{aligned}$$

for $l=0,1,2$ with $H_0\equiv H$. The differential operator A_0 was introduced before equation (47). It is a fourth order differential operator. From the structure of this operator and from (132) it is clear that it is enough to estimate

$$I \triangleq \frac{h}{2}\sum_{j=0}^{n-1} h\int\sum_{i=0}^{j-1} h\int_{ih}^{jh}\chi_\nu(s,v)ds\int p(0,ih,x,z)D_z^{\nu+\mu}\widetilde{p}(ih,jh,z,v)dz\Phi(jh,T,v,y)dv \tag{133}$$

for $|\nu|=3, |\mu|=3$. To estimate (133) we consider three possible cases: a) $jh>T/2, ih\le jh/2 \Longrightarrow jh-ih>T/4$ b) $jh>T/2, ih>jh/2 \Longrightarrow ih>T/4$ c) $jh<T/2 \Longrightarrow T-jh>T/2$. In the case a) we apply integration by parts. This transfers two derivatives to $p(0,ih,x,z)$. This gives

$$\begin{aligned}
&\left|\frac{h}{2}\sum_{i=0}^{j-1} h\int_{ih}^{jh}\chi_\nu(s,v)ds\int D_z^{e_k+e_l}p(0,ih,x,z)D_z^{\nu+\mu-e_k-e_l}\widetilde{p}(ih,jh,z,v)dz\right|\\
&\le Ch^{1-2\varepsilon}\int_0^{jh}\frac{1}{u^{1-\varepsilon}}\frac{du}{(jh-u)^{1-\varepsilon}}\phi_{\sqrt{jh}}(v-x)\\
&\le C(\varepsilon)h^{1-2\varepsilon}(jh)^{2\varepsilon-1}\phi_{\sqrt{jh}}(v-x)
\end{aligned}$$

and

$$\begin{aligned}
|I| &\le C(\varepsilon)h^{1-2\varepsilon}\int_0^T\frac{du}{u^{1-2\varepsilon}(T-u)^{1/2}}\phi_{\sqrt{T}}(y-x)\\
&\le C(\varepsilon)h^{3/4}T^{1/2}n^{-(1/4-2\varepsilon-\varkappa/2)}\phi_{\sqrt{T}}(y-x)\\
&\le C(\varepsilon)T^{1/2-\delta}h^{3/4+\delta}\phi_{\sqrt{T}}(y-x), \qquad (134)
\end{aligned}$$

where $\delta=(1/4-2\varepsilon-\varkappa/2)>0$ if $\varkappa<1/2-4\varepsilon$, $0<\varepsilon<0,05$ (see the condition (B2)). In the case b) we apply integration by parts and transfer four derivatives to $p(0,ih,x,z)$. This gives the same estimate as in (134). At last, in the case c) we make an integration by parts and transfer three derivatives to $\Phi(jh,T,v,y)$ and one derivative to $p(0,ih,x,z)$. This gives the same estimate as in (134). To pass from $D_z^\mu\widetilde{p}(ih,jh,z,v)$ to $D_v^\mu\widetilde{p}(ih,jh,z,v)$ we use the following estimate

$$|D_z^\mu\widetilde{p}(ih,jh,z,v)+D_v^\mu\widetilde{p}(ih,jh,z,v)|\le C\phi_{\sqrt{jh-ih}}(v-z).$$

Clearly, the same estimate (134) holds true for the other summands in the right hand side of (132). This gives

$$\begin{aligned}
&\frac{h}{2}\left|\sum_{j=0}^{n-1}h\int\sum_{i=0}^{j-1}h\int_{ih}^{jh}\chi_\nu(s,v)ds\int p(0,ih,x,z)D_z^\nu H_1(ih,jh,z,v)dz\Phi(jh,T,v,y)dv\right|\\
&\le C(\varepsilon)T^{1/2-\delta}h^{3/4+\delta}\phi_{\sqrt{T}}(y-x),\\
&\frac{h}{2}\sum_{i=0}^{j-1}h\int\chi_\nu(ih,v)\sum_{i=0}^{j-1}h\int p(0,ih,x,z)D_z^\nu H(ih,jh,z,v)dz\Phi(jh,T,v,y)dv\\
&\le C(\varepsilon)T^{1/2-\delta}h^{3/4+\delta}\phi_{\sqrt{T}}(y-x).
\end{aligned}$$

We now estimate the second summand in the right hand side of (131). Similarly as in (50) we obtain

$$\begin{aligned}
&\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2du\int_0^1(1-\delta)\int\lambda''(s)\mid_{s=s_i}dzd\delta du\\
&=\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\sum_{k=1}^4\int_s^{jh}\chi_\nu(\tau,v)d\tau\int p(0,s,x,z)D_z^\nu A_k(s,jh,z,v)\mid_{s=s_i}dzd\delta du\\
&\quad-\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\chi_\nu(s,v)\int p(0,s,x,z)D_z^\nu A_0(s,jh,z,v)\mid_{s=s_i}dzd\delta du\\
&\quad-\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\chi_\nu(s,v)\int p(0,s,x,z)D_z^\nu H_1(s,jh,z,v)\mid_{s=s_i}dzd\delta du\\
&\quad-\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\frac{\partial\chi_\nu(s,v)}{\partial s}\int p(0,s,x,z)D_z^\nu H(s,jh,z,v)\mid_{s=s_i}dzd\delta du, \qquad (135)
\end{aligned}$$

where the operators $A_i, i = 1, 2, 3, 4$, are defined as follows:

$$\begin{aligned}A_1(s,jh,z,v) &= (L^3 - 3L^2\widetilde{L} + 3L\widetilde{L}^2 - \widetilde{L}^3)\widetilde{p}(s,jh,z,v),\\ A_2(s,jh,z,v) &= (L_1H + 2LH_1)(s,jh,z,v),\\ A_3(s,jh,z,v) &= [(L-\widetilde{L})\widetilde{L}_1 + 2(L_1 - \widetilde{L}_1)\widetilde{L}]\widetilde{p}(s,jh,z,v),\\ A_4(s,jh,z,v) &= H_2(s,jh,z,v).\end{aligned}$$

The operator A_1 was introduced in (51). For this operator it is enough to estimate for fixed p, q, r, l

$$\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\sum_{k=1}^{4}\int_s^{jh}\chi_\nu(\tau,v)d\tau\int p(0,s,x,z)D_z^\nu\left(\frac{\partial^4\widetilde{p}(s,jh,z,v)}{\partial z_p\partial z_q\partial z_l\partial z_r}\right)|_{s=s_i}\,dzd\delta du.$$

As in (56) we obtain that this term does not exceed

$$C(\varepsilon)h^{3/2-\varepsilon}(jh)^{2\varepsilon-1}\phi_{\sqrt{jh}}(v-x). \tag{136}$$

It follows from the explicit form of these operators that the same estimate (136) holds for A_2, A_3 and A_4. The other three terms in the right hand side of (135) do not contain the factor $\int_s^{jh}\chi_\nu(\tau,v)d\tau$ and they can be estimated separately. Clearly, it is enough to estimate the term containing A_0. The remaining two summands are less singular. From the explicit form of A_0 (compare also (46)) we obtain that it is enough to estimate for fixed q, l, r

$$\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\chi_\nu(s,v)\int p(0,s,x,z)D_z^\nu\left(\frac{\partial^3\widetilde{p}(s,jh,z,v)}{\partial z_q\partial z_l\partial z_r}\right)(s,jh,z,v)\,|_{s=s_i}\,dzd\delta du.$$

Analogously to (47) we get that this term does not exceed

$$C(\varepsilon)h^{1-2\varepsilon}(jh)^{2\varepsilon-1}\phi_{\sqrt{jh}}(v-x). \tag{137}$$

Now from (131), (134), (135), (136) and (137) we obtain that

$$\begin{aligned}&\left|[p\otimes_h(L-\widetilde{L})\widetilde{\pi}_1]\otimes_h\Phi(0,T,x,y) - p\otimes[(L-\widetilde{L})\widetilde{\pi}_1\otimes_h'\Phi](0,T,x,y)\right|\\ &\qquad\le Ch^{3/4+\delta}\phi_{\sqrt{T}}(y-x)\end{aligned} \tag{138}$$

for some $\delta > 0$. The other replacements can be shown analogously. Thus we come to the following representation

$$\begin{aligned}&p_h(0,T,x,y) - p(0,T,x,y)\\ &= \sqrt{h}\,[\widetilde{\pi}_1\otimes_h'\Phi(0,T,x,y) + p\otimes(\Re_1\otimes_h'\Phi)(0,T,x,y)]\\ &\quad +h\,[\widetilde{\pi}_2\otimes_h'\Phi(0,T,x,y) + p\otimes(\Re_2\otimes_h'\Phi)(0,T,x,y) + p\otimes_h(\Re_3\otimes_h'\Phi)(0,T,x,y)]\\ &\quad +h\,[\widetilde{\pi}_1\otimes_h'\Phi + p\otimes(\Re_1\otimes_h'\Phi)]\otimes_h'(\Re_1\otimes_h'\Phi)(0,T,x,y)\\ &\quad +\frac{h}{2}p\otimes(L_\star^2 - L^2)p(0,T,x,y) - \frac{h}{2}p\otimes(L'-\widetilde{L}')p(0,T,x,y) + O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)).\end{aligned} \tag{139}$$

We now further simplify our expansion of $p_h - p$. We start by showing the following expansion

$$\begin{aligned}&p_h(0,T,x,y) - p(0,T,x,y)\\ &= \sqrt{h}(p\otimes\mathcal{F}_1[p_\Delta])(0,T,x,y) + h\,(p\otimes\mathcal{F}_2[p_\Delta])\,(0,T,x,y)\\ &\quad +h\,(p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p_\Delta]])\,(0,T,x,y)\\ &\quad +\frac{h}{2}p\otimes(L_\star^2 - L^2)p(0,T,x,y) - \frac{h}{2}p\otimes(L'-\widetilde{L}')p(0,T,x,y) + O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)),\end{aligned} \tag{140}$$

where for $s \in [0, t-h], t \in \{h, 2h, ..., T\}$

$$\begin{aligned} p_\Delta(s,t,z,y) &= (\widetilde{p} \otimes_h' \Phi)(s,t,z,y) \\ &= \widetilde{p}(s,t,z,y) + \sum_{s \le jh \le t-h} h \int \widetilde{p}(s,jh,z,v)\Phi_1(jh,t,v,y)dv. \end{aligned}$$

Here $\Phi_1 = H + H \otimes_h' H + H \otimes_h' H \otimes_h' H + ...$ We now treat the term $p \otimes \widetilde{L}\widetilde{\pi}_1(s,t,x,y)$.

$$\begin{aligned} p \otimes \widetilde{L}\widetilde{\pi}_1(s,t,x,y) &= \int_s^t d\tau \int p(s,\tau,x,v)(t-\tau) \sum_{|\nu|=3} \frac{\overline{\chi}_\nu(\tau,t,y)}{\nu!} D_v^\nu(\widetilde{L}_v\widetilde{p}(\tau,t,v,y))dv \\ &= -\sum_{|\nu|=3} \frac{1}{\nu!} \int dv \left[\int_s^t p(s,\tau,x,v) \left(\int_\tau^t \chi_\nu(u,y)du \right) \frac{\partial}{\partial\tau} D_v^\nu \widetilde{p}(\tau,t,v,y)d\tau \right] \\ &= -\sum_{|\nu|=3} \frac{1}{\nu!} \int dv \int_s^{\frac{s+t}{2}} ... - \sum_{|\nu|=3} \frac{1}{\nu!} \int dv \int_{\frac{s+t}{2}}^t ... \\ &= I + II. \end{aligned} \tag{141}$$

By integrating by parts w.r.t. the time variable we obtain for I.

$$\begin{aligned} I &= -\sum_{|\nu|=3} \frac{1}{\nu!} \int dv \Bigg[p(s,\tau,x,v) \left(\int_\tau^t \chi_\nu(u,y)du \right) D_v^\nu \widetilde{p}(\tau,t,v,y) \,|_{\tau=s}^{\tau=(s+t)/2} \\ &\qquad - \int_s^{\frac{s+t}{2}} D_v^\nu \widetilde{p}(\tau,t,v,y) \left(\frac{\partial p(s,\tau,x,v)}{\partial \tau} \int_\tau^t \chi_\nu(u,y)du - p(s,\tau,x,v)\chi_\nu(\tau,y) \right) d\tau \Bigg] \\ &= -\sum_{|\nu|=3} \frac{1}{\nu!} \int dv \Bigg[p(s,\frac{s+t}{2},x,v) \left(\int_{\frac{s+t}{2}}^t \chi_\nu(u,y)du \right) D_v^\nu \widetilde{p}(\frac{s+t}{2},t,v,y) \\ &\qquad + \sum_{|\nu|=3} \frac{1}{\nu!} \left(\int_s^t \chi_\nu(u,y)du \right) D_v^\nu \widetilde{p}(s,t,x,y) \Bigg] \\ &\qquad + \sum_{|\nu|=3} \frac{1}{\nu!} \int_s^{\frac{s+t}{2}} d\tau \left(\int_\tau^t \chi_\nu(u,y)du \right) \int L^T p(s,\tau,x,v) D_v^\nu \widetilde{p}(\tau,t,v,y)dv \\ &\qquad - \sum_{|\nu|=3} \frac{1}{\nu!} \int_s^{\frac{s+t}{2}} \chi_\nu(\tau,y)d\tau \int p(s,\tau,x,v) D_v^\nu \widetilde{p}(\tau,t,v,y)dv. \end{aligned} \tag{142}$$

For the second term we get

$$\begin{aligned} II &= \sum_{|\nu|=3} \frac{1}{\nu!} \int p(s,\frac{s+t}{2},x,v) \left(\int_{\frac{s+t}{2}}^t \chi_\nu(u,y)du \right) D_v^\nu \widetilde{p}(\frac{s+t}{2},t,v,y)dv \\ &\qquad + \sum_{|\nu|=3} \frac{1}{\nu!} \int_{\frac{s+t}{2}}^t d\tau \left(\int_\tau^t \chi_\nu(u,y)du \right) \int L^T p(s,\tau,x,v) D_v^\nu \widetilde{p}(\tau,t,v,y)dv \\ &\qquad - \sum_{|\nu|=3} \frac{1}{\nu!} \int_{\frac{s+t}{2}}^t \chi_\nu(\tau,y)d\tau \int p(s,\tau,x,v) D_v^\nu \widetilde{p}(\tau,t,v,y)dv. \end{aligned} \tag{143}$$

From (141)- (143) we have

$$p \otimes \widetilde{L}\widetilde{\pi}_1(s,t,x,y) = \widetilde{\pi}_1(s,t,x,y) + p \otimes L\widetilde{\pi}_1(s,t,x,y) - p \otimes \widetilde{M}_1(s,t,x,y).$$

This shows that

$$\begin{aligned}
&\widetilde{\pi}_1(s,t,x,y) + p \otimes \Re_1(s,t,x,y) = \widetilde{\pi}_1(s,t,x,y) \\
&\quad + p \otimes L\widetilde{\pi}_1(s,t,x,y) - p \otimes \widetilde{L}\widetilde{\pi}_1(s,t,x,y) + p \otimes M_1(s,t,x,y) - p \otimes \widetilde{M}_1(s,t,x,y) \\
&= p \otimes M_1(s,t,x,y). \qquad (144)
\end{aligned}$$

It follows from (144) and the definitions of the operations $\otimes$ and $\otimes_h'$ that

$$\begin{aligned}
&\sqrt{h}\,[\widetilde{\pi}_1 \otimes_h' \Phi(s,t,x,y) + (p \otimes \Re_1) \otimes_h' \Phi(s,t,x,y)] \\
&= \sqrt{h}(\widetilde{\pi}_1 + p \otimes \Re_1) \otimes_h' \Phi(s,t,x,y) \\
&= \sqrt{h}(p \otimes M_1) \otimes_h' \Phi(s,t,x,y) \\
&= \sqrt{h} \sum_{0 \le jh \le t-h} h \int (p \otimes M_1)(s,jh,x,z)\Phi(jh,t,z,y)dz \\
&= \sqrt{h} \sum_{0 \le jh \le t-h} h \int \left[\int_s^{jh} du \int p(s,u,x,v)M_1(u,jh,v,z)dv\right] \Phi(jh,t,z,y)dz \\
&= \sqrt{h} \sum_{0 \le jh \le t-h} h \int \left[\int_s^{t} du\chi[s,jh] \int p(s,u,x,v)M_1(u,jh,v,z)dv\right] \Phi(jh,t,z,y)dz \\
&= \sqrt{h} \int_s^t du \int p(s,u,x,v) \sum_{|\nu|=3} \frac{\chi_\nu(u,v)}{\nu!} \times D_v^\nu \left[\sum_{0 \le jh \le t-h} h\chi[s,jh] \int \widetilde{p}(u,jh,v,z)\Phi(jh,t,z,y)dz\right] dv \\
&= \sqrt{h} \int_s^t du \int p(s,u,x,v) \times \sum_{|\nu|=3} \frac{\chi_\nu(u,v)}{\nu!} D_v^\nu p_\Delta(u,t,v,y)dv \\
&= \sqrt{h}(p \otimes \mathcal{F}_1)[p_\Delta](s,t,x,y). \qquad (145)
\end{aligned}$$

Here, $\chi[s,jh]$ denotes the indicator of the interval $[s,jh]$. Using similar arguments as in the proof of (145) one can show that

$$\begin{aligned}
&h\,[\widetilde{\pi}_2 \otimes_h' \Phi(s,t,x,y) + (p \otimes \Re_2) \otimes_h' \Phi(s,t,x,y) \;+ p \otimes_h (\Re_3 \otimes_h' \Phi)(s,t,x,y)] \\
&= h(p \otimes \mathcal{F}_2)[p_\Delta](s,t,x,y) + hp \otimes \Pi_1 \otimes_h' \Phi(s,t,x,y). \qquad (146)
\end{aligned}$$

For the first two terms in the right hand side of (139) we obtain from (145) and (146)

$$\begin{aligned}
&\sqrt{h}\,[\widetilde{\pi}_1 \otimes_h' \Phi(0,T,x,y) + (p \otimes \Re_1) \otimes_h' \Phi(0,T,x,y)] \\
&\quad + h\,[\widetilde{\pi}_2 \otimes_h' \Phi(0,T,x,y) + p \otimes (\Re_2 \otimes_h' \Phi)(0,T,x,y) + p \otimes_h (\Re_3 \otimes_h' \Phi)(0,T,x,y)] \\
&= \sqrt{h}(p \otimes \mathcal{F}_1)[p_\Delta](0,T,x,y) + h(p \otimes \mathcal{F}_2)[p_\Delta](s,t,x,y) + hp \otimes \Pi_1 \otimes_h' \Phi(s,t,x,y). \qquad (147)
\end{aligned}$$

Using (145) we get

$$\begin{aligned}
&h\,[\widetilde{\pi}_1 \otimes_h' \Phi + p \otimes (\Re_1 \otimes_h' \Phi)] \otimes_h' (\Re_1 \otimes_h' \Phi)(0,T,x,y) \\
&= h(p \otimes \mathcal{F}_1[p_\Delta]) \otimes_h' (\Re_1 \otimes_h' \Phi)(0,T,x,y) \\
&= hp \otimes \mathcal{F}_1\,[p_\Delta \otimes_h' (\Re_1 \otimes_h' \Phi)]\,(0,T,x,y).
\end{aligned}$$

Note that

$$hp \otimes \Pi_1 \otimes'_h \Phi(s,t,x,y) \;=\; h\int_s^t du \int p(s,u,x,v) \sum_{|\nu|=3} \frac{\chi_\nu(u,v)}{\nu!} D_v^\nu[\widetilde{\pi}_1 \otimes'_h \Phi](u,t,v,y)$$
$$= hp \otimes \mathcal{F}_1[\widetilde{\pi}_1 \otimes'_h \Phi](s,t,x,y).$$

For the proof of (140) it remains to show that

$$hp \otimes \mathcal{F}_1[\widetilde{\pi}_1 \otimes'_h \Phi + p_\Delta \otimes'_h (\Re_1 \otimes'_h \Phi)](0,T,x,y)$$
$$= h\left(p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p_\Delta]]\right)(0,T,x,y) + O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)). \tag{148}$$

We will show that

$$hp \otimes \mathcal{F}_1[(p-p_\Delta) \otimes'_h (\Re_1 \otimes'_h \Phi)](0,T,x,y) = O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)), \tag{149}$$
$$hp \otimes \mathcal{F}_1[p \otimes'_h (\Re_1 \otimes'_h \Phi)](0,T,x,y) - hp \otimes \mathcal{F}_1[p \otimes (\Re_1 \otimes'_h \Phi)](0,T,x,y)$$
$$= O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)). \tag{150}$$

Claim (148) follows from (149), (150) and (145). The estimate (150) can be shown similarly as in the proof of (138). An additional singularity arising from the derivatives in the operator $\mathcal{F}_1[\cdot]$ can be treated by using the additional factor h in (150). To estimate (149) note that from the definition of $\Re_1$ and Φ

$$|(\Re_1 \otimes'_h \Phi)(jh,T,x,y)| \le C(\varepsilon)h^{-\varepsilon}(T-jh)^{\varepsilon-1}\phi_{\sqrt{T-jh}}. \tag{151}$$

Then we use the following estimate which can be proved by the same method as in the treatment of T_1, where an estimate for $(p-p^d)(0,jh,x,y)$ was obtained.

$$|(p-p_\Delta)(u,jh,v,z)| \le Ch^{1/2}\phi_{\sqrt{jh-u}}(z-v). \tag{152}$$

From (151) and (152)

$$|(p-p_\Delta) \otimes'_h (\Re_1 \otimes'_h \Phi)(u,T,v,y)| \le C(\varepsilon)h^{1/2-\varepsilon}(T-u)^\varepsilon \phi_{\sqrt{T-u}}(y-v) \tag{153}$$

For an estimate (149) it is enough to estimate a typical summand of the sum of the detailed representation of the left hand side of (149). E.g., for $|\nu| = 3$ we have to estimate

$$h\int_0^T du \int p(0,u,x,v)\frac{\chi_\nu(u,v)}{\nu!} D_v^\nu[\sum_{\{j: u \le jh \le T-h\}} h\int (p-p_\Delta)(u,jh,v,z)$$
$$\times(\Re_1 \otimes'_h \Phi)(jh,T,z,y)dz]dv$$
$$= h\int_0^{T/2} ... + h\int_{T/2}^T ...$$
$$= I + II.$$

For an estimate of II we apply integration by parts and transfer three derivatives to $p(0,u,x,v)\frac{\chi_\nu(u,v)}{\nu!}$. Using (153) we obtain the following estimate

$$|II| \;\le\; C(\varepsilon)h^{3/2-\varepsilon}\int_{T/2}^T \frac{(T-u)^\varepsilon}{u^{3/2}} du\, \phi_{\sqrt{T}}(y-x)$$
$$\le\; C(\varepsilon)h^{3/2-\varepsilon}T^\varepsilon \phi_{\sqrt{T}}(y-x). \tag{154}$$

For the treatment of I we consider two cases: a) $jh-u\geq T/4$ and b) $jh-u\leq T/4 \Longrightarrow T-jh\geq T/4$. Similarly as in (42) the difference $h(p-p_\Delta)$ can be represented as

$$\begin{aligned}
&h(p-p_\Delta)(u,jh,v,z)=h(p\otimes H-p\otimes_h' H)(u,jh,v,z)\\
&\qquad +h(p\otimes H-p\otimes_h' H)\otimes_h'\Phi_1(u,jh,v,z)\\
&=h\int_u^{j^\star h}d\tau\int p(u,\tau,v,z')H(\tau,jh,z',z)dz'\\
&\qquad +h\sum_{i=j^\star}^{j-1}\int_{ih}^{(i+1)h}d\tau\int(\lambda(\tau,z')-\lambda(ih,z'))dz'+h(p\otimes H-p\otimes_h' H)\otimes_h'\Phi_1(u,jh,v,z)\\
&=I'+II'+III',
\end{aligned} \tag{155}$$

where $\lambda(\tau,z')=p(u,\tau,v,z')H(\tau,jh,z',z),\Phi_1(ih,jh,z'z)=H(ih,jh,z'z)+H\otimes_h' H(ih,jh,z'z)+...,j^\star=j^\star(u)=[\frac{u}{h}]+1$. Here $[x]$ is equal to the integral part for noninteger x and equal to $x-1$ for integer x. For I', case a), we have $jh-\tau>T/5$ for n large enough. With the substitution $v+v'=z'$ we obtain

$$\begin{aligned}
&\left|D_v^\nu h\int_u^{j^\star h}d\tau\int p(u,\tau,v,z')H(\tau,jh,z',z)dz'\right|\\
&=\left|D_v^\nu h\int_u^{j^\star h}d\tau\int p(u,\tau,v,v+v')H(\tau,jh,v+v',z)dv'\right|\\
&\leq Ch\int_u^{j^\star h}\frac{d\tau}{(jh-\tau)^2}\phi_{\sqrt{jh-u}}(z-v)\leq Ch^2T^{-2}\phi_{\sqrt{jh-u}}(z-v)\\
&\leq Cn^{-2}\phi_{\sqrt{jh-u}}(z-v)=CT^2h^2\phi_{\sqrt{jh-u}}(z-v)
\end{aligned} \tag{156}$$

For the proof of (156) we used the following estimate from Friedman (1964) (Theorem 7, page 260)

$$|D_v^\nu p(u,\tau,v,v+v')|\leq C(\tau-u)^{-d/2}\exp\left(\frac{C\,|v'|}{\tau-u)}\right).$$

For $\int I'(u,jh,v,z)(\Re_1\otimes_h'\Phi)(jh,T,z,y)dz$, case b), it is enough to estimate for $|\nu|=3$

$$\begin{aligned}
&h\int_u^{j^\star h}d\tau\int[p(u,\tau,v,v+v')(\sigma_{lk}(\tau,v+v')-\sigma_{lk}(\tau,z))]D_{v'}^\nu\frac{\partial^2\widetilde{p}(\tau,jh,v+v',z)}{\partial v_l'\partial v_k'}dv'\\
&\qquad\times(\Re_1\otimes_h'\Phi)(jh,T,z,y)dz.
\end{aligned} \tag{157}$$

For an estimate of this term we transfer five derivatives from $\widetilde{p}$ to $(\Re_1\otimes_h'\Phi)(jh,T,z,y)$ and we use the following estimate for $|\mu|=5$

$$\begin{aligned}
&|D_{v'}^\mu\widetilde{p}(\tau,jh,v+v',z)+D_z^\mu\widetilde{p}(\tau,jh,v+v',z)|\\
&\qquad\leq C(jh-\tau)^{-d/2}\phi_{\sqrt{jh-\tau}}(z-v-v').
\end{aligned}$$

We obtain that (157) does not exceed

$$\begin{aligned}
C(j^\star h-\tau)hT^{-7/2}\phi_{\sqrt{T-u}}(y-v) &\leq Ch^{1+\delta}T^{1-\delta}n^{-(1-\delta-7\varkappa/2)}\phi_{\sqrt{T-u}}(y-v)\\
&= o(h^{1+\delta}T^{1-\delta})\phi_{\sqrt{T-u}}(y-v).
\end{aligned} \tag{158}$$

We used that for any $0<\delta<1$ it holds that $\varkappa<\frac{2-2\delta}{7}$, see condition (B2). For an estimate of $\int II'(u,jh,v,z)(\Re_1\otimes_h'\Phi)(jh,T,z,y)dz$ we use the decomposition (43). For getting an estimate for the terms in II' that contain the first derivatives $\lambda'(ih, z')$ we use the identity (45) and similar arguments as already used in the estimation of $\int I'(u, jh, v, z)(\Re_1 \otimes'_h \Phi)(jh, T, z, y)dz$. The estimate for terms in II' containing second derivatives $\lambda''(ih, z')$ follows from (53) and (54). Finally, for III' the same estimates hold because of smoothing properties of the convolution $...\otimes'_h \Phi_1(u, jh, v, z)$. This implies (149) and, hence, the expansion (140).

Asymptotic replacement of p_Δ by p. Now, we compare $hp \otimes \mathcal{F}_2[p_\Delta](0, T, x, y)$ with $hp \otimes \mathcal{F}_2[p](0, T, x, y)$. Note that for $2\varkappa < \delta < \frac{2}{5}, |\nu| = 4$

$$\left| h \int_0^{h^\delta} du \int p(0, u, x, z)\chi_\nu(u, z) D_z^\nu p(u, T, z, y)dz \right| \leq Ch^{1+\delta}(T - h^\delta)^{-2}\phi_{\sqrt{T}}(y - x)$$

$$\leq Ch^{1+\delta} \frac{n^{2\varkappa}}{(Tn^\varkappa - n^\varkappa h^\delta)^2} \leq Ch^{1+(\delta-2\varkappa)}T^{2\varkappa}\phi_{\sqrt{T}}(y - x), \tag{159}$$

$$\left| h \int_{T-h^\delta}^{T} du \int D_z^\nu[p(0, u, x, z)\chi_\nu(u, z)]p(u, T, z, y)dz \right| \leq Ch^{1+(\delta-2\varkappa)}T^{2\varkappa}\phi_{\sqrt{T}}(y - x). \tag{160}$$

The same estimates hold for $p_\Delta(u, T, z, y)$. Hence, it suffices to consider $u \in [h^\delta, T - h^\delta]$. We now treat

$$h \int_{h^\delta}^{T-h^\delta} du \int p(0, u, x, z)\chi_\nu(u, z) D_z^\nu (p - p_\Delta)(u, T, z, y)dz$$

$$= h \int_{h^\delta}^{T/2} ... + h \int_{T/2}^{T-h^\delta} ... = I + II. \tag{161}$$

By using (152) we get

$$\begin{aligned} |II| &= \left| h \int_{T/2}^{T-h^\delta} du \int D_z^\nu[p(0, u, x, z)\chi_\nu(u, z)](p - p_\Delta)(u, T, z, y)dz \right| \\ &\leq Ch^{3/2}n^\varkappa \phi_{\sqrt{T}}(y - x) = Ch^{3/2-\varkappa}T^\varkappa \phi_{\sqrt{T}}(y - x) \\ &= Ch^{1+\gamma}\phi_{\sqrt{T}}(y - x), \gamma > 0 \end{aligned} \tag{162}$$

For $u \in [h^\delta, T/2]$ it holds that

$$\begin{aligned} |I| &= \left| h \int_{h^\delta}^{T/2} du \int D_z^\nu[p(0, u, x, z)\chi_\nu(u, z)](p - p_\Delta)(u, T, z, y)dz \right| \\ &\leq Ch^{3/2-\delta}\phi_{\sqrt{T}}(y - x). \end{aligned} \tag{163}$$

Note that the condition $\delta < \frac{2}{5}$ implies that $3/2 - \delta > 1$. It follows from (159)-(163) that

$$hp \otimes \mathcal{F}_2[p_\Delta](0, T, x, y) - hp \otimes \mathcal{F}_2[p](0, T, x, y) = O(h^{1+\gamma}\phi_{\sqrt{T}}(y - x)). \tag{164}$$

For the proof of

$$hp \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p - p_\Delta]] = O(h^{1+\delta}\phi_{\sqrt{T}}(y - x))$$

we consider a typical summand for fixed $\nu, |\nu| = 3$,

$$h \int_0^T du \int p(0, u, x, z)\chi_\nu(u, z) D_z^\nu \left[\int_u^T d\tau \int p(u, \tau, z, v)\chi_\nu(\tau, v) D_v^\nu (p - p_\Delta)(\tau, T, v, y)dv \right] dz. \tag{165}$$

As before it suffices to consider the integrals for $u \in [h^\delta, T-h^\delta]$. The integral in (165) is a sum of four integrals

$$\begin{aligned}
I_1 &= h\int_{h^\delta}^{T/2} du \int ...D_z^\nu \int_u^{(T+u)/2} d\tau \int ...,\\
I_2 &= h\int_{h^\delta}^{T/2} du \int ...D_z^\nu \int_{(T+u)/2}^{T} d\tau \int ...,\\
I_3 &= h\int_{T/2}^{T-h^\delta} du \int ...D_z^\nu \int_u^{(T+u)/2} d\tau \int ...,\\
I_4 &= h\int_{T/2}^{T-h^\delta} du \int ...D_z^\nu \int_{(T+u)/2}^{T} d\tau \int .... \qquad (166)
\end{aligned}$$

Note that in the integrand in I_2 it holds that $\tau - u \geq T/4$. By applying integration by parts w.r.t. v and (152) we get

$$|I_2| \leq Ch^{3/2-\varkappa}T^{\varkappa}\phi_{\sqrt{T}}(y-x)). \qquad (167)$$

Furthermore, in the integrand in I_4 it holds that $u \geq T/2, \tau - u \geq h^\delta/2, T-u \geq h^\delta$. Using integration by parts w.r.t. z we obtain

$$|I_4| \leq Ch^{3/2-\delta}T^{-1/2}\phi_{\sqrt{T}}(y-x)) \leq CT^{\varkappa/2}h^{3/2-\varkappa/2-\delta}\phi_{\sqrt{T}}(y-x)), \qquad (168)$$

where, by our choice of δ, $3/2-\varkappa/2-\delta > 1$. For an estimate of I_3 we use the representation

$$\begin{aligned}
&(p-p_\Delta)(\tau,T,v,y) = (p\otimes H - p\otimes_h' H)(\tau,T,v,y)\\
&\quad +(p\otimes H - p\otimes_h' H)\otimes_h'\Phi_1(\tau,T,v,y)\\
&= \int_\tau^{j^\star h} ds \int p(\tau,s,v,w)H(s,T,w,y)dw\\
&\quad +\frac{h}{2}[p\otimes_h'(H_1+A_0)](\tau,T,v,y)\\
&\quad +\frac{1}{2}\sum_{i=j^\star}^{n-1}\int_{ih}^{(i+1)h}(t-ih)^2\int_0^1(1-\gamma)\sum_{k=1}^{4}\int p(\tau,s,v,w)A_k(s,T,w,y)\mid_{s=ih+\gamma(t-ih)} dwd\gamma dt,\\
&\quad +(p\otimes H - p\otimes_h' H)\otimes_h'\Phi_1(\tau,T,v,y), \qquad (169)
\end{aligned}$$

where $j^\star = j^\star(\tau) = [\tau/h]+1$. As above $[x]$ denotes the integer part for nonintegers x and it is equal to $x-1$ for integers x. The quantities H_1 and $A_k, k=0,1,2,3,4$, have been defined (52) and

$$\Phi_1(ih,i'h,z,z') = H(ih,i'h,z,z') + H\otimes_h' H(ih,i'h,z,z') + ...$$

To estimate $D_v^\nu(p-p_\Delta)(\tau,T,v,y)$ we note that

$$\begin{aligned}
&h\left|D_v^\nu \int_\tau^{j^\star h} ds \int p(\tau,s,v,w)H(s,T,w,y)dw\right|\\
&= h\left|D_v^\nu \int_\tau^{j^\star h} ds \int p(\tau,s,v,v+w')H(s,T,v+w',y)dw'\right|\\
&\leq Ch\int_\tau^{j^\star h}\frac{ds}{(T-s)^2}\phi_{\sqrt{T-\tau}}(y-v) \leq Ch^{2-2\delta}\phi_{\sqrt{T-\tau}}(y-v). \qquad (170)
\end{aligned}$$

Furthermore,

$$\left|\frac{h^2}{2}D_v^\nu[p\otimes_h' H_1](\tau,T,v,y)\right| = \left|\frac{h^2}{2}D_v^\nu\sum_{\tau\le jh\le T-h} h\int p(\tau,jh,v,w)H_1(jh,T,w,y)dw\right|$$

$$\le \frac{h^2}{2}\left|\sum_{\tau\le jh\le T-h^\delta/2} h\int D_v^\nu[p(\tau,jh,v,v+w')H_1(jh,T,v+w',y)]dw'\right|$$

$$+\frac{h^2}{2}\left|C\sum_{i,k=1}^{d}\sum_{T-h^\delta/2<jh\le T-h} h\int D_{w'}^{\nu+e_i+e_k}[p(\tau,jh,v,v+w')]\widetilde{p}(jh,T,v+w',y)dw'\right|$$

$$\le Ch^{2-2\delta}\phi_{\sqrt{T-\tau}}(y-v)+Ch^{2-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v). \quad (171)$$

Because of the structure of the operator A_0 it is enough to estimate for fixed i,l,k

$$\frac{h^2}{2}D_v^\nu\sum_{\tau\le jh\le T-h} h\int D_{w'}^{e_k}p(\tau,jh,v,v+w')\frac{\partial^2\widetilde{p}(jh,T,v+w',y)}{\partial w_i'\partial w_l'}dw'. \quad (172)$$

With the same decomposition as in (171) we obtain that (172) does not exceed

$$Ch^{2-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v). \quad (173)$$

From (171) and (173) we obtain that

$$\frac{h^2}{2}\left|D_v^\nu[p\otimes_h'(H_1+A_0)](\tau,T,v,y)\right|\le Ch^{1+\gamma}\phi_{\sqrt{T-\tau}}(y-v) \quad (174)$$

for some $\gamma>0$. It remains to estimate the last summand in (169). It follows from the structure of the operators $A_k, k=1,2,3,4$, that it is enough to estimate

$$\sum_{i=j^\star}^{n-1}\int_{ih}^{(i+1)h}(t-ih)^2\int_0^1(1-\gamma)\sum_{k=1}^{4}\int D_v^\nu\left[p(\tau,s,v,v+w')\frac{\partial^4\widetilde{p}(s,T,v+w',y)}{\partial w_i'\partial w_l'\partial w_p'\partial w_q'}\right]|_{s=ih+\gamma(t-ih)}\,dw'd\gamma dt \quad (175)$$

for fixed i,j,p,q. As above, we obtain that (175) does not exceed

$$Ch^2\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2\int_0^1(1-\gamma)\sum_{i=j^\star}^{n-1} h\frac{1}{[(ih-\tau)+\gamma hz]^{3/2}}\frac{1}{[(n-\gamma z)h-ih]^2}d\gamma dz$$

$$=Ch^2\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2\int_0^1(1-\gamma)\sum_{\{i:j^\star h\le ih\le\tau+h^\delta/4\}}\dots \quad (176)$$

$$+Ch^2\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2\int_0^1(1-\gamma)\sum_{\{i:\tau+h^\delta/4<ih\le T-h\}}\dots$$

$$=I''+II''. \quad (177)$$

Now,

$$\begin{aligned}|I''| &\le Ch^2h^{-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2\int_0^1(1-\gamma)\sum_{\{i:j^\star h\le ih\le\tau+h^\delta/4\}} h\frac{1}{[(ih-\tau)+\gamma hz]}d\gamma dz\\ &\le Ch^{2-\varepsilon}h^{-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^{2-\varepsilon}\int_0^1\frac{(1-\gamma)}{\gamma^\varepsilon}\sum_{\{i:j^\star h\le ih\le\tau+h^\delta/4\}} h\frac{1}{[(ih-\tau)+\gamma hz]^{1-\varepsilon}}d\gamma dz\\ &\le C(\varepsilon)h^{2-\varepsilon-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v). \quad (178)\end{aligned}$$

Using inequality $(h-\gamma z)h - ih = (n-i)h - \gamma zh \geq h(1-\gamma z) \geq h(1-\gamma)$ we obtain that

$$\begin{aligned} |II''| &\leq Chh^{-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2 \int_0^1 d\gamma \sum_{\{i:\tau+h^\delta/4<ih\leq T-h\}} h \\ &\leq Ch^{1-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v). \end{aligned} \tag{179}$$

Now from (169), (170), (174), (175), (178) and (178) we obtain that

$$|D_v^\nu(p\otimes H - p\otimes_h' H)(\tau,T,v,y)| \leq Ch^\gamma\phi_{\sqrt{T-\tau}}(y-v) \tag{180}$$

for some positive γ. The last summand in the right hand side of (169) admits the same estimate (180) because of the smoothing properties of the operation $\otimes_h'$. Hence,

$$|D_v^\nu(p-p_\Delta)(\tau,T,v,y)| \leq Ch^\gamma\phi_{\sqrt{T-\tau}}(y-v). \tag{181}$$

Making the change of variables $v = z+v'$ into (165) we get that the integral w.r.t. v is equal to

$$D_z^\nu\left[\int_u^T d\tau \int p(u,\tau,z,z+v')\chi_\nu(\tau,v)D_v^\nu(p-p_\Delta)(\tau,T,z+v',y)dv'\right]. \tag{182}$$

Taking into account (181) and applying integration by parts in (182) we obtain that (182) does not exceed

$$Ch^\gamma\int_u^T \frac{d\tau}{(\tau-u)^{3/2}}\phi_{\sqrt{T-u}}(y-z) \leq \frac{Ch^\gamma}{\sqrt{T-u}}\phi_{\sqrt{T-u}}(y-z). \tag{183}$$

From (165) and (183) we obtain that

$$|I_3| \leq Ch^{1+\gamma}\phi_{\sqrt{T}}(y-x).$$

The estimate for I_1 can be proved analogously to the estimate for I_3. Thus, we proved that

$$hp\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p-p_\Delta]] = O(h^{1+\delta}\phi_{\sqrt{T}}(y-x)).$$

The estimate

$$h^{1/2}p\otimes\mathcal{F}_1[p-p_\Delta] = O(h^{1+\delta}\phi_{\sqrt{T}}(y-x))$$

can be proved by using the same decomposition of $p-p_\Delta$. This completes the proof of Theorem 1.

References.

1. Bally V., Talay D. (1996 a). The law of the Euler scheme for stochastic differential equations: I. Convergence rate of the distribution function. *Probability Theory and Related Fields*, **104**, 43-60.

2. Bally V., Talay D. (1996 b). The law of the Euler scheme for stochastic differential equations: II. Convergence rate of the density. *Monte Carlo Methods Appl.*, **2**, 93-128.

3. Bertail, P. and Clémençon (2004). Edgeworth expansions of suitably normalized sample mean statistics for atomic Markov chains. *Probability Theory and Related Fields*, **130**, 388-414.

4. Bertail, P. and Clémençon (2006). Regenerative block bootstrap for Markov chains. *Bernoulli*, **12**, 689-712.

5. Bhattacharya R. and Rao R. (1976). *Normal approximations and asymptotic expansions.* John Wiley & Sons, New York.

6. Bolthausen, E. (1980). The Berry-Esseen theorem for functionals of discrete Markov chains. *Z. Wahrsch. verw. Geb.*, **54**, 59-73.

7. Bolthausen, E. (1982). The Berry-Esseen theorem for strongly mixing Harris recurrent Markov chains. *Z. Wahrsch. verw. Geb.*, **60**, 283-289.

8. Friedman A. (1964). *Partial differential equations of parabolic type.* Prentice-Hall, Englewood Cliffs, New Jersey.

9. Fukasawa, M. (2006a). Edgeworth expansion for ergodic diffusions. *Preprint.*

10. Fukasawa, M. (2006b). Regenerative block bootstrap for ergodic diffusions. *Preprint.*

11. Götze, F. (1989). Edgeworth expansions in functional limit theorems. *Ann. Probab.*, **17**, 4, 1602-1634.

12. Götze, F. and Hipp, C. (1983). Asymptotic expansions for sums of weakly dependent random vectors. *Z. Wahrsch. verw. Geb.*, **64**, 211-239.

13. Guyon J. (2006). Euler scheme and tempered distributions. *Stoch. Proc. Appl.* **116**, 877-904.

14. Jacod J. (2004). The Euler scheme for Levy driven stochastic differential equations: limit theorems. *Ann. Probab.*, **32**, 1830-1872.

15. Jacod J., Protter P. (1998). Asymptotic error distributiions for the Euler method for stochastic differential equations. *Ann. Probab.*, **26**, 267-307.

16. Jacod J., Kurtz T., Meleard S., Protter P. (2005). The approximate Euler method for Levy driven stochastic differential equations. *Ann. de l'I.H.P.* , **41**, 523-558.

17. Jensen, J.L. (1989). Asymptotic expansions for strongly mixing Harris recurrent Markov chains. *Scand. J. Statist.*, **16**, 47-63.

18. Konakov V., Molchanov S. (1984). On the convergence of Markov chains to diffusion processes. *Teoria veroyatnostei i matematiceskaya statistika*, **31**, 51-64 (in russian) [English translatiion in *Theory Probab. Math. Stat.* (1985), **31**, 59-73].

19. Konakov V., Mammen E. (2000). Local limit theorems for transition densities of Markov chains converging to diffusions. *Probab. Theory Relat. Fields.* **117**, 551-587.

20. Konakov V., Mammen E. (2002). Edgeworth type expansions for Euler schemes for stochastic differential equations. *Monte Carlo Methods Appl.*, **8**, 271-286.

21. Konakov V., Mammen E. (2005) . Edgeworth-type expansions for transition densities of Markov chains converging to diffusions. *Bernoulli*, **11**, 4, 591-641.

22. Kusuoka, S. and Yoshida, N. (2000). Malliavin calculus, geometric mixing, and expansion of diffusion functionals. *Probability Theory and Related Fields*, **116**, 457-484.

23. Ladyzenskaya O.A., Solonnikov V.A., Ural'ceva N. (1968). *Linear and quasi-linear equations of parabolic type.* Amer. Math. Soc., Providence, Phode Island.

24. Malinovskii, V.K. (1987). Limit theorems for Harris Markov chains, 1. . *Theory Probab. Appl.*, **31**, 269-285.

25. Mykland, P.A.(1992). Aymptotic expansions and bootstrapping distributions for dependent variables: A martingale approach. *Ann. Statist.*, **20**, 623-654.

26. McKean H.P., Singer I.M. (1967). Curvature and the eigenvalues of the Laplacian. *J. Diff. Geometry*, **1**, 43-69.

27. Protter P., Talay D. (1997). The Euler scheme for Levy driven stochastic differential equations. *Ann.Probab.*, **25**, 393-323.

28. Skorohod A.V. (1965). *Studies in the theory of random processes.* Addison-Wesley. Reading, Massachussetts. [English translation of Skorohod A. V. (1961). *Issledovaniya po teorii sluchainykh processov.* Kiev University Press].

29. Stroock D.W., Varadhan S.R. (1979). *Multidimensional diffusion processes.* Springer, Berlin, Heidelberg, New York.

30. Yoshida, N. (2004). Partial mixing and Edgeworth expansion. *Probability Theory and Related Fields*, **129**, 559-624.