

Solution of Time-Fractional Korteweg–de Vries Equation in Warm Plasma

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Abstract

The reductive perturbation method has been employed to derive the Korteweg-de Vries (KdV) equation for small but finite amplitude ion-acoustic waves. The Lagrangian of the time fractional KdV equation is used in similar form to the Lagrangian of the regular KdV equation. The variation of the functional of this Lagrangian leads to the Euler-Lagrange equation that leads to the time fractional KdV equation. The Riemann-Liouville definition of the fractional derivative is used to describe the time fractional operator in the fractional KdV equation. The variational-iteration method given by He is used to solve the derived time fractional KdV equation. The calculations of the solution with initial condition $A_0 \operatorname{sech}(cx)^2$ are carried out. The result of the present investigation may be applicable to some plasma environments, such as ionosphere.

Keywords : Ion-acoustic waves; Euler-Lagrange equation, Riemann-Liouville fractional derivative, fractional KdV equation, He's variational-iteration method.

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1 Introduction

The ion-acoustic solitary wave is one of the fundamental nonlinear wave phenomena appearing in fluid dynamics [1] and plasma physics [2, 3]. There

are different methods to study nonlinear systems. Washimi and Taniti [4] were the first to use reductive perturbation method to study the propagation of a slow modulation of a quasi-monochromatic waves through plasma. And then the attention has been focused by many authors [5, 6]. The evolution of small but finite-amplitude solitary waves, studied by means of the Korteweg–deVries (KdV) equation, is of considerable interest in plasma dynamics.

Because most classical processes observed in the physical world are non-conservative, it is important to be able to apply the power of variational methods to such cases. A method used a Lagrangian that leads to an Euler-Lagrange equation that is, in some sense, equivalent to the desired equation of motion. Hamilton's equations are derived from the Lagrangian and are equivalent to the Euler-Lagrange equation. If a Lagrangian is constructed using noninteger-order derivatives, then the resulting equation of motion can be nonconservative. It was shown that such fractional derivatives in the Lagrangian describe nonconservative forces [7, 8]. Further study of the fractional Euler-Lagrange can be found in the work of Agrawal [9, 10], Baleanu and coworkers [11, 12] and Tarasov and Zaslavsky [13, 14]. During the last decades, Fractional Calculus has been applied to almost every field of science, engineering and mathematics. Some of the areas where Fractional Calculus has been applied include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory [15].

To the author's knowledge, the problem of time fractional KdV equation in collisionless plasma has not been addressed in the literature before. So, our motive here is to study the effects of time fractional parameter on the electrostatic structures for a system of collisionless plasma consisting of a mixture of warm ion-fluid and isothermal electrons. We expect that the inclusion of time fractional parameter will change the properties as well as the regime of existence of solitons.

Several methods have been used to solve fractional differential equations such as: the Laplace transform method, the Fourier transform method, the iteration method and the operational method [16]. Recently, there are some papers deal with the existence and multiplicity of solution of nonlinear fractional differential equation by the use of techniques of nonlinear analysis [17-18]. In this paper, the resultant fractional KdV equation will be solved using a variational-iteration method (VIM) firstly used by He [19].

This paper is organized as follows: Section 2 is devoted to describe the formulation of the time-fractional KdV (FKdV) equation using the variational

Euler-Lagrange method. In section 3, variational-Iteration Method (VIM) is discussed. The resultant time-FKdV equation is solved approximately using VIM. Section 5 contains the results of calculations and discussion of these results.

2 Basic Equations and KdV Equation

Consider a collisionless ionization-free unmagnetized plasma consisting of a mixture of warm ion-fluid and isothermal electrons. The basic equations describing the system in dimensionless variables is governed by [6] :

$$\frac{\partial n(x, t)}{\partial t} + \frac{\partial [n(x, t)u(x, t)]}{\partial x} = 0, \quad (1.a)$$

$$\left(\frac{\partial u(x, t)}{\partial t} + u(x, t)\frac{\partial u(x, t)}{\partial x}\right) + \frac{\sigma}{n(x, t)}\frac{\partial p(x, t)}{\partial x} + \frac{\partial \phi(x, t)}{\partial x} = 0, \quad (1.b)$$

$$\left[\frac{\partial}{\partial t} + u(x, t)\frac{\partial}{\partial x}\right]p(x, t) + 3p(x, t)\frac{\partial u(x, t)}{\partial x} = 0, \quad (1.c)$$

$$\frac{\partial^2 \phi(x, t)}{\partial x^2} + n(x, t) - n_e(x, t) = 0, \quad (1.d)$$

$$n_e(x, t) - \exp[\phi(x, t)] = 0. \quad (1.e)$$

In the earlier equations $n(x, t)$ and $n_e(x, t)$ are the densities of ions and electrons respectively, $u(x, t)$ is the ion flow velocity, $p(x, t)$ is the ion pressure, $\phi(x, t)$ is the electric potential, x is the space co-ordinate and t is the time variable. $\sigma = T_i/T_e \ll 1$ is the ratio of the ion temperature T_i to the electron temperature T_e . All these quantities are dimensionless, being normalized in terms of the following characteristic quantities: $n(x, t)$ and $n_e(x, t)$ by the unperturbed electron density n_0 , $u(x, t)$ by the sound velocity $(KT_e/m_i)^{1/2}$; $p(x, t)$ and $\phi(x, t)$ by n_0KT_i and KT_e/e , respectively. The time variable t and the space x by the inverse of the plasma frequency $\omega_{pi}^{-1} = (4\pi e^2 n_0/m_i)^{-1/2}$ and the electron Debye length $\lambda_D = (KT_e/4\pi e^2 n_0)^{1/2}$, respectively. K is the Boltzmann's constant and m_i is the mass of plasma ion.

According to the general method of reductive perturbation theory, we introduce the stretched variables

$$\tau = \epsilon^{3/2}t, \quad \xi = \epsilon^{1/2}(x - vt), \quad (2)$$

where v is the unknown phase velocity. All the physical quantities appeared in (1) are expanded as power series in ϵ about the equilibrium values as

$$n(\xi, \tau) = 1 + \epsilon n_1(\xi, \tau) + \epsilon^2 n_2(\xi, \tau) + \dots, \quad (3a)$$

$$u(\xi, \tau) = \epsilon u_1(\xi, \tau) + \epsilon^2 u_2(\xi, \tau) + \dots, \quad (3b)$$

$$p(\xi, \tau) = 1 + \epsilon p_1(\xi, \tau) + \epsilon^2 p_2(\xi, \tau) + \dots, \quad (3c)$$

$$\phi(\xi, \tau) = \epsilon \phi_1(\xi, \tau) + \epsilon^2 \phi_2(\xi, \tau) + \dots, \quad (3d)$$

where ϵ represents the amplitude of the perturbation. We impose the boundary conditions that as $|\xi| \rightarrow \infty$, $n = n_e = p = 1$, $u = \phi = 0$. Substituting (2) and (3) into the system of equations (1) and equating coefficients of like powers of ϵ . Then, from the lowest, second-order equations in ϵ and by elimination the second order perturbed quantities $n_2(\xi, \tau)$, $u_2(\xi, \tau)$, $p_2(\xi, \tau)$ and $\phi_2(\xi, \tau)$, we obtain the following KdV equation for the first-order perturbed potential:

$$\frac{\partial}{\partial \tau} \phi_1(\xi, \tau) + A \phi_1(\xi, \tau) \frac{\partial}{\partial \xi} \phi_1(\xi, \tau) + B \frac{\partial^3}{\partial \xi^3} \phi_1(\xi, \tau) = 0, \quad (4.a)$$

where both the nonlinear coefficient A and the dispersion coefficient B are given by

$$A = \frac{6\sigma + 1}{\sqrt{3\sigma + 1}}, \quad B = \frac{1}{2\sqrt{3\sigma + 1}}. \quad (4.b)$$

In equation (4.a), $\phi_1(\xi, \tau)$ is a field variable, ξ is a space coordinate in the propagation direction of the field and $\tau \in T (= [0, T_0])$ is the time variable. The resultant KdV equation (4.a) can be converted into time-fractional KdV equation as follows:

Using a potential function $V(\xi, \tau)$ where $\phi_1(\xi, \tau) = V_\xi(\xi, \tau)$ gives the potential equation of the regular KdV equation (1) in the form

$$V_{\xi\tau}(\xi, \tau) + A V_\xi(\xi, \tau) v_{\xi\xi}(\xi, \tau) + B V_{\xi\xi\xi\xi}(\xi, \tau) = 0, \quad (5)$$

where the subscripts denote the partial differentiation of the function with respect to the parameter. The Lagrangian of this regular KdV equation (4.a) can be defined using the semi-inverse method [20, 21] as follows.

The functional of the potential equation (5) can be represented by

$$J(V) = \int_R d\xi \int_T d\tau \{V(\xi, \tau)[c_1 V_{\xi\tau}(\xi, \tau) + c_2 AV_{\xi}(\xi, \tau)v_{\xi\xi}(\xi, \tau) + c_3 BV_{\xi\xi\xi\xi}(\xi, \tau)]\}, \quad (6)$$

where c_1 , c_2 and c_3 are constants to be determined. Integrating by parts and taking $V_{\tau}|_R = V_{\xi}|_R = V_{\xi}|_T = 0$ lead to

$$J(V) = \int_R d\xi \int_T d\tau \{V(\xi, \tau)[-c_1 V_{\xi}(\xi, \tau)V_{\tau}(\xi, \tau) - \frac{1}{2}c_2 AV_{\xi}^3(\xi, \tau) + c_3 BV_{\xi\xi}^2(\xi, \tau)]\}. \quad (7)$$

The unknown constants c_i ($i = 1, 2, 3$) can be determined by taking the variation of the functional (7) to make it optimal. Taking the variation of this functional, integrating each term by parts and make the variation optimum give the following relation

$$2c_1 V_{\xi\tau}(\xi, \tau) + 3c_2 AV_{\xi}(\xi, \tau)V_{\xi\xi}(\xi, \tau) + 2c_3 BV_{\xi\xi\xi\xi}(\xi, \tau) = 0. \quad (8)$$

As this equation must be equal to equation (5), the unknown constants are given as

$$c_1 = 1/2, c_2 = 1/3 \text{ and } c_3 = 1/2. \quad (9)$$

Therefore, the functional given by (7) gives the Lagrangian of the regular KdV equation as

$$L(V_{\tau}, V_{\xi}, V_{\xi\xi}) = -\frac{1}{2}V_{\xi}(\xi, \tau)V_{\tau}(\xi, \tau) - \frac{1}{6}AV_{\xi}^3(\xi, \tau) + \frac{1}{2}BV_{\xi\xi}^2(\xi, \tau). \quad (10)$$

Similar to this form, the Lagrangian of the time-fractional version of the KdV equation can be written in the form

$$F({}_0D_{\tau}^{\alpha}V, V_{\xi}, V_{\xi\xi}) = -\frac{1}{2}[{}_0D_{\tau}^{\alpha}V(\xi, \tau)]V_{\xi}(\xi, \tau) - \frac{1}{6}AV_{\xi}^3(\xi, \tau) + \frac{1}{2}BV_{\xi\xi}^2(\xi, \tau), \quad (11)$$

$$0 \leq \alpha < 1,$$

where the fractional derivative is represented, using the left Riemann-Liouville fractional derivative definition as [16]

$$\begin{aligned} {}_a D_t^\alpha f(t) &= \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \left[\int_a^t d\tau (t-\tau)^{k-\alpha-1} f(\tau) \right], \\ k-1 &\leq \alpha \leq 1, t \in [a, b]. \end{aligned} \quad (12)$$

The functional of the time-FKdV equation can be represented in the form

$$J(V) = \int_R d\xi \int_T d\tau F({}_0 D_\tau^\alpha V, V_\xi, V_{\xi\xi}), \quad (13)$$

where the time-fractional Lagrangian $F({}_0 D_\tau^\alpha V, V_\xi, V_{\xi\xi})$ is defined by (11).

Following Agrawal's method [9, 10], the variation of functional (13) with respect to $V(\xi, \tau)$ leads to

$$\delta J(V) = \int_R d\xi \int_T d\tau \left\{ \frac{\partial F}{\partial {}_0 D_\tau^\alpha V} \delta {}_0 D_\tau^\alpha V + \frac{\partial F}{\partial V_\xi} \delta V_\xi + \frac{\partial F}{\partial V_{\xi\xi}} \delta V_{\xi\xi} \right\}. \quad (14)$$

The formula for fractional integration by parts reads [9, 16]

$$\int_a^b dt f(t) {}_a D_t^\alpha g(t) = \int_a^b dt g(t) {}_t D_b^\alpha f(t), \quad f(t), g(t) \in [a, b], \quad (15)$$

where ${}_t D_b^\alpha$, the right Riemann-Liouville fractional derivative, is defined by [16]

$$\begin{aligned} {}_t D_b^\alpha f(t) &= \frac{(-1)^k}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \left[\int_t^b d\tau (\tau-t)^{k-\alpha-1} f(\tau) \right], \\ k-1 &\leq \alpha \leq 1, t \in [a, b]. \end{aligned} \quad (16)$$

Integrating the right-hand side of (14) by parts using formula (15) leads to

$$\delta J(V) = \int_R d\xi \int_T d\tau \left[{}_\tau D_{T_0}^\alpha \left(\frac{\partial F}{\partial {}_0 D_\tau^\alpha V} \right) - \frac{\partial}{\partial \xi} \left(\frac{\partial F}{\partial V_\xi} \right) + \frac{\partial^2}{\partial \xi^2} \left(\frac{\partial F}{\partial V_{\xi\xi}} \right) \right] \delta V, \quad (17)$$

where it is assumed that $\delta V|_T = \delta V|_R = \delta V_\xi|_R = 0$.

Optimizing this variation of the functional $J(V)$, i. e; $\delta J(V) = 0$, gives the Euler-Lagrange equation for the time-FKdV equation in the form

$${}_\tau D_{T_0}^\alpha \left(\frac{\partial F}{\partial_0 D_\tau^\alpha V} \right) - \frac{\partial}{\partial \xi} \left(\frac{\partial F}{\partial V_\xi} \right) + \frac{\partial^2}{\partial \xi^2} \left(\frac{\partial F}{\partial V_{\xi\xi}} \right) = 0. \quad (18)$$

Substituting the Lagrangian of the time-FKdV equation (11) into this Euler-Lagrange formula (18) gives

$$-\frac{1}{2} {}_\tau D_{T_0}^\alpha V_\xi(\xi, \tau) + \frac{1}{2} {}_0 D_\tau^\alpha V_\xi(\xi, \tau) + AV_\xi(\xi, \tau)V_{\xi\xi}(\xi, \tau) + BV_{\xi\xi\xi}(\xi, \tau) = 0. \quad (19)$$

Substituting for the potential function, $V_\xi(\xi, \tau) = \phi_1(\xi, \tau) = \Phi(\xi, \tau)$, gives the time-FKdV equation for the state function $\Phi(\xi, \tau)$ in the form

$$\frac{1}{2} [{}_0 D_\tau^\alpha \Phi(\xi, \tau) - {}_\tau D_{T_0}^\alpha \Phi(\xi, \tau)] + A\Phi(\xi, \tau)\Phi_\xi(\xi, \tau) + B\Phi_{\xi\xi\xi}(\xi, \tau) = 0, \quad (20)$$

where the fractional derivatives ${}_0 D_\tau^\alpha$ and ${}_\tau D_{T_0}^\alpha$ are, respectively the left and right Riemann-Liouville fractional derivatives and are defined by (12) and (16).

The time-FKdV equation represented in (20) can be rewritten by the formula

$$\frac{1}{2} {}_0^R D_\tau^\alpha \Phi(\xi, \tau) + A \Phi(\xi, \tau)\Phi_\xi(\xi, \tau) + B \Phi_{\xi\xi\xi}(\xi, \tau) = 0, \quad (21)$$

where the fractional operator ${}_0^R D_\tau^\alpha$ is called Riesz fractional derivative and can be represented by [10, 16]

$$\begin{aligned} {}_0^R D_t^\alpha f(t) &= \frac{1}{2} [{}_0 D_t^\alpha f(t) + (-1)^k {}_t D_{T_0}^\alpha f(t)] \\ &= \frac{1}{2} \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \left[\int_a^t d\tau |t-\tau|^{k-\alpha-1} f(\tau) \right], \\ k-1 &\leq \alpha \leq 1, t \in [a, b]. \end{aligned} \quad (22)$$

The nonlinear fractional differential equations have been solved using different techniques [16-20]. In this paper, a variational-iteration method (VIM) [21, 22] has been used to solve the time-FKdV equation that formulated using Euler-Lagrange variational technique.

3 Variational-Iteration Method

Variational-iteration method (VIM) [21] has been used successfully to solve different types of integer nonlinear differential equations [22, 23]. Also, VIM is used to solve linear and nonlinear fractional differential equations [24, 25]. This VIM has been used in this paper to solve the formulated time-FKdV equation.

A general Lagrange multiplier method is constructed to solve non-linear problems, which was first proposed to solve problems in quantum mechanics [21]. The VIM is a modification of this Lagrange multiplier method [22]. The basic features of the VIM are as follows. The solution of a linear mathematical problem or the initial (boundary) condition of the nonlinear problem is used as initial approximation or trail function. A more highly precise approximation can be obtained using iteration correction functional.

Considering a nonlinear partial differential equation consists of a linear part $\hat{L}U(x, t)$, nonlinear part $\hat{N}U(x, t)$ and a free term $f(x, t)$ represented as

$$\hat{L}U(x, t) + \hat{N}U(x, t) = f(x, t), \quad (23)$$

where \hat{L} is the linear operator and \hat{N} is the nonlinear operator. According to the VIM, the $(n + 1)$ th approximation solution of (23) can be given by the iteration correction functional as [21, 22]

$$U_{n+1}(x, t) = U_n(x, t) + \int_0^t d\tau \lambda(\tau) [\hat{L}U_n(x, \tau) + \hat{N}\hat{U}_n(x, \tau) - f(x, \tau)],$$

$$n \geq 0, \quad (24)$$

where $\lambda(\tau)$ is a Lagrangian multiplier and $\hat{U}_n(x, \tau)$ is considered as a restricted variation function, i. e; $\delta\hat{U}_n(x, \tau) = 0$. Extreme the variation of the correction functional (24) leads to the Lagrangian multiplier $\lambda(\tau)$. The initial iteration can be used as the solution of the linear part of (23) or the initial value $U(x, 0)$. As n tends to infinity, the iteration leads to the exact solution of (23), i. e;

$$U(x, t) = \lim_{n \rightarrow \infty} U_n(x, t). \quad (25)$$

For linear problems, the exact solution can be given using this method in only one step where its Lagrangian multiplier can be exactly identified.

4 Solution of the time-FKdV equation

The time-FKdV equation represented by (21) can be solved using the VIM by the iteration correction functional (24) as follows:

Affecting from left by the fractional operator on (21) leads to

$$\begin{aligned} \frac{\partial}{\partial \tau} \Phi(\xi, \tau) &= {}_0^R D_\tau^{\alpha-1} \Phi(\xi, \tau) |_{\tau=0} \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\quad - {}_0^R D_\tau^{1-\alpha} [A \Phi(\xi, \tau) \frac{\partial}{\partial \xi} \Phi(\xi, \tau) + B \frac{\partial^3}{\partial \xi^3} \Phi(\xi, \tau)], \\ 0 &\leq \alpha \leq 1, \tau \in [0, T_0], \end{aligned} \quad (26)$$

where the following fractional derivative property is used [16]

$$\begin{aligned} {}_a^R D_b^\alpha [{}_a^R D_b^\beta f(t)] &= {}_a^R D_b^{\alpha+\beta} f(t) - \sum_{j=1}^k {}_a^R D_b^{\beta-j} f(t) |_{t=a} \frac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \\ k-1 &\leq \beta < k. \end{aligned} \quad (27)$$

As $\alpha < 1$, the Riesz fractional derivative ${}_0^R D_\tau^{\alpha-1}$ is considered as Riesz fractional integral ${}_0^R I_\tau^{1-\alpha}$ that is defined by [10, 16]

$$\begin{aligned} {}_0^R I_\tau^\alpha f(t) &= \frac{1}{2} [{}_0 I_\tau^\alpha f(t) + {}_\tau I_b^\alpha f(t)] \\ &= \frac{1}{2} \frac{1}{\Gamma(\alpha)} \int_a^b d\tau |t-\tau|^{\alpha-1} f(\tau), \quad \alpha > 0, \end{aligned} \quad (28)$$

where ${}_0 I_\tau^\alpha f(t)$ and ${}_\tau I_b^\alpha f(t)$ are the left and right Riemann-Liouville fractional integrals, respectively [16].

The iterative correction functional of equation (26) is given as

$$\begin{aligned} \Phi_{n+1}(\xi, \tau) &= \Phi_n(\xi, \tau) + \int_0^\tau d\tau' \lambda(\tau') \left\{ \frac{\partial}{\partial \tau'} \Phi_n(\xi, \tau') \right. \\ &\quad \left. - {}_0^R I_{\tau'}^{1-\alpha} \Phi_n(\xi, \tau') |_{\tau'=0} \frac{\tau'^{\alpha-2}}{\Gamma(\alpha-1)} \right. \\ &\quad \left. + {}_0^R D_{\tau'}^{1-\alpha} [A \tilde{\Phi}_n(\xi, \tau') \frac{\partial}{\partial \xi} \tilde{\Phi}_n(\xi, \tau') + B \frac{\partial^3}{\partial \xi^3} \tilde{\Phi}_n(\xi, \tau')] \right\}, \end{aligned} \quad (29)$$

where $n \geq 0$ and the function $\Phi_n(\xi, \tau)$ is considered as a restricted variation function, i. e; $\delta\Phi_n(\xi, \tau) = 0$. The extreme of the variation of (29) using the restricted variation function leads to

$$\begin{aligned}\delta\Phi_{n+1}(\xi, \tau) &= \delta\Phi_n(\xi, \tau) + \int_0^\tau d\tau' \lambda(\tau') \delta \frac{\partial}{\partial \tau'} \Phi_n(\xi, \tau') \\ &= \delta\Phi_n(\xi, \tau) + \lambda(\tau) \delta\Phi_n(\xi, \tau) - \int_0^\tau d\tau' \frac{\partial}{\partial \tau'} \lambda(\tau') \delta\Phi_n(\xi, \tau') = 0.\end{aligned}$$

This relation leads to the stationary conditions $1 + \lambda(\tau) = 0$ and $\frac{\partial}{\partial \tau'} \lambda(\tau') = 0$, which leads to the Lagrangian multiplier as $\lambda(\tau') = -1$. Therefore, the correction functional (29) is given by the form

$$\begin{aligned}\Phi_{n+1}(\xi, \tau) &= \Phi_n(\xi, \tau) - \int_0^\tau d\tau' \left\{ \frac{\partial}{\partial \tau'} \Phi_n(\xi, \tau') \right. \\ &\quad \left. - {}^R I_{\tau'}^{1-\alpha} \Phi_n(\xi, \tau') \Big|_{\tau'=0} \frac{\tau'^{\alpha-2}}{\Gamma(\alpha-1)} \right. \\ &\quad \left. + {}^R D_{\tau'}^{1-\alpha} [A \Phi_n(\xi, \tau') \frac{\partial}{\partial \xi} \Phi_n(\xi, \tau') + B \frac{\partial^3}{\partial \xi^3} \Phi_n(\xi, \tau')] \right\} \quad (30)\end{aligned}$$

where $n \geq 0$.

In Physics, if τ denotes the time-variable, the right Riemann-Liouville fractional derivative is interpreted as a future state of the process. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development [9]. Therefore, the right-derivative is used equal to zero in the following calculations.

The zero order correction of the solution can be taken as the initial value of the state variable, which is taken in this case as

$$\Phi_0(\xi, \tau) = \Phi(\xi, 0) = A_0 \sec h^2(c\xi). \quad (31)$$

where $A_0 = \frac{3v}{A}$ and $c = \frac{1}{2} \sqrt{\frac{v}{B}}$ are constants.

Substituting this zero order approximation into (30) and using the definition of the fractional derivative (22) lead to the first order approximation as

$$\begin{aligned}
\Phi_1(\xi, \tau) = & A_0 \sec h^2(c\xi) \\
& + 2A_0c \sinh(c\xi) \sec h^3(c\xi) \\
& * [4c^2B + (A_0A - 12c^2B) \sec h^2(c\xi)] \frac{\tau^\alpha}{\Gamma(\alpha + 1)}. \quad (32)
\end{aligned}$$

Substituting this equation into (30), using the definition (22) and the Maple package lead to the second order approximation in the form

$$\begin{aligned}
\Phi_2(\xi, \tau) = & A_0 \sec h^2(c\xi) \\
& + 2A_0c \sinh(c\xi) \sec h(c\xi)^3 \\
& * [4c^2B + (A_0A - 12c^2B) \sec h(c\xi)^2] \frac{\tau^\alpha}{\Gamma(\alpha + 1)} \\
& + 2A_0c^2 \sec h(c\xi)^2 \\
& * [32c^4B^2 + 16c^2B(5A_0A - 63c^2B) \sec h(c\xi)^2 \\
& + 2(3A_0^2A^2 - 176A_0c^2AB + 1680c^4B^2) \sec h(c\xi)^4 \\
& - 7(A_0^2A^2 - 42A_0c^2AB + 360c^4B^2) \sec h(c\xi)^6] \frac{\tau^{2\alpha}}{\Gamma(2\alpha + 1)} \\
& + 4A_0^2c^3 \sinh(c\xi) \sec h(c\xi)^5 \\
& * [32c^4B^2 + 24c^2B(A_0A - 14c^2B) \sec h(c\xi)^2 \\
& + 4(A_0^2A^2 - 32A_0c^2AB + 240c^4B^2) \sec h(c\xi)^4 \\
& - 5(A_0^2A^2 - 24A_0c^2AB + 144c^4B^2) \sec h(c\xi)^6] \\
& * \frac{\Gamma(2\alpha + 1)}{[\Gamma(\alpha + 1)]^2} \frac{\tau^{3\alpha}}{\Gamma(3\alpha + 1)} \quad (33)
\end{aligned}$$

The higher order approximations can be calculated using the Maple or the Mathematica package to the appropriate order where the infinite approximation leads to the exact solution.

5 Results and calculations

For a small amplitude ion-acoustic solitary wave in unmagnetized collisionless plasma consisting of a mixture of warm ion-fluid and isothermal electrons, we have obtained the Korteweg-de Vries equation by using the reductive

perturbation method. The time fractional Korteweg-de Vries equation has been derived using the variational technique [9, 20]. The Riemann-Liouville fractional derivative is used to describe the time fractional operator in the time-FKdV equation. He's variational-iteration method [21, 22] is used to solve the derived time-FKdV equation. The calculations in this work is carried out using the fourth order approximation of the VIM.

However, since one of our motivations was to study effects of the ratio of the ion temperature to the electron temperature σ and the time fractional order α on the existence of solitary waves. The present system admits a solitary wave solution for any order approximation. In Fig (1), a profile of the solution of the time FKdV equation as a solitary pulse is obtained. Figure (2) shows that the both the amplitude and the width of the solitary wave decrease with the increase of the temperatures ratio σ . Also, the time fractional order α and the velocity v increase the soliton amplitude as shown in Fig (3). In summery, it has been found that amplitude of the ion-acoustic waves as well as parametric regime where the solitons can exist is sensitive to the temperatures ratio σ and the time fractional order α . Moreover, the time fractional order α plays a role of higher order perturbation theory in increasing the soliton amplitude. The application of our model might be particularly interesting in some plasma environments, such as ionosphere.

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6 Figure Captions

Fig. (1): The electrostatic potential $\Phi(\xi, \tau)$ against the position ξ and the time τ for $\sigma = 0.5$, $v = 0.4$ and $\alpha = 0.5$.

Fig. (2): The electrostatic potential $\Phi(\xi, \tau)$ against the position ξ and the temperatures ratio σ at $\tau = 1$, $v = 0.4$ and $\alpha = 0.5$.

Fig. (3): The electrostatic potential $\Phi(\xi, \tau)$ against the velocity v and the fractional order α at $\xi = 0$, $\tau = 1$ and $\sigma = 0.5$.