Nonlinear Two-Dimensional Green's Function in Smectics

E A. Brener[1], V. I. Marchenko[2,3] and D. Pilipenko[1],
[1] Institut für Festkörperforschung, Forschungszentrum Jülich, D-52425 Jülich, Germany
[2] Kapitza Institute for Physical Problems, Russian Academy of Sciences, ul. Kosygina 2, Moscow, 119334 Russia
[3] Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudnyi, Moscow region, 141700 Russia
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The problem of the strain of smectics subjected to a force distributed over a line in the basal plane has been solved.

The asymptotic expressions for strains around isolated defects in smectics at long distances are characterized by the exponent α [1]. If $\alpha < 0$, the linear theory is applicable. If $\alpha = 0$, (edge dislocation [2, 3], Greens function [4]), the linear solution is valid at small action amplitudes, whereas nonlinear effects become important at larger amplitudes. If $\alpha > 0$, nonlinear effects should be taken into account even for extremely weak actions. In this paper, we report the solution of the nonlinear problem of the two-dimensional Greens function ($\alpha = 1/2$) [1]. Let us consider a smectic sample with the thickness L that is sandwiched between solid undeformable walls parallel to smectic layers (see figure). A force uniformly distributed along the y axis (normal to the figure plane) with the linear density F is applied at the center ($x = z = 0$) of the smectic sample. The energy of the small strains of the smectic sample is given by the expression (see Eq. (44.13) in [5])

$$E = \frac{A}{2}\int \left\{ \left(\partial_z u - \frac{(\partial_x u)^2}{2} \right)^2 + \lambda^2 (\partial_x^2 u)^2 \right\} dV, \qquad (1)$$

where u is the displacement of the layers along the smectic axis z, A is the elastic modulus, and λ is the microscopic length parameter. In our problem, the maximum displacement u_0 is reached in the force application line. In terms of the new function $f = u/u_0$ and new coordinates $\tilde{z} = z/L$ and $\tilde{x} = x/L$, where $\varepsilon = u_0/L$, Eq. (1) is represented in the form

$$E = 2^{-1} A L^2 \varepsilon^{5/2} \int \left(\sigma^2 + \beta (f'')^2 \right) d\tilde{z} d\tilde{x}, \qquad (2)$$

where $\beta = (\lambda/\varepsilon L)^2$, $\sigma = \dot{f} - (f')^2/2$, and the dot and prime mean differentiations with respect to $\tilde{z}$ and $\tilde{x}$ respectively. Thus, it is necessary to determine the function f that provides the minimum of energy (2), is equal to unity at $\tilde{x} = \tilde{z} = 0$, and is equal to zero at the edges of the smectic layer $\tilde{z} = \pm 1/2$. The force is given by the expression $F = F_+ - F_-$, where

$$F_\pm = \mp \int \sigma_{zz} dS = \mp A L \varepsilon^{3/2} \int \sigma d\tilde{x}. \qquad (3)$$

In the macroscopic problem, curvature $\propto \beta$ can be neglected even for the case of a negligibly small force $F \sim \lambda A \sqrt{\lambda/L}$, where the amplitude u_0 is larger than the distance between smectic layers $\sim \lambda \ll L$. It is interesting that integrals (2) and (3) can be calculated in this case even without the complete solution, because the strain field is divided into two regions. In the first region, where the material is compressed, the problem can be solved analytically. In the second region, owing to the Helfrish instability (see the problem in Section 44 in [5]), rotary states almost without stresses appear instead of tension. Here, noticeable stresses exist only inside microscopically thin twin boundaries [6].

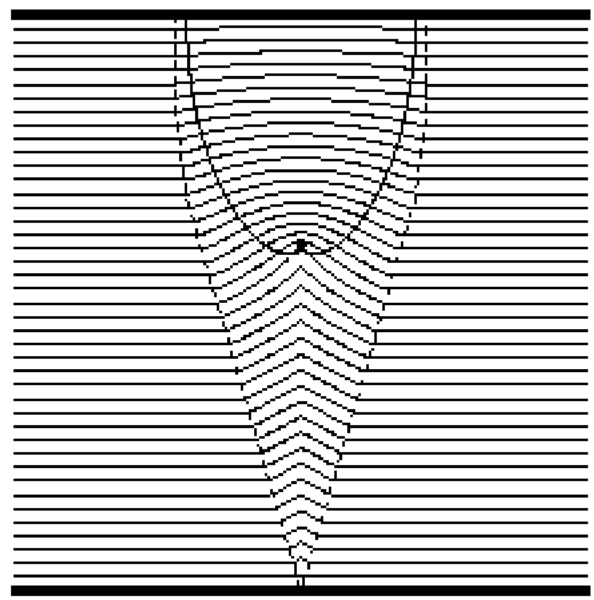

FIG. 1: Strain of the smectic layer subjected to the linearly distributed force($\varepsilon = 0.1$)applied at the thick point. The dashed line is the boundary of the region, where a noticeable strain appears. The thin line is the boundary of the compression region($\sigma < 0$).

The figure shows the strain pattern in the smectic layer obtained by numerically solving the problem. In the compression region, the function f is a quadratic function of the variable $\tilde{x}$: $f(\tilde{x},\tilde{z}) = f_0(\tilde{z}) + f_2(\tilde{z})\tilde{x}^2$. It is easy to verify that the equilibrium equation $\dot{\sigma} - (f'\sigma)' = 0$ corresponding to the extremum of energy (2) has such an exact solution satisfying the necessary conditions. It can be represented in the parametric form

$$f_0 = 1 - \frac{2\varphi}{\pi}, f_2 = -\frac{\pi\cos\varphi}{2\sin^3\varphi}, \tilde{z} = \frac{\varphi - \sin\varphi\cos\varphi}{\pi}. \quad (4)$$

The compression region $\sigma < 0$ corresponds to the range $-\tilde{x}_m < \tilde{x} < \tilde{x}_m$, at $\tilde{z} > 0$, where $\tilde{x}_m = 2\sin^2\varphi/\pi$. The thin line in the figure is the boundary taking into account the displacement of smectic layers, where σ is zero. Under this line, down to the dashed line, strains result in the appearance of two twins [6] in which inhomogeneous strains increase in the direction to the force application level. Here, stresses are small and are likely caused by the finiteness of the grid used for the calculation.

When the tension of the twin boundaries is disregarded (at $\beta = 0$) [6], $F_- = 0$. Then, $F = F_+ = (8/3\pi)AL\varepsilon^{3/2}$. Correspondingly,

$$u_0 = \left(\frac{3\pi a}{8}\right)^{2/3} L^{1/3}, \, a = \frac{F}{A}. \quad (5)$$

At $\tilde{z} \ll 1$, according to (4), $u = u_0 + \delta u$, where

$$\delta u = -3\left(\frac{a^2 z}{16}\right)^{1/3} - \frac{x^2}{3z}, \, \delta u \ll u_0. \quad (6)$$

Let us assume that the displacement near the force application point has the form $\delta u = -z^{1/3}a^{2/3}\psi(v)$, where $v = xa^{-1/3}z^{-2/3}$. Note that $\delta u$is independent of L. In this case, the equilibrium equation is the ordinary differential equation

$$\left(\left(2\psi - 4v\psi' + 3(\psi')^2\right)\left(3\psi' - 2v\right)\right)' = 0, \quad (7)$$

where the prime means differentiation with respect to v, and has a trivial first integral. The integration constant is zero, because the expression in the second parentheses in Eq. (7) is zero for solution (6). The expression in the first parentheses is proportional to stress σ. It is convenient to represent the solution of the equation $\sigma = 0$ in the parametric form $\psi = (4+t^3)/2t$, and $v = (1+t^3)/t^2$. The intervals $-1 < t < 0$ and $0 < t < 2^{1/3} \to z > 0$ correspond to the regions $z < 0$ and $z > 0$ respectively. The integration constant is chosen from the condition $\psi\left(3/2^{2/3}\right) = 3/2^{1/3}$ at $t = 2^{1/3}$ (matching with solution (6) on the line $x_m(z) \simeq 3(az^2/4)^{1/3}$). Near the $z = 0$ line ($|v| \to \infty$),

$$\delta u = -2\sqrt{a|x|} + \frac{az}{2|x|}.$$

At the $x = 0$ line at $z < 0$ ($|v| \to 0$)

$$\delta u = -\frac{3}{2}a^{2/3}|z|^{1/3} - \frac{a^{1/3}|x|}{|z|^{1/3}}.$$

If $a \ll \lambda$, onlinear asymptotic expressions are valid at $|z| \gg \lambda^3/a^2$, and $|x| \gg \lambda^2/a$, and the Greens function of the linear approximation is applicable at smaller distances [1]. If $a \gg \lambda$, small-strain approximation (1) is violated at $|z| \sim |x| \sim a$.

Stresses in the problem under consideration exist only in the smectic compression region and rotary adjustment in a certain bounded region occurs instead of tension. Such a character of nonlinear response implies that the compression field and, correspondingly, the u_0 value (with the change $L \to 2L_+$), as well as the asymptotic expression for δu, remain unchanged in the general case, where the force is applied not to the center, but at any distance L_+ in the direction of the force action from the undeformable wall. The boundary conditions on the opposite side and, generally speaking, beyond the compression region affect only the adjustment structure at distances of about the sample sizes. This work was supported by the German-Israeli Foundation and the Russian Foundation for Basic Research (project no. 09-02-00483).

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