On the effect of random inhomogeneities in Kerr-media modelled by non-linear Schrödinger equation

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Abstract.

We consider propagation of optical pulses under the interplay of dispersion and Kerr non-linearity in optical fibres with impurities distributed at random uniformly on the fibre. By using a model based on the non-linear Schrödinger equation we clarify how such inhomogeneities affect different aspects such as the number of solitons present and the intensity of the signal. We also obtain the mean distance for the signal to dissipate to a given level.

PACS numbers: 42.81.Dp, 42.65.Sf, 02.50.Ey

1. Introduction

In this article we consider the evolution of a complex electric field $u(x,t)$ in a non-linear Kerr media which has constant dispersion and losses and, in addition, impurities at certain points x_n, $x_n < x_{n+1}$, which *occur randomly* on the fibre. We suppose that these loss elements cause the "input" signal $u(x_n^-,t)$ to abruptly decrease to an "output" value $u(x_n^+,t) = e^{-\gamma_n} u(x_n^-,t)$, where $e^{-\gamma_n} < 1$ measures the dimming ratio and $u(x_n^-,t)$, say, denotes the limit value from the left. Assuming the validity of the self-focusing non-linear Schrödinger (NLS) equation as a model of ideal transmission [1] we find that the above situation must be described by a perturbed NLS equation which written in dimensionless units reads

$$iu_x + u_{tt} + 2|u|^2 u = i\left[-\Gamma u + \sum_n (e^{-\gamma_n} - 1)\delta(x - x_n)u(x_n^-,t)\right], \qquad (1)$$

where the Dirac-delta terms account precisely for the amplitude decrease at impurities; further $\Gamma \geq 0$ is the normalized loss coefficient. For the sake of avoiding extra mathematical difficulties we do not consider a compensated loss mechanism; this will be the subject of a future publication. We also remark that with minor changes our results may be applicable to other physically interesting systems such as Bose-Einstein condensates.

It appears that while the effect of *continuous* random noise —or white noise— on NLS solitons has been well studied in the literature (see [2, 3, 4, 5, 6]) far less is known as regards the effects of sudden, discrete random perturbations. We intend to clarify how these inhomogeneities —which may be relevant for long-distance fibre-optic communication systems— affect the evolution of the pulse. We remark that perturbations involving delta masses also appear related to erbium-doped amplifiers and dispersion management, see [7, 8, 9]. In such a context, the positions of the amplifiers x_n are *deterministic* and periodically disposed, $x_n \equiv nx_1$, while the strengths are constant and negative, $\gamma_n = -\Gamma x_1$. Kodama and Hasegawa [10] generalize the latter ideas to a random context but, unlike us, maintain the amplifier interpretation and consider the distribution of the "intensity" of the signal only in the limit when both Δ_n (here $\Delta_n \equiv x_n - x_{n-1} > 0$ is the distance between impurities) and γ_n tend to zero. Thus while these ideas have some bearing with our work both the physical interpretation and the mathematical model are quite different.

We will start our analysis of equation (1) by considering that there are no deterministic losses, $\Gamma = 0$, since this case is simpler from a mathematical viewpoint: We show that upon performing a change of dependent variable the resulting formula can be piecewise related to the unperturbed NLS equation. Let us recall here that the classical NLS equation

$$i\Theta_x + \Theta_{tt} + 2|\Theta|^2\Theta = 0, \ \ \Theta(0,t) = \varphi(t), \qquad (2)$$

was first derived by Zakharov [11] as an equation of slowly varying wave packets of small amplitude. He showed that despite its non-linear character the corresponding initial value problem (IVP) can be reduced to a linear problem (the Zakharov-Shabat spectral problem) by the so called inverse scattering transform (IST) —see [12, 13] for general background on NLS equation and the IST method. Its interest has been further underlined by the realization that it also models the evolution of the complex amplitude of an optical pulse in a non-linear fibre [1]. Applications of NLS equation to optical communications and photonics are nowadays standard [1, 7, 14, 15]. We devote section 2 to the study of the non-linear dynamics of the classical solitary waves within this regime, and we will show how impurities result in the appearance of radiation and general broadening of the signal. In particular, we find that solitons may be destroyed by the action of just one impurity.

When $\Gamma \neq 0$ equation (1) is no longer solvable in analytic way by IST; however we find —see section 3— that the evolution of intensity, momentum and position of the pulse can be described precisely and that, under certain natural assumptions, their average values *decrease exponentially* due to the "impurities": concretely, we suppose that positions and strengths of impurities are statistically independent between themselves; we also suppose that in any interval $[0, x]$ impurities are uniformly distributed (provided its number is given). Nevertheless the frequency and position of the pulse are not affected.

In section 4 we study the mean distance for the signal's intensity to attenuate to a given level due to the impurities. In applications, this level could be a recommended threshold value for reliability, say. To this end we formulate a linear integral equation that this distance satisfies and, by means of a Laplace transform, solve it. Results are discussed.

2. Method of solution and the loss-less case

Here we solve (1) given arbitrary sequences x_n and γ_n with $0 < x_n < x_{n+1}$ and $\gamma_n > 0$. We perform the change of variable $u(x,t) = \zeta(x)v(x,t)$ where we require that $\zeta(x)$ depends *only on space and has jump discontinuities* at x_n and that $v(x,t)$ *be continuous.* By substitution we find that these functions must solve the equations

$$iv_x + v_{tt} + \zeta^2|v|^2 v = 0, \tag{3}$$

$$\frac{d\zeta(x)}{dx} + \Gamma\zeta(x) + \sum_n (1 - e^{-\gamma_n})\delta(x - x_n)\zeta(x_n^-) = 0. \tag{4}$$

It follows that $\zeta(x)$ is continuous on the intervals (x_n, x_{n+1}) wherein it solves equation (4) with no delta terms; further, it has jump discontinuities at the random points $x = x_n$ at which $\zeta(x_n^+) = e^{-\gamma_n}\zeta(x_n^-)$. Hence if $N(x)$ is the number of defects on $[0, x]$ we have that

$$\zeta(x) = e^{-S(x)}, \text{ where } S(x) \equiv s + \Gamma x + \sum_{j=1}^{N(x)} \gamma_j. \tag{5}$$

Alternatively, $S(x) = s + \Gamma x + \sum_{j=1}^{n} \gamma_j$, if $x_n \leq x < x_{n+1}$. (By contrast, for erbium-doped fibre amplifiers $S(x) = -\Gamma x - n\Gamma x_1$ if $nx_1 \leq x < (n+1)x_1$, see [8].) Thus $S(x)$ is a piece-wise linear function with initial value s and jumps at the random points $x = x_n$, i.e., a pure random point process with drift, well known in the physics literature. For convenience we take $s = 0$ hereafter, and until section 4. There we will need to consider a more general situation where the starting value $S(0)$ is free. In figure 1 we plot a sample of both $S(x)$ and $\zeta(x)$ for a particular choice of the parameter set under this assumption.

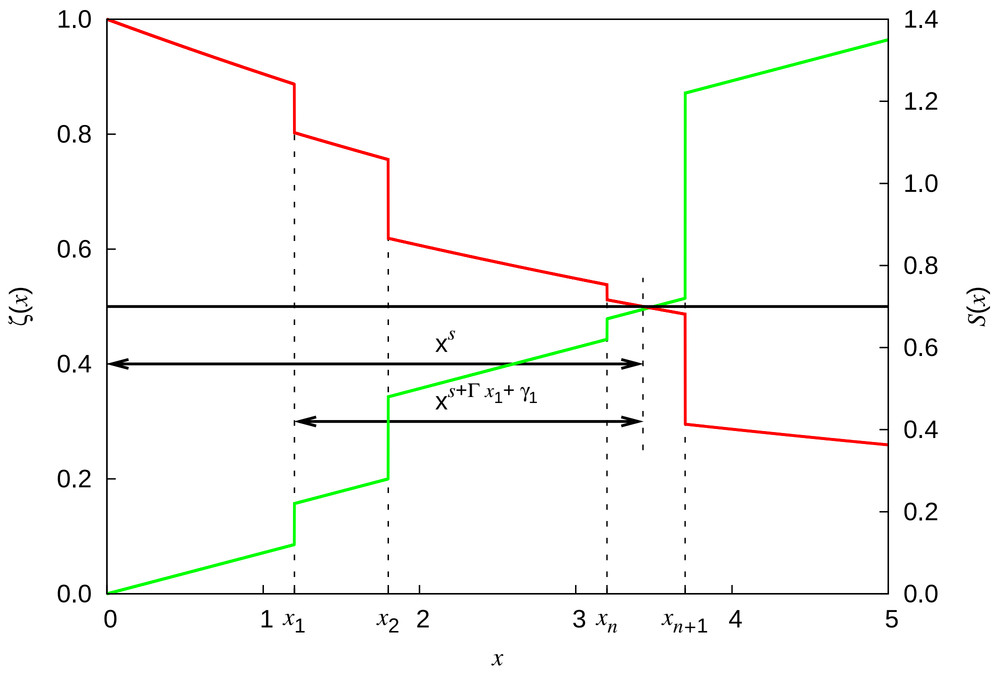

Figure 1. A sample path of $\zeta(x)$ (in red) and $S(x)$ (green line) versus distance (in Km) showing the distance **x** for the energy to dissipate to half its initial value. We take a fibre with mean impurities distance $< \Delta_n >= \lambda^{-1} = 1$ km, a loss rate 0.02dB/Km and dispersion distance 50 km, i.e., $\Gamma = 0.1$ —which accounts for the seemingly linear behaviour between jumps.

We shall now focus our attention in equation (3). We first consider the simpler case when the loss vanishes: $\Gamma = 0$. It turns out that, even though the resulting equation has random discontinuous coefficients, it can be piecewise reduced to an integrable equation whereupon we show how to obtain the evolution of an initial pulse (see [16] for related considerations). The reasoning in the rest of this section is essentially independent of the sequences $\Delta_n \equiv x_n - x_{n-1}$ and γ_n. Nevertheless we shall suppose that both are sequences of *positive, independent, equally distributed* random variables and that Δ_n and γ_m are also independent for all n, m. Note that all these assumptions are physically well founded as they imply, say, that the knowledge of the position of a given impurity does not provide any information on the location of the remaining ones. The further assumption that Δ_n is *exponentially distributed:* $\Pr\Big(\Delta_n \geq x\Big) = e^{-\lambda x}$ where $\lambda \equiv < \Delta_n >^{-1}$ is a certain parameter, is natural from physical principles. It has several fruitful consequences as then there follows that the number $N(x)$ of impurities that occur on $[0, x]$ has Poisson distribution with parameter λx and that they are uniformly distributed on the interval. It further implies the memory-less property: the distribution of impurities on $(x, x + \Delta x]$ remains unaffected given that none was observed on $[0, x]$. By contrast, we consider here a general probability density function (PDF) $h(y)$ of γ_n: $\Pr(y < \gamma_n \leq y + dy) = h(y)dy$.

For the sake of being specific let us consider the case when the initial data is just a solitary wave pulse: $\upsilon(0, t) = 2\eta \operatorname{sech}(2\eta t)\, e^{2i\xi t} \equiv \varphi^{(0)}(t)$ where the real parameters η and ξ give, up to a constant, the wave's amplitude and the carrier velocity. † Note that up to the first impurity $\upsilon^{(0)}(x, t) \equiv \upsilon(x, t), 0 \leq x \leq x_1$, solves the IVP

$$i\upsilon_x^{(0)} + \upsilon_{tt}^{(0)} + 2|\upsilon^{(0)}|^2\upsilon^{(0)} = 0,\ \upsilon^{(0)}(0, t) = \varphi^{(0)}(t). \tag{6}$$

This is the standard IVP for NLS equation and hence the solution for $0 \leq x \leq x_1$ is the classical soliton

$$u(x, t) = \upsilon^{(0)}(x, t) = 2\eta \operatorname{sech}(2\eta(t - 4\xi x))\, e^{i\left[2\xi t + 4(\eta^2 - \xi^2)x\right]}. \tag{7}$$

As commented, we continue this solution to the interval $x_1 \leq x \leq x_2$ by requiring $\upsilon(x, t)$ to be continuous at $x = x_1$. This requirement fixes $\upsilon^{(1)}(x, t) \equiv \upsilon(x, t)$, $x_1 \leq x \leq x_2$, to satisfy the non-linear partial differential equation

$$\begin{aligned} &i\upsilon_x^{(1)} + \upsilon_{tt}^{(1)} + 2e^{-2\gamma_1}|\upsilon^{(1)}|^2\upsilon^{(1)} = 0,\ \text{with} \\ &\upsilon^{(1)}(x_1, t) = 2\eta \operatorname{sech}(2\eta(t - 4\xi x_1))\, e^{i\left[2\xi t + 4(\eta^2 - \xi^2)x_1\right]}. \end{aligned}$$

Remarkably *this equation can be reduced again to NLS*: by using both temporal and translational invariance of NLS equation one can prove that

$$e^{\gamma_1} u(x, t) = \upsilon^{(1)}(x, t) = e^{4i(\eta^2 + \xi^2)x_1 + \gamma_1}\Theta(x - x_1, t - 4\xi x_1),$$

where $\Theta(\cdot, \cdot)$ is the solution to the NLS equation (2), with data at $x = 0$ given by $\Theta(0, t) = e^{-\gamma_1}\varphi^{(0)}(t)$. Notice that, unlike $\upsilon(x, t)$, $u(x, t)$ is not continuous at $x = x_1$.

The determination of the specific form of the solution requires solving a linear spectral problem. The procedure is awkward but fortunately the solution's main features may to a large extent be determined avoiding these complexities. We note that if $\gamma_1 \neq 0$ the solution that evolves from data $\Theta(0, t) = e^{-\gamma_1}\varphi^{(0)}(t)$ is no longer a soliton but a complicated pulse that may contain radiation, in addition to the soliton. The former component has a much weaker rate of decay than the later; concretely, it decays as the corresponding solution for the linearized Schrödinger equation (i.e. as $t^{-1/2}$, see [12]). Further, if $\gamma_1 \gtrsim 1.41$ the arriving soliton at $x = x_1$ —cf. equation (7)— is destroyed by the action of the fist impurity after x_1. ‡ Hence the resulting configuration for $x > x_1$ consists *solely of radiation*. To be specific, suppose that the jump PDF $h(\cdot)$ has exponential distribution with mean $1/\sigma$. Then, after the first impurity the soliton disappears with probability bounded below by $\Pr(\gamma_1 \geq y) = e^{-\sigma y}$, where $y = 1.41$.

† We adopt the convention and terminology of standard NLS theory wherein t is space and x a temporal variable, a situation opposite to that that occurs in Optics.

‡ This stems from the fact that the condition $\Xi^2 I_0(2\Xi) < 1$ on the initial data guarantees that no solitons will be formed upon evolution [12, 17]. Here $I_0(\cdot)$ denotes the modified Bessel function of zero order and $\Xi \equiv \int_{-\infty}^{\infty} |\Theta(0, t)|dt = \pi e^{-\gamma_1}$.

Finally, we mention that by using similar ideas one can extend the solution to $x > x_n$ by solving (3) with data $\upsilon^{(n-1)}(x_n^-, t)$, where as before $\upsilon^{(n)}(x,t)$ denotes the general solution $\upsilon(x,t)$ restricted on $x_{n-1} \leq x \leq x_n$. Translation invariance allows one to reduce this to NLS equation with new data which involves a contraction factor $e^{-(\gamma_1+\ldots+\gamma_n)}$. Eventually, this dimming of the initial signal results in a disappearance of the starting solitons into radiation, an indication that, as a result of impurities, broadening of the signal takes place. We skip the mathematical details.

3. General case with deterministic loss and impurities

When $\Gamma > 0$ equation (3) can be mapped into the so called dispersion-managed NLS equation, which, unfortunately, is not solvable in analytic way, neither by using IST nor by any other method. It is then remarkable that *the evolution of the main physically observable functionals can be discerned in an exact way.* Consider the following quantities

$$
\begin{aligned}
M(x) &\equiv \int_{-\infty}^{\infty} |u(x,t)|^2 \, dt, \\
P(x) &\equiv i \int_{-\infty}^{\infty} \bar{u}(x,t) u_t(x,t) \, dt, \text{ and} \\
Q(x) &\equiv \int_{-\infty}^{\infty} t |u(x,t)|^2 \, dt,
\end{aligned}
$$

where $M(x)$ and $P(x)$ are the (accumulated) intensity and momentum of the signal at a position x, while $Q(x)/M(x) \equiv T(x)$ is the pulse position. The functional $P(x)/M(x) \equiv \Omega(x)$ is interpreted as the pulse-centre frequency. The singular nature of the delta terms prevent us from determining the relevant evolution by manipulating equation (1). Nevertheless, one can rely again in the decomposition $u(x,t) = \zeta(t)\upsilon(x,t)$ and use equation (3). Then, proper manipulation of the latter expressions yields that

$$
\begin{aligned}
M(x) &= M(0) e^{-2S(x)}, \\
P(x) &= P(0) e^{-2S(x)}, \text{ and} \\
Q(x) &= [Q(0) - 2P(0)x] \, e^{-2S(x)}.
\end{aligned}
$$

Thus the effect of the presence of impurities results in the addition of a multiplicative *random factor* $e^{-2S(x)}$ in both intensity and momentum. Note however that $\Omega(x) = \Omega(0)$ and $T(x) = [T(0) - 2\Omega(0)x]$, and hence that inhomogeneities have no effect whatsoever on position and frequency, a fact that accords with the physical intuition.

It is therefore of interest to evaluate the mean amplitude's decrease. We do so by first assuming that previously n defects have occurred: $N(x) = n$. Let $\mathbb{E}$ denote statistical averaging and $\mathbb{E}\Big(\zeta^2(x) | N(x) = n\Big)$ be the mean value of $\zeta^2(x)$ knowing that exactly n jumps have occurred on $[0, x]$. Note that given this information one has $S(x) = \Gamma x + \sum_{j=1}^n \gamma_j$: i.e., only the uncertainty regarding the value of the γ_j's remains but not that associated with the number of summands $N(x)$. In view of the assumed statistical independence we have that the mean factorizes as

$$\mathbb{E}\left(\zeta^2(x)\big|\, N(x)=n\right) = \mathbb{E}\left(e^{-2\Gamma x}\prod_{j=1}^{n} e^{-2\gamma_j}\right)$$

$$= e^{-2\Gamma x}\prod_{j=1}^{n}\mathbb{E}\Big(e^{-2\gamma_j}\Big) = e^{-2\Gamma x}Q_2^n,$$

where $Q_r \equiv \mathbb{E}\left[\exp\left(-r\gamma_j\right)\right] = \int_0^\infty e^{-ry}h(y)dy < 1$ is the Laplace Transform of the jump-size PDF. The mean intensity is obtained by further averaging with respect to the number of impurities:

$$\mathbb{E}[M(x)] = M_0\mathbb{E}[\zeta^2(x)]$$

$$= M_0\sum_{n=0}^{\infty}\frac{(\lambda x)^n e^{-\lambda x}}{n!}\mathbb{E}\left(\zeta^2(x)\big|\, N(x)=n\right) = M_0 e^{-[2\Gamma+\lambda(1-Q_2)]x}, \tag{8}$$

where $M(0) \equiv M_0$ and we used that if Δ_j has exponential distribution, i.e., if $\Pr(\Delta_j \geq x) = e^{-\lambda x}$ for some $\lambda > 0$, then $N(x)$, the number of defects on $[0, x]$, is Poisson distributed: $\Pr\left(N(x) = n\right) = (\lambda x)^n e^{-\lambda x}/n!$. Hence we obtain that the existence of defects implies an additional exponential decrease in the field's intensity and momentum at a rate $2\lambda(1-Q_2)$, an effect which might result in the degradation of the bit patterns.

4. Mean half life

A natural related problem of interest is determining the distance $\mathbf{x}$ at which $M(x)$ dissipates from a starting value M_0 to a given level M_1, i.e., such that $M(\mathbf{x}) = M_1$. For convenience we set $M_1 \equiv M_0 e^{-2b}$ and hence require $S(\mathbf{x}) = b$. This distance could be considered as a threshold value below which the signal is no longer reliable (it gives the mean half life of the signal if $M_0 = 2M_1$). In the deterministic case ($\lambda = 0$) this distance follows inverting $M_1 = M_0 \exp\left(-2\Gamma\mathbf{x}\right)$ as $\mathbf{x} = \frac{1}{2\Gamma}\log\frac{M_0}{M_1}$. When inhomogeneities are present $\mathbf{x}$ is a random variable whose mean *is not obtained* by inverting equation (8) —as it might have been naively thought. Instead, we reason as follows: call $\mathbf{x}^s$, see figure 1, the (random) distance that takes for the generalized process $S(x)$ with initial value $S(0) = s$ —cf. equation (5)— to go beyond the level b. It turns out that $\mathbb{X}(s) \equiv \mathbb{E}(\mathbf{x}^s)$ satisfies the *linear integral equation*

$$\mathbb{X}(s) = \frac{1-e^{-\lambda\varrho}}{\lambda} + \frac{\lambda}{\Gamma}\int_0^{b-s} dl e^{\frac{\lambda}{\Gamma}(s+l-b)}\int_0^l dy\mathbb{X}(y+b-l)h(y), \tag{9}$$

where $\varrho \equiv \frac{b-s}{\Gamma}$ and we recall that $h(x)$ is the density of γ_n. *

* We sketch the derivation of this integral equation (see [18] for a similar derivation in a financial context). With $S(0) = s$ there are three possibilities for the future evolution: If the first jump satisfies $x_1 > \varrho$ then S reaches the level b at $x = \varrho$. If this is not the case and if the jump at x_1 satisfies $s + \Gamma x_1 + \gamma_1 \geq b$ then the process goes past b at $x = x_1$. Otherwise the process still remains within $[0, b)$ at $x = x_1$ and starts afresh with an initial value $S(x_1) = s + \Gamma x_1 + \gamma_1 < b$ (hence the process will

This equation can be solved in a closed form by Laplace transformation. We consider again the case corresponding to a jump PDF also exponential with mean $\sigma^{-1} \equiv < \gamma_n >$, i.e., $h(x) = \sigma e^{-\sigma x}$ where $\sigma > 0$. If $\kappa = \lambda + \sigma\Gamma$, Laplace transformation yields the solution to (9) as

$$\mathbb{X}(s) = \frac{\sigma\Gamma\varrho}{\kappa} + \frac{\lambda}{\kappa^2}\Big(1 - e^{-\kappa\varrho}\Big).$$

The mean distance for the amplitude to decrease to M_1 follows letting $s = 0$ and $b = \frac{1}{2}\log\frac{M_0}{M_1}$ as

$$\mathbb{E}(\mathbf{x}) \equiv \mathbb{X}(0) = \frac{1}{2(\Gamma + \lambda/\sigma)}\log\frac{M_0}{M_1} + \frac{\lambda}{\kappa^2}\left[1 - \left(\frac{M_1}{M_0}\right)^{\frac{\kappa}{2\Gamma}}\right]. \quad (10)$$

If $\lambda = 0$ we recover the deterministic limit above: $\mathbf{x} = \frac{1}{2\Gamma}\log\frac{M_0}{M_1}$. Note how the incorporation of impurities corrects this formula in a significant way, cf. equation (10). Another interesting limit is that of vanishing deterministic loss rate, $\Gamma = 0$. The mean attenuation distance can only be accounted to the presence of impurities and reads $\mathbb{E}(\mathbf{x}) = \frac{1}{\lambda} + \frac{\sigma}{2\lambda}\log\frac{M_0}{M_1}$. The first term is the mean time for the first jump at x_1 to happen; the logarithmic correction corresponds to the mean time to go beyond the level b after the first jump. Actually, this rate rules the mean dissipation distance whenever $M_0 >> M_1$ and $\lambda/\sigma >> \Gamma$. In figure 2 we perform a plot of this function. Note how, by contrast, the distance implied inverting equation (8), namely

$$\mathbb{E}(M(x)) = M_0 \exp\left[-2x\left(\Gamma + \frac{\lambda}{\sigma + 2}\right)\right], \text{ and therefore} \quad (11)$$

$$\mathbf{x} = \left[2\left(\Gamma + \frac{\lambda}{\sigma + 2}\right)\right]^{-1}\log\frac{M_0}{M_1}, \quad (12)$$

deviates from the correct result, equation (10), and fails to capture the sharp behaviour occurring for $M_1 \approx M_0$. The error increases as Γ decreases.

5. Conclusions

We have analyzed how the existence of randomly distributed impurities affects the evolution of an optical pulse in a non-linear Kerr media with constant dispersion and loss. We suppose that the unperturbed situation is described by NLS equation. When the deterministic loss vanishes it is shown by changing the dependent variable that the resulting equation can be piecewise related to the unperturbed NLS equation. The effect of impurities in the non-linear propagation is pinpointed. In particular we address the issue of how they affect the initial solitons and the possibility to dissipate them

exit $[0, b)$ at $x_1 + \mathbf{x}^{s+\Gamma x_1+\gamma_1}$). Upon appropriate rearrangement this reasoning leads to

$$\mathbf{x}^s = \varrho\theta(x_1 - \varrho) + x_1\theta(\varrho - x_1) + \mathbf{x}^{s+\Gamma x_1+\gamma_1}\theta(b - s - \Gamma x_1 - \gamma_1)\theta(\varrho - x_1).$$

Averaging this relationship yields with further manipulations equation (9).

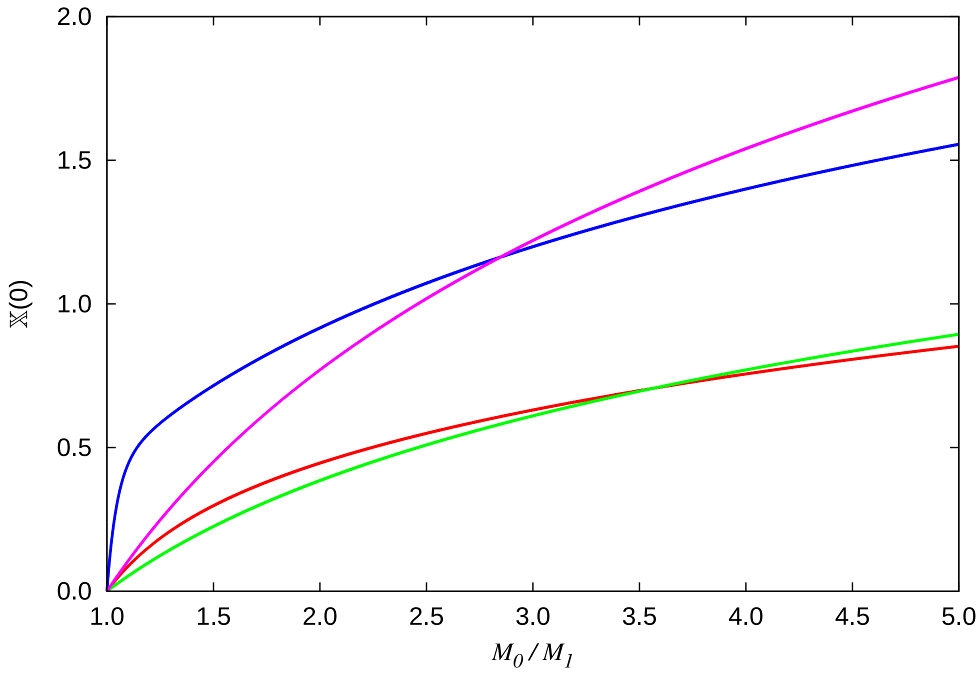

Figure 2. Mean distance in terms of M_0/M_1 for $\lambda = 2.0$, $\sigma = 3.0$ while $\Gamma = 0.5$ (red line) and $\Gamma = 0.05$ (blue one) as follows from (10). Note how in the latter case $\mathbb{X}(0)$ jumps an amount $< \Delta_n >= 0.5$ right after the origin. The green and magenta curves are the (incorrect) mean distances implied by equation (12) with the above parameters.

into radiation. In the general, non-solvable $\Gamma \neq 0$ case we show that while impurities do not influence the frequency and position of the signal they induce an exponential decrease of the main physical observables intensity and momentum and hence a general degradation. We also determine the mean half life or mean distance for the signal to dissipate to a given threshold value. We find that this distance satisfies a certain integral equation. Its analysis shows that impurities result in an important decrease in the mean dissipation distance. To overcome these effects the addition of amplifiers is in order. The introduction of such a device and the relevant statistical implications will be considered in a future publication.

Acknowledgments

We appreciate conversations with Prof. Y. Kodama and M. A. Ablowitz, which have helped enhance this article. The authors acknowledge support from MICINN under contracts No. FIS2008-01155-E, FIS2009-09689, and MTM2009-09676; from Junta de Castilla-León, SA034A08; and Generalitat de Catalunya, 2009SGR417.

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